KYUNGPOOK Math. J. 50(2010), 315-327

Hyers-Ulam Stability of Cubic Mappings in Non-Archimedean Normed Spaces

Alireza Kamel Mirmostafaee
Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad, Iran
e-mail : mirmostafaei@ferdowsi.um.ac.ir

Abstract. We give a fixed point approach to the generalized Hyers-Ulam stability of the cubic equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

in non-Archimedean normed spaces. We will give an example to show that some known results in the stability of cubic functional equations in real normed spaces fail in non-Archimedean normed spaces. Finally, some applications of our results in non-Archimedean normed spaces over p-adic numbers will be exhibited.

1. Introduction

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940, S.M. Ulam [28] posed the first stability problem. In the next year, D. H. Hyers [8] gave a partial affirmative answer to the Ulam's problem. The theorem of Hyers was generalized by T. Aoki [1] and Bourgin [3]. In 1978, Th. M. Rassias [27] provided a remarkable generalization of Hyers's result by allowing the Cauchy difference to be unbounded. In 1994, a generalization of Rassias' theorem was obtained by P. Găvruta [6] by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x,y)$. Several stability results have been recently obtained for various equations, also for mappings with more general domains and ranges (see e.g. [9, 10]).

The functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

is called the *cubic functional equation* because $f(x) = cx^3$ is a solution of the equation. Every solution of the cubic functional equation is said to be a *cubic mapping*.

Received October 29, 2009; accepted April 23, 2010.
2000 Mathematics Subject Classification: 39B82; Secondary 39B52, 46S10, 39B62.
Key words and phrases: Hyers-Ulam-Rassias stability, Cubic functional equation, fixed point alternative, non-Archimedean normed space.
This research was supported by a grant from Ferdowsi University of Mashhad; (No. MP88114MIM).

Jun and Kim in [13] have shown that a function $f : X \to Y$ satisfies (1.1) if and only if there exists a function $B : X^3 \to Y$ such that $f(x) = B(x, x, x)$, where B is symmetric for each fixed variable and it is additive for fixed two variables. They also proved the following:

Theorem 1.1. *Let X be a real vector space and Y be a Banach space, let*

$$||f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)|| \le \varphi(x,y) \quad (x \in X)$$

where $\varphi : X^2 \to [0,\infty)$ satisfies $\sum_{i=0}^{\infty} \frac{\varphi(2^i x,0)}{8^i} < \infty$ and $\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{8^n} = 0$ for all $x, y \in X$. Then $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n}$ defines a unique cubic mapping from X to Y which satisfies (1.1) *and the inequality*

$$||f(x) - T(x)|| \le \frac{1}{16} \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 0)}{8^i} \quad (x \in X).$$

The stability of some types of the cubic equations has been considered by some mathematicians [11, 14, 17, 22, 26].

In 1897, Hensel [7] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. He indeed introduced a field with a valuation normed which does not have the Archimedean property. Although many results in the classical normed space theory have a non-Archimedean counterpart, but their proofs are different and require a rather new kind of intuition. One may note that $|n| \le 1$ in each valuation field, every triangle is isosceles and there may be no unit vector in a non-Archimedean normed space; cf. [23]. These facts show that the non-Archimedean framework is of special interest.

L. M. Arriola and W. A. Beyer in [2] initiated the stability of functional equations in non-Archimedean spaces. In fact they established stability of Cauchy functional equations over p-adic fields. The stability of some other functional equations in non-Archimedean normed spaces have been investigated by some mathematicians (see e.g. [12, 15, 16, 19, 20]). In particular, Moslehian et al. [21] have proved the stability of cubic functional equations in non-Archimedean normed spaces. In fact they showed that:

Theorem 1.2. *Let X be an abelian (additive) group and let Y be a complete non-Archimedean normed space. Let $\varphi : X \times X \to [0,\infty)$ be a function such that $\lim_{n\to\infty} \frac{\varphi(2^n x,\ 2^n y)}{|8|^n} = 0$ for each $x, y \in X$ and $f : X \to Y$ be a mapping satisfying*

$$||f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)|| \le \varphi(x,y) \quad (x,y) \in X \times X.$$

Then $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n}$ defines a unique cubic mapping such that

$$||f(x) - T(x)|| \le \frac{1}{|16|} \sup\{\frac{\varphi(2^j x, 0)}{|8|^j} : j \in \mathbb{N}\} \quad (x \in X).$$

In 2003, Radu [25] employed the fixed point method to prove the stability of Cauchy additive functional equations. Since then several authors have applied this method to investigate the stability of some functional equations, see e. g. [4, 11, 18, 24].

