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On a Bilateral Hilbert-Type Inequality with a Homogeneous Kernel of 0-Degree

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Abstract. By introducing a homogeneous kernel of 0-degree with an independent parameter and estimating the weight coefficient, a bilateral form of the Hilbert-type series inequality with a best constant factor is established.

1. Introduction

If $a_n, b_n \geq 0, 0 < \sum\limits_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum\limits_{n=1}^{\infty} b_n^2 < \infty$, then(see [1])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well known as Hilbert's inequality. Soon after, inequality (1.1) had been generalized by Hardy-Riesz as(see [1]): If $a_n b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.2) is named of Hardy-Hilbert's integral inequality (see [1]). It is important in analysis and its applications. It was studied extensively and refinements, generalizations and numerous variants appeared in the literature (see [1]- [6]). Under the same condition of (1.2), we have the Hardy-Hilbert's type inequality (see [1], Th. 341, Th. 342) as follows

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}; \tag{1.3}$$

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$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\log(m/n)}{m-n}a_m b_n < \pi^2 \csc^2\frac{\pi}{p}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{1.4}$$

where the constant factors pq and $\pi^2\csc^2\frac{\pi}{p}$ are both the best possible.

In 2008, Yang (see [7]) gave a bilateral inequality as follows: If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1, 0 < \lambda \leq 2, a, b, c \geq 0, a + bc > 0$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1}a_n^p < \infty, 0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1}b_n^q < \infty$, then

$$\begin{aligned} H &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\max\{m^\lambda, n^\lambda\}+bm^\lambda+cn^\lambda} \\ &< C_\lambda(a,b,c)\left\{\sum_{n=1}^{\infty}n^{p(1-\frac{\lambda}{2})-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right\}^{1/q}, \end{aligned}\tag{1.5}$$

where the constant factor $C_\lambda(a,b,c)$ is the best possible. In addition, for $0 < p < 1$, Yang got the reverse inequality as follows

$$\begin{aligned} H &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\max\{m^\lambda, n^\lambda\}+bm^\lambda+cn^\lambda} \\ &> C_\lambda(a,b,c)\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(a,b,c,n)]n^{p(1-\frac{\lambda}{2})-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right\}^{1/q}, \end{aligned}\tag{1.6}$$

where $\theta_\lambda(a,b,c,m) := \frac{1}{C_1(a,b,c)}\int_0^{1/m^\lambda}\frac{1}{a+b+cu}u^{-1/2}\mathbf{d}u = O(\frac{1}{m^{\lambda/2}}) \in (0,1)$, and the constant factor $C_\lambda(a,b,c)$ is the best possible. By the way, in recent years, the reverse form of the Hardy-Hilbert's inequality has been studied by Zhong(see [8]), Zhao(see [9]) and so on.

Until now, we only focus on the Hilbert's inequality with negative number homogeneous and non-homogeneous kernel, but we are just at the beginning of the study on the real number homogeneous kernel. The main purpose of this article is to establish the bilateral form of the Hilbert's type inequality concerning series with the mixed homogeneous kernel of 0-degree.

2. Main results

Lemma 2.1. *Set $\lambda > 0$, define the weight function $\varpi_\lambda(m)$ as*

$$\varpi_\lambda(m) := \sum_{n=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda+n^\lambda}\cdot\min\{m^\lambda, n^\lambda\}\cdot\frac{1}{n}\ (m\in\mathbb{N}), \tag{2.1}$$

then we have the following inequality:

$$\frac{\pi^2}{6\lambda^2}[1-\theta_\lambda(m)] < \varpi_\lambda(m) < \frac{\pi^2}{6\lambda^2}, \tag{2.2}$$

where $0 < \theta_\lambda(m) := \frac{24}{\pi^2}\int_0^{m^{-\frac{\lambda}{2}}} \frac{-t\ln t}{1+t^2}\mathrm{d}t = O(\frac{1}{m^{\lambda/2}}) \in (0,1)$ $(m \to \infty)$.

Proof. On one hand, setting $t = (y/m)^{\lambda/2}$, by monotonicity, we have

$$
\begin{aligned}
\varpi_\lambda(m) &< \int_0^\infty \frac{|\ln(m/y)|}{m^\lambda + y^\lambda} \cdot \min\{m^\lambda, y^\lambda\} \cdot \frac{1}{y}\mathrm{d}y \\
&= \frac{4}{\lambda^2}\int_0^\infty \frac{|\ln t|}{1+t^2} \cdot \min\{1, t^2\} \cdot t^{-1}\mathrm{d}t \\
&= \frac{4}{\lambda^2}\int_0^1 \frac{-t\ln t}{1+t^2}\mathrm{d}t + \int_1^\infty \frac{\ln t}{t(1+t^2)}\mathrm{d}t \\
&= \frac{-8}{\lambda^2}\int_0^1 \frac{t\ln t}{1+t^2}\mathrm{d}t = \frac{\pi^2}{6\lambda^2}.
\end{aligned} \tag{2.3}
$$