In this paper, we apply non-Archimedean fixed point alternative theorem to give a new approach to the Hyers-Ulam stability of cubic functional equations in non-Archimedean normed spaces. The theme of this papers goes as follows.
In Section 2, we introduce some preliminaries results which will be used in the sequel. In Section 3, we use fixed point method to prove stability of the cubic functional equation in non-Archimedean normed spaces. Finally, in Section 4, some applications of our results have been illustrated. In particular, we give an example to show that our method is different from the one used in [21]. This example also shows that the exact form of some results about the stability of cubic functions in real normed spaces may fail in non-Archimedean normed spaces.

2. Preliminaries

We begin with the definition of a non-Archimedean field and a non-Archimedean normed linear space. Then we give the non-Archimedean version of fixed point alternative principle.

Definition 2.1. Let $\mathbb{K}$ be a field. A non-Archimedean valuation on $\mathbb{K}$ is a function $|\ |:\mathbb{K}\to\mathbb{R}$ such that for any $a,b\in\mathbb{K}$ we have

(i) $|a|\geq 0$ and equality holds if and only if $a=0$,

(ii) $|ab|=|a||b|$,

(iii) $|a+b|\leq\max\{|a|,|b|\}$.

The condition (iii) is called the strong triangle inequality. By (ii), we have $|1|=|-1|=1$. Thus, by induction, it follows from (iii) that $|n|\leq 1$ for each integer n. We always assume in addition that $|\ |$ is non trivial, i.e.,

(iv) there is an $a_0\in\mathbb{K}$ such that $|a_0|\neq 0,1$.

Definition 2.2. Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\ .\ |$. A function $||\ .\ ||:X\to\mathbb{R}$ is a non-Archimedean norm (valuation) if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely,

$$||x+y||\leq\max\{||x||,||y||\}\quad(x,y\in X).$$

Then $(X,||\ .\ ||)$ is called a *non-Archimedean space.*

By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Remark 2.3. Thanks to the inequality

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all x and $y > 0$, there exists an integer n such that $x < ny$.

Example 2.4. Let p be a prime number. For any nonzero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p, define the p-adic absolute value $|a|_p = p^{-r}$. Then $|\ |$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\ |$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field.
Note that if $p \geq 3$, then $|2^n| = 1$ in for each integer n.

Definition 2.5. Let X be a nonempty set and $d : X \times X \to [0, \infty]$ satisfy the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$(symmetry),

(iii) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$(strong triangle inequality),

for all $x, y, z \in X$. Then (X, d) is called a generalized non-Archimedean metric space. (X, d) is called complete if every d-Cauchy sequence in X is d-convergent.

Example 2.6. Let X and Y be two non-Archimedean spaces over a non-Archimedean field $\mathbb{K}$. If Y has a complete non-Archimedean norm over $\mathbb{K}$ and $\psi : X \to [0, \infty)$, for each $f, g : X \to Y$, define

$$d(f, g) = \inf\{\alpha > 0 : ||f(x) - g(x)|| \leq \alpha\psi(x)\ \forall x \in X\}.$$

Then an easy computation, similar to the proof of [4, Theorem 2.5], shows that d defines a generalized non-Archimedean complete metric on $\mathcal{F} = \{f | f : X \to Y;\ f(0) = 0\}$.

Theorem 2.7. (Non-Archimedean Alternative Contraction Principle) *If (X, d) is a non-Archimedean generalized complete metric space and $J : X \to X$ a strictly contractive mapping (that is $d(J(x), J(y)) \leq Ld(y, x)$, for all $x, y \in X$ and a Lipschitz constant $L < 1$), then either*

(i) *$d(J^n(x), J^{n+1}(x)) = \infty$ for all $n \geq 0$, or*

(ii) *there exists some $n_0 \geq 0$ such that $d(J^n(x), J^{n+1}(x)) < \infty$ for all $n \geq n_0$;*

the sequence $\{J^n(x)\}$ is convergent to a fixed point x^* of J; x^* is the unique fixed point of J in the set

$$\mathcal{Y} = \{y \in X : d(J^{n_0}(x), y) < \infty\}$$

and $d(y, x^*) \leq d(y, J(y))$ *for all* y *in this set.*

Proof. The proof of similar theorem in [5] can be applied to show that in case (ii), $J|\mathcal{Y}$ has a unique fixed point x^* in $\mathcal{Y}$ such that for each $y \in \mathcal{Y}$, $\{J^n(y)\}$ converges to x^*. By the strong triangle inequality for all $y \in \mathcal{Y}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y, J^n(y)) &\leq \max\{d(y, J(y)), \ldots, d(J^{n-1}(y), J^n(y))\} \\ &\leq \max\{d(y, J(y), \ldots, L^{n-1}d(y, J(y))\} \\ &= d(y, J(y)). \end{aligned}$$

From this the last inequality of the Theorem follows. □

3. Non-Archimedean stability of cubic functional equation

Hereafter, we will assume that X and Y are non-Archimedean spaces over a non-Archimedean field $\mathbb{K}$.
Let $f : X \to Y$ be a mapping, we define

$$Cf(x, y) = f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x) \quad (x, y \in X).$$

Suppose that $\varphi : X \times X \to [0, \infty)$, then $f : X \to Y$ is said to be φ-approximately cubic if

$$||Cf(x, y)|| \leq \varphi(x, y) \quad (x, y \in X). \tag{3.1}$$

Let θ map X into $[0, \infty)$, we call a mapping $f : X \to Y$, θ-approximately odd if

$$||f(x) + f(-x)|| \leq \theta(x) \quad (x \in X).$$

By putting $x = 0$ in (3.1), we get to the following observation:

Lemma 3.1. *Let* $\varphi : X \times X \to [0, \infty)$. *If* $f : X \to Y$ *is a* φ*-approximately cubic function such that* $f(0) = 0$, *then* f *is* $\varphi(0, .)$*-approximately odd.*

Theorem 3.2. *Let* $\varphi : X \times X \to [0, \infty)$ *and* $f : X \to Y$ *be a* φ*-approximately cubic function. If* Y *is complete and for some integer* $k \in \mathbb{K}$ *and* $0 < L < 1$,

$$|k|^3\varphi(k^{-1}x, k^{-1}y) \leq L\ \varphi(x, y) \tag{3.2}$$

for all $x, y \in X$. *Then there exists a unique cubic mapping* $c : X \to Y$ *such that*

$$||f(x) - f(0) - c(x)|| \leq \Psi_k(k^{-1}x) \quad (x \in X), \tag{3.3}$$

where

$$\Psi_2(x) = \frac{1}{|2|}\max\{\varphi(x, 0), \varphi(0, 0)\}, \quad \Psi_3(x) = \max\{\varphi(x, x), \varphi(x, 0), \varphi(0, 0)\} \tag{3.4}$$

and for $j > 3$,

$$\Psi_j(x) = \max\Big\{\varphi(x,(j-2)x),\ldots,\varphi(x,x),\varphi(x,0),\varphi(0,0),\varphi(0,x),\ldots,\varphi(0,(j-3)x)\Big\} \tag{3.5}$$

for all $x \in X$.

Proof. Let $f_1(x) = f(x) - f(0)$ for each $x \in X$. We will show that for every $j \geq 2$,

$$||f_1(jx) - j^3 f_1(x)|| \leq \Psi_j(x) \quad (x \in X). \tag{3.6}$$

Replacing f by f_1 in (3.1), we obtain

$$||Cf_1(x,y)|| \leq \max\{\varphi(x,y),\varphi(0,0)\} \quad (x \in X). \tag{3.7}$$

Put $y = 0$ in (3.7) to obtain

$$||2f_1(2x) - 16 f_1(x)|| \leq \max\{\varphi(x,0),\varphi(0,0)\} \quad (x \in X). \tag{3.8}$$

Hence for each $x \in X$, $||f_1(2x) - 2^3 f_1(x)|| \leq \Psi_2(x)$. Replacing y by x in (3.7), we get to the following inequality

$$||f_1(3x) + f_1(x) - 2f_1(2x) - 12 f_1(x)|| \leq \max\{\varphi(x,x),\varphi(0,0)\} \quad (x \in X). \tag{3.9}$$