On the other hand, similarly, setting $t = (y/m)^{\lambda/2}$, we get

$$
\begin{aligned}
\varpi_\lambda(m) &> \int_1^\infty \frac{|\ln(m/y)|}{m^\lambda + y^\lambda} \cdot \min\{m^\lambda, y^\lambda\} \cdot \frac{1}{y}\mathrm{d}y = \frac{4}{\lambda^2}\int_{m^{-\frac{\lambda}{2}}}^\infty \frac{|\ln t|}{1+t^2} \cdot \min\{1, t^2\} \cdot t^{-1}\mathrm{d}t \\
&= \frac{4}{\lambda^2}\left[\int_0^\infty \frac{|\ln t|}{1+t^2} \cdot \min\{1, t^2\} \cdot t^{-1}\mathrm{d}t - \int_0^{m^{-\frac{\lambda}{2}}} \frac{|\ln t|}{1+t^2} \cdot \min\{1, t^2\} \cdot t^{-1}\mathrm{d}t\right] \\
&= \frac{\pi^2}{6\lambda^2} - \frac{4}{\lambda^2}\int_0^{m^{-\frac{\lambda}{2}}} \frac{|\ln t|}{1+t^2} \cdot \min\{1, t^2\} \cdot t^{-1}\mathrm{d}t \\
&= \frac{\pi^2}{6\lambda^2}\left(1 - \frac{24}{\pi^2}\int_0^{m^{-\frac{\lambda}{2}}} \frac{-t\ln t}{1+t^2}\mathrm{d}t\right) = \frac{\pi^2}{6\lambda^2}[1 - \theta_\lambda(m)].
\end{aligned}
$$

It is obvious that $0 < \theta_\lambda(m) := \frac{24}{\pi^2}\int_0^{m^{-\frac{\lambda}{2}}} \frac{-t\ln t}{1+t^2}\mathrm{d}t < 1$. Since

$$
\begin{aligned}
&\int_0^{m^{-\frac{\lambda}{2}}} \frac{-t\ln t}{1+t^2}\mathrm{d}t = -\int_0^{m^{-\frac{\lambda}{2}}} t\ln t \sum_{k=0}^\infty (-t^2)^k \mathrm{d}t \\
&= \sum_{k=0}^\infty (-1)^{k+1}\int_0^{m^{-\frac{\lambda}{2}}} t^{2k+1}\ln t\mathrm{d}t = \sum_{k=0}^\infty \frac{(-1)^{k+1}}{2k+2}\int_0^{m^{-\frac{\lambda}{2}}} \ln t\mathrm{d}t^{2k+2} \\
&= \frac{1}{m^\lambda}\sum_{k=0}^\infty \frac{(-1)^{k+1}}{2k+2}\left(\frac{-\lambda}{2} \cdot \frac{\ln m}{m^{\lambda k}} - \frac{1}{2k+2} \cdot \frac{1}{m^{\lambda k}}\right) = O(\frac{1}{m^\lambda}) \ (m \to \infty),
\end{aligned}
$$

then $\theta_\lambda(m) = O(\frac{1}{m^\lambda})$. Hence (2.2) is valid. The Lemma is proved. □

Lemma 2.2. *If* $p > 0, p \neq 1$, $\frac{1}{p} + \frac{1}{q} = 1, \lambda > 0$ *and* $0 < \varepsilon < \frac{p\lambda}{2}$, *define*

$$
J(\varepsilon) = \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{|\ln(m/n)|}{m^\lambda + n^\lambda} \cdot \min\{m^\lambda, n^\lambda\} \cdot m^{-1-\frac{\varepsilon}{p}} n^{-1-\frac{\varepsilon}{q}}, \tag{2.4}
$$

then for $\varepsilon \to 0^+$, *we have*

$$\left(\frac{\pi^2}{6\lambda^2} - o(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < J(\varepsilon) < \left(\frac{\pi^2}{6\lambda^2} + \widetilde{o}(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}. \tag{2.5}$$