Since

$$f_1(3x) - 3^3 f_1(x) = f_1(3x) + f_1(x) - 2f_1(2x) - 12f_1(x) + 2f_1(2x) - 16 f_1(x) \quad (x \in X),$$

it follows from (3.8) and (3.9) that

$$||f_1(3x) - 3^3 f_1(x)|| \leq \max\{\varphi(x,x),\varphi(x,0),\varphi(0,0)\} = \Psi_3(x) \quad (x \in X). \tag{3.10}$$

Let (3.6) hold for $j = 3,\ldots,i$. Put $y = (i-1)x$ in (3.7) to obtain

$$\begin{aligned} &||Cf_1\big(x,(i-1)x\big)|| \\ &= ||f_1\big((i+1)x\big) + f_1\big(-(i-3)x\big) - 2f_1(ix) - 2f_1\big(-(i-2)x\big) - 12 f_1(x)|| \\ &\leq \max\{\varphi\big(x,(i-1)x\big),\varphi(0,0)\} \end{aligned} \tag{3.11}$$

for every x in X. Since for each $x \in X$,

$$\begin{aligned} &f_1\big((i+1)x\big) - f_1\big((i-3)x\big) - 2f_1(ix) + 2f_1\big((i-2)x\big) - 12 f_1(x) \\ &\quad = f_1\big((i+1)x\big) - (i+1)^3 f_1(x) - [f_1\big((i-3)x\big) - (i-3)^3 f_1(x)] \\ &\qquad - 2[f_1(ix) - i^3 f_1(x)] + 2[f_1\big((i-2)x\big) - (i-2)^3 f_1(x)] \end{aligned}$$

and by Lemma 3.1

$$f_1\big(-(i-3)x\big) - f_1\big((i-3)x\big)|| \leq \max\{\varphi\big(0,(i-3)x\big),\varphi(0,0)\},$$

$$f_1\big(-(i-2)x\big) - f_1\big((i-2)x\big)|| \leq \max\{\varphi\big(0,(i-2)x\big), \varphi(0,0)\},$$

(3.6) follows from (3.11) for $i+1$. Hence, by induction on i, (3.6) is proved. In particular,

$$||f_1(kx) - k^3 f_1(x)|| \leq \Psi_k(x) \quad (x \in X).$$

This is equivalent to

$$||f_1(x) - k^3 f_1(k^{-1}x)|| \leq \Psi_k(k^{-1}x) \quad (x \in X).$$

For every $g, h : X \to Y$, define

$$d(g,h) = \inf\{\alpha > 0 : ||g(x) - h(x)|| \leq \alpha\ \Psi_k(k^{-1}x), \quad \forall x \in X\}. \tag{3.12}$$

By Example 2.6, d defines a complete generalized non-Archimedean metric on $\mathcal{F} = \{g|g : X \to Y; g(0) = 0\}$. Let $J : \mathcal{F} \to \mathcal{F}$ be defined by $J(g)(x) = k^3 g(k^{-1}x)$ for all $x \in X$ and $g \in \mathcal{F}$. If for some $g, h \in \mathcal{F}$ and $\alpha > 0$,

$$||g(x) - h(x)|| \leq \alpha\ \Psi_k(k^{-1}x), \quad (x \in X),$$

then

$$\begin{aligned} ||J(g)(x) - J(h)(x)|| &= |k|^3 ||g(k^{-1}x) - h(k^{-1}x)|| \\ &\leq \alpha\ |k|^3 \Psi_k(k^{-2}x) \leq \alpha\ L\Psi_k(k^{-1}x) \quad (x \in X). \end{aligned}$$

Therefore, $d(J(g), J(h)) \leq Ld(g,h)$. Hence d is a strictly contractive mapping on $\mathcal{F}$ with Lipschitz constant L. Let $\mathcal{G} = \{g \in \mathcal{F} : d(f_1, g) < \infty\}$, since

$$||J(f_1)(x) - J^0 f_1(x)|| = ||k^3 f_1(k^{-1}x) - f_1(x)|| \leq \Psi_k(k^{-1}x) \quad (x \in X),$$

$d(J(f_1), J^0(f_1)) \leq 1$. By Theorem 2.7 (ii), J has a unique fixed point $c \in \mathcal{G}$ which is defined by

$$c(x) = \lim_{n\to\infty} J^n(f_1)(x) = \lim_{n\to\infty} k^{3n} f_1\big(k^{-n}x\big) \quad (x \in X).$$

The inequality

$$\begin{aligned} ||c(2x+y)+c(2x-y)-2c(x+y)-2c(x-y)-12c(x)|| &= \lim_{n\to\infty} |k|^{3n} ||Cf(k^{-n}x, k^{-n}y)|| \\ &\leq \lim_{n\to\infty} |k|^{3n} \varphi(k^{-n}x, k^{-n}y) \\ &\leq \lim_{n\to\infty} L^n \varphi(x,y) = 0 \end{aligned}$$

for each $x, y \in X$, implies that c is cubic. By Theorem 2.7 (ii), $d(f_1, c) \leq 1$. This means that

$$||f(x) - f(0) - c(x)|| = ||f_1(x) - c(x)|| \leq \Psi_k(k^{-1}x) \quad (x \in X).$$

If $c' : X \to Y$ is another cubic mapping such that

$$||f(x) - f(0) - c'(x)|| \leq \Psi_k(k^{-1}x) \quad (x \in X),$$

then c' is a fixed point of J in $\mathcal{G}$. The uniqueness of the fixed point of J in $\mathcal{G}$ implies that $c = c'$. □

Imitating the proof of Theorem 3.2, we get to the following result:

Corollary 3.3. *Let $\varphi : X \times X \to [0, \infty)$ and $f : X \to Y$ be an add function which is φ-approximately cubic. If Y is complete and for some integer $k \in \mathbb{K}$ and $0 < L < 1$,*

$$|k|^3 \varphi(k^{-1}x, k^{-1}y) \leq L\varphi(x, y) \quad (x, y \in X). \tag{3.13}$$

Then there exists a unique cubic mapping $c : X \to Y$ such that

$$||f(x) - c(x)|| \leq \Phi_k(k^{-1}x) \quad (x \in X), \tag{3.14}$$

where

$$\Phi_2(x) = \frac{1}{|2|}\varphi(x, 0) \quad \Phi_3(x) = \max\{\varphi(x, x), \varphi(x, 0)\} \tag{3.15}$$

and for $j > 3$,

$$\Phi_j(x) = \max\Big\{\varphi(x, (j-2)x), \ldots, \varphi(x, x), \varphi(x, 0)\Big\} \tag{3.16}$$

for all $x \in X$.

The following result can be considered as the dual version of Theorem 3.2.

Theorem 3.4. *Let $\varphi : X \times X \to [0, \infty)$ and $f : X \to Y$ be a φ-approximately cubic function. If Y is complete and for some integer $k \in \mathbb{K}$ and $0 < L < 1$,*

$$|k|^{-3} \varphi(kx, ky) \leq L\ \varphi(x, y) \tag{3.17}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $c : X \to Y$ such that

$$||f(x) - f(0) - c(x)|| \leq |k|^{-3}\Psi_k(x) \quad (x \in X), \tag{3.18}$$

where Ψ_j is defined either by (3.4) or (3.5).

Proof. Let $f_1(x) = f(x) - f(0)$ for each $x \in X$. The first part of the proof of Theorem 3.2 shows that

$$||f_1(kx) - k^3 f_1(x)|| \leq \Psi_k(x) \quad (x \in X). \tag{3.19}$$

Let $\mathcal{F} = \{g | g : X \to Y, g(0) = 0\}$, then

$$d(g, h) = \inf\{\alpha > 0 : ||g(x) - h(x)|| \leq \alpha\ \Psi_k(x), \quad \forall x \in X\} \quad (g, h \in \mathcal{F}) \tag{3.20}$$

defines a complete generalized non-Archimedean metric on $\mathcal{F}$. Define $J : \mathcal{F} \to \mathcal{F}$ by $J(g)(x) = k^{-3}g(kx)$ for all $x \in X$ and $g \in \mathcal{F}$. If for some $g, h \in \mathcal{F}$ and $\alpha > 0$,

$$||g(x) - h(x)|| \leq \alpha \ \Psi_k(x), \quad \forall x \in X,$$

then

$$||J(g)(x)-J(h)(x)|| = |k|^{-3}||g(kx)-h(kx)|| \leq \alpha \, |k|^{-3}\Psi_k(kx) \leq \alpha \, L\Psi_k(x) \quad (x \in X).$$