Proof. Setting $t = (\frac{x}{n})^{\lambda/2}$ in the following, in view of Lemma 2.1, we get

$$\begin{aligned} J(\varepsilon) &< \sum_{n=1}^{\infty} n^{-1-\frac{\varepsilon}{q}}\left(\int_0^{\infty}\frac{|\ln(x/n)|}{x^{\lambda}+n^{\lambda}}\cdot\min\{x^{\lambda},n^{\lambda}\}\cdot x^{-1-\frac{\varepsilon}{p}}\mathrm{d}x\right) \\ &= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left(\frac{4}{\lambda^2}\int_0^{\infty}\frac{|\ln t|}{1+t^2}\cdot\min\{1,t^2\}\cdot t^{-1-\frac{2\varepsilon}{p\lambda}}\mathrm{d}t\right) \\ &= \left(\frac{\pi^2}{6\lambda^2}+\widetilde{o}(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} \quad (\varepsilon\to 0^+); \end{aligned}$$

$$\begin{aligned} J(\varepsilon) &> \sum_{n=1}^{\infty} n^{-1-\frac{\varepsilon}{q}}\left(\int_1^{\infty}\frac{|\ln(x/n)|}{x^{\lambda}+n^{\lambda}}\cdot\min\{x^{\lambda},n^{\lambda}\}x^{-1-\frac{\varepsilon}{p}}\mathrm{d}x\right) \\ &= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left(\frac{4}{\lambda^2}\int_{n^{-\frac{\lambda}{2}}}^{\infty}\frac{|\ln t|}{1+t^2}\cdot\min\{1,t^2\}\cdot t^{-1-\frac{2\varepsilon}{p\lambda}}\mathrm{d}t\right) \\ &> \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left(\frac{4}{\lambda^2}\int_0^{\infty}\frac{|\ln t|}{1+t^2}\cdot\min\{1,t^2\}\cdot t^{-1-\frac{2\varepsilon}{p\lambda}}\mathrm{d}t\right) \\ &\quad -\frac{4}{\lambda^2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^{n^{-\frac{\lambda}{2}}}\frac{-\ln t}{1+t^2}\mathrm{d}t\right) \\ &= \left(\frac{\pi^2}{6\lambda^2}+\widetilde{o}(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} - \frac{4}{\lambda^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\int_0^{n^{-\frac{\lambda}{2}}}\frac{-\ln t}{1+t^2}\mathrm{d}t\right) \quad (\varepsilon\to 0^+). \end{aligned}$$

Since

$$\begin{aligned} &\sum_{n=1}^{\infty}\left(\frac{1}{n}\int_0^{n^{-\frac{\lambda}{2}}}\frac{-\ln t}{1+t^2}\mathrm{d}t\right) = \sum_{n=1}^{\infty}\left[-\frac{1}{n}\int_0^{n^{-\frac{\lambda}{2}}}\ln t\sum_{k=0}^{\infty}(-t^2)^k\mathrm{d}t\right] \\ &= \sum_{k=0}^{\infty}\left[\frac{(-1)^{k+1}}{n}\int_0^{n^{-\frac{\lambda}{2}}}t^{2k}\ln t\mathrm{d}t\right] = \sum_{k=0}^{\infty}\left[\frac{(-1)^{k+1}}{n(2k+1)}\int_0^{n^{-\frac{\lambda}{2}}}\ln t\mathrm{d}t^{2k+1}\right] \\ &= \frac{1}{n^{1+\lambda}}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\left(\frac{-\lambda}{2}\cdot\frac{\ln n}{n^{\lambda k}} - \frac{1}{2k+1}\cdot\frac{1}{n^{\lambda k}}\right) = O\left(\frac{1}{n^{1+\lambda}}\right). \end{aligned}$$

In view of the above inequalities, we have

$$
\begin{aligned}
J(\varepsilon) &> \left(\frac{\pi^2}{6\lambda^2}+\widetilde{o}(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}-\frac{4}{\lambda^2}\cdot O\left(\frac{1}{n^{1+\lambda}}\right)\\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[\left(\frac{\pi^2}{6\lambda^2}+\widetilde{o}(1)\right)-\frac{4}{\lambda^2}\cdot O\left(\frac{1}{n^{1+\lambda}}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\right]\\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left(\frac{\pi^2}{6\lambda^2}-o(1)\right)\ (\varepsilon\to 0^+).
\end{aligned}
$$

The Lemma is proved. □

Theorem 2.3. *If $p>1, \frac{1}{p}+\frac{1}{q}=1, \lambda>0, a_n, b_n\geq 0$ such that $0<\sum_{n=1}^{\infty}n^{p-1}a_n^p<\infty$ and $0<\sum_{n=1}^{\infty}n^{q-1}b_n^q<\infty$, then we have the following inequality*

$$
\begin{aligned}
I &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda+n^\lambda}\cdot\min\{m^\lambda,n^\lambda\}a_m b_n \\
&< \frac{\pi^2}{6\lambda^2}\left\{\sum_{n=1}^{\infty}n^{p-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q-1}b_n^q\right\}^{1/q},
\end{aligned}
\tag{2.6}
$$

where the constant factor $\frac{\pi^2}{6\lambda^2}$ is the best possible.