Therefore $d(J(g), J(h)) \leq Ld(g, h)$. That is d is a strictly contractive mapping on $\mathcal{F}$ with Lipschitz constant L. Since by (3.19),

$$||J(f_1)(x) - J^0 f_1(x)|| = ||k^{-3}f_1(kx) - f_1(x)|| \leq |k|^{-3}\Psi_k(x) \quad (x \in X),$$

$d(J(f_1), J^0(f_1)) \leq |k|^{-3}$. By Theorem 2.7 (ii), J has a unique fixed point c in the set $\{g \in \mathcal{F} : d(f_1, g) < \infty\}$, which satisfies the relation

$$c(x) = \lim_{n\to\infty} J^n(f_1)(x) = \lim_{n\to\infty} k^{-3n} f_1\big(k^n x\big) \quad (x \in X).$$

The inequality

$$\begin{aligned} ||c(2x+y) + c(2x-y) - 2c(x+y) - 2c(x-y) - 12c(x)|| &= \lim_{n\to\infty} |k|^{-3n}||Cf(k^n x, k^n y)|| \\ &\leq \lim_{n\to\infty} |k|^{-3n}\varphi(k^n x, k^n y) \\ &\leq \lim_{n\to\infty} L^n\varphi(x, y) = 0 \end{aligned}$$

for each $x, y \in X$, implies that c is cubic. By Theorem 2.7 (ii), $d(f_1, c) \leq |k|^{-3}$. This means that

$$||f(x) - f(0) - c(x)|| = ||f_1(x) - c(x)|| \leq |k|^{-3}\Psi_k(x) \quad (x \in X).$$

The proof for the uniqueness assertion of c is similar to the end part of the proof of Theorem 3.2. □

Whenever f is an odd function, by imitating the proof of Theorem 3.4, we get to the following estimation.

Corollary 3.5. *Let $\varphi : X \times X \to [0, \infty)$ and $f : X \to Y$ be an add function which is φ-approximately cubic. If Y is complete and for some integer $k \in \mathbb{K}$ and $0 < L < 1$,*

$$|k|^{-3}\varphi(x, y) \leq L\varphi(x, y) \quad (x, y \in X). \tag{3.21}$$

Then there exists a unique cubic mapping $c : X \to Y$ such that

$$||f(x) - c(x)|| \leq |k|^{-3}\Phi_k(x) \quad (x \in X), \tag{3.22}$$

where Φ_j is defined either by (3.15) *or* (3.16)*.*

4. Applications

In this section, we give some applications of our results. In fact, we give non-Archimedean versions of some known results about the stability of cubic functional equations in real normed spaces.

We begin with the following example to show the difference between our result with the one obtained in [21]. This example also shows that there is no counterpart non-Archimedean version of Theorem 1.1.

Example 4.1. Let $p > 2$ be a prime number and $X = Y = \mathbb{Q}_p$. Define $f : X \to Y$ by $f(x) = x^3 + 1$ for all $x \in X$. Since $|2| = 1$,

$$|Cf(x,y)| = |-14| = |2|.|7| = |7| \leq 1 \quad (x \in X).$$

Hence for $\varphi(x,y) = |7|$, the conditions of Theorem 1.1 hold. However, for each natural number n, we have

$$|\frac{f(2^n x)}{8^n} - \frac{f(2^{n+1}x)}{8^{n+1}}| = |\frac{1}{8^n} - \frac{1}{8^{n+1}}| = \frac{|7|}{|8^{n+1}|} = |7| \quad (x \in X).$$

Therefore for each $x \in X$, $\lim_{n\to\infty} \frac{f(2^n x)}{8^n}$ does not exist. This shows that in general, Theorem 1.1 has no non-Archimedean interpretation. Moreover, Theorem 1.2 can not be applied, since for $\varphi(x,y) = |7|$,

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|8|^n} = \lim_{n\to\infty} |7| = |7| \neq 0.$$

However, the conditions of Theorem 3.2 are satisfied. In fact, for $k = p$, we have

$$p^{3n} f(p^{-n}x) = x^3 + p^{3n} \quad (x \in X),$$

and

$$|p^{3n} f(p^{-n}x) - x^3| = |p^{3n}| = \frac{1}{p^{3n}} \quad (x \in X).$$

Therefore $\lim_{n\to\infty} p^{3n} f(p^{-n}x) = x^3$ defines a cubic function on X. It is easy to see that $\Psi_p(x) = |7|$ for each $x \in X$ and

$$0 = |f(x) - f(0) - c(x)| < \Psi_p(x) \quad (x \in X).$$

Hereafter, unless otherwise stated, we will assume that X and Y are non-Archimedean normed spaces over $\mathbb{Q}_p$, where $p > 2$ is a prime number and Y is complete.

Proposition 4.2. *Let $f : X \to Y$ satisfy the following inequality*

$$||Cf(x,y)|| \leq \varepsilon_0\big(||x||^r ||y||^s\big) \quad (x, y \in X),$$

where ε_0 is a positive number. Let Y be complete and $r, s > 0$.

(a) *If $r+s<3$, then there exists a unique cubic mapping $c : X \to Y$ such that*

$$||f(x) - c(x)|| \leq \varepsilon_0 p^{r+s} ||x||^{r+s} \quad (x \in X).$$

(b) *If $r+s>3$, then there exists a unique cubic mapping $c : X \to Y$ such that*

$$||f(x) - c(x)|| \leq \varepsilon_0 p^3 ||x||^{r+s} \quad (x \in X).$$

Proof. Let $\varphi(x,y) = \varepsilon_0\big(||x||^r||y||^s\big)$. By Lemma 3.1, f is an odd function. Since

$$|p|^3\varphi(p^{-1}x, p^{-1}y) = \frac{1}{p^{3-(r+s)}}\varphi(x,y) \quad (x,y \in X),$$

if $r+s<3$, the conditions of Theorem 3.2 for $L = \frac{1}{p^{3-(r+s)}}$ are satisfied. It is easy to see that in this case $\Psi_p(x) = \varepsilon_0||x||^{r+s}$ for all $x, y \in X$. Hence and by Theorem 3.2, (a) holds. Let $r+s>3$. The relation

$$|p|^{-3}\varphi(px, py) = \frac{1}{p^{r+s-3}}\varphi(x,y) \quad (x,y \in X),$$

shows that for $0 < L = p^{3-(r+s)} < 1$, the conditions of Theorem 3.4 are fulfilled. Hence (b) holds. □

Proposition 4.3. *Let Y be complete and $f : X \to Y$ satisfy the following inequality*

$$||Cf(x,y)|| \leq 2\varepsilon_0\big(||x||^r + ||y||^r\big) \quad (x,y \in X), \tag{4.1}$$

where ε_0 is a positive number and $r>0$.

(a) *If $r<3$, then there exists a unique cubic mapping $c : X \to Y$ such that*

$$||f(x) - c(x)|| \leq 2\varepsilon_0 p^r ||x||^r \quad (x \in X).$$

(b) *If $r>3$, then there exists a unique cubic mapping $c : X \to Y$ such that*

$$||f(x) - c(x)|| \leq \varepsilon_0 p^3 ||x||^r \quad (x \in X).$$

Proof. Put $x=y=0$ in (4.1) to obtain $f(0)=0$. Let $\varphi(x,y) = \varepsilon_0\big(||x||^r + ||y||^r\big)$. Since

$$|p|^3\varphi(p^{-1}x, p^{-1}y) = p^{r-3}\varphi(x,y) \quad (x,y \in X),$$

if $r<3$, the conditions of Theorem 3.2 are satisfied. It is easy to see that in this case $\Psi_p(x) = 2\varepsilon_0||x||^r$ for all $x \in X$. This together with Theorem 3.2, proves (a).

Suppose that $r>3$. The relation

$$|p|^{-3}\varphi(px, py) = p^{3-r}\varphi(x,y) \quad (x,y \in X),$$

shows that for $0 < L = p^{3-r} < 1$, therefore the conditions of Theorem 3.4 are fulfilled. Hence (b) holds. □

Proposition 4.4. *Let $f : X \to Y$ satisfy the condition*

$$||Cf(x,y)|| \leq \varepsilon \quad (x, y \in X)$$

for some $\varepsilon > 0$. Then there is a unique cubic mapping $c : X \to Y$ such that

$$||f(x) - f(0) - c\ (x)|| \leq \varepsilon \quad (x \in X).$$

Proof. The result follows from Theorem 3.2 for $\varphi(x,y) = \varepsilon$ for each $x, y \in X$. □

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