Proof. By Hölder's inequality with weight[10], we obtain

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda+n^\lambda}\cdot\min\{m^\lambda,n^\lambda\}a_m b_n\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda+n^\lambda}\cdot\min\{m^\lambda,n^\lambda\}\left[\frac{m^{1/q}}{n^{1/p}}a_m\right]\left[\frac{n^{1/p}}{m^{1/q}}b_n\right]\\
&\leq \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda+n^\lambda}\cdot\min\{m^\lambda,n^\lambda\}\frac{m^{p-1}}{n}a_m^p\right\}^{1/p}\\
&\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda+n^\lambda}\cdot\min\{m^\lambda,n^\lambda\}\frac{n^{q-1}}{m}b_n^q\right\}^{1/q}\\
&= \left\{\sum_{m=1}^{\infty}\varpi_\lambda(m)m^{p-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\varpi_\lambda(n)n^{q-1}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

In view of (2.2), we have (2.6).

For $0<\varepsilon<\frac{\lambda p}{2}$, setting $\widetilde{a}_m=m^{-1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n=n^{-1-\frac{\varepsilon}{q}}$ $(m,n\in\mathbb{N})$, by (2.4), we have

$$
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda+n^\lambda}\cdot\min\{m^\lambda,n^\lambda\}\cdot m^{-1-\frac{\varepsilon}{p}}n^{-1-\frac{\varepsilon}{q}}=J(\varepsilon),
$$

Assume that the constant factor $\frac{\pi^2}{6\lambda^2}$ in (2.6) is not the best possible, then there exists a positive number k with $0 < k \leq \frac{\pi^2}{6\lambda^2}$, such that (2.6) is still correct by changing $\frac{\pi^2}{6\lambda^2}$ to k, then, in particular, by (2.5), we have

$$\left(\frac{\pi^2}{6\lambda^2} - o(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < J(\varepsilon) < k\left\{\sum_{n=1}^{\infty} n^{p-1}\widetilde{a}_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q-1}\widetilde{b}_n^q\right\}^{1/q} = k\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}.$$

It follows that $\frac{\pi^2}{6\lambda^2} - o(1) < k$, so $\frac{\pi^2}{6\lambda^2} \leq k(\varepsilon \to 0^+)$. Hence the constant factor $\frac{\pi^2}{6\lambda^2}$ in (2.6) is the best possible. This completes the proof. □

Theorem 2.4. *If $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, a_n, b_n \geq 0$ such that $0 < \sum_{n=1}^{\infty} n^{p-1}a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q-1}b_n^q < \infty$, then we have the following inequality*

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda + n^\lambda}\cdot\min\{m^\lambda, n^\lambda\}a_m b_n \\ &> \frac{\pi^2}{6\lambda^2}\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(n)]n^{p-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q-1}b_n^q\right\}^{1/q}, \end{aligned} \tag{2.7}$$

where $0 < \theta_\lambda(m) := \frac{24}{\pi^2}\int_0^{m^{-\frac{\lambda}{2}}}\frac{-t\ln t}{1+t^2}\mathrm{d}t = O(\frac{1}{m^{\lambda/2}}) \in (0,1)(m \to \infty)$ and the constant factor $\frac{\pi^2}{6\lambda^2}$ is the best possible.

Proof. By the reverse Hölder's inequality with weight[10] and in view of (2.1), we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda + n^\lambda}\cdot\min\{m^\lambda, n^\lambda\}\left[\frac{m^{1/q}}{n^{1/p}}a_m\right]\left[\frac{n^{1/p}}{m^{1/q}}b_n\right] \\ &\geq \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda + n^\lambda}\cdot\min\{m^\lambda, n^\lambda\}\frac{m^{p-1}}{n}a_m^p\right\}^{1/p} \\ &\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda + n^\lambda}\cdot\min\{m^\lambda, n^\lambda\}\frac{n^{q-1}}{m}b_n^q\right\}^{1/q} \\ &= \left\{\sum_{m=1}^{\infty}\varpi_\lambda(m)m^{p-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\varpi_\lambda(n)n^{q-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

By (2.2), in view of $q < 0$, we have (2.7).

For $0 < \varepsilon < \frac{\lambda p}{2}$, setting $\widetilde{a}_m = m^{-1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{-1-\frac{\varepsilon}{q}}$ $(m, n \in \mathbb{N})$, by (2.4), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m^\lambda + n^\lambda}\cdot\min\{m^\lambda, n^\lambda\}\cdot m^{-1-\frac{\varepsilon}{p}}n^{-1-\frac{\varepsilon}{q}} = J(\varepsilon).$$

Assume that the constant factor $\frac{\pi^2}{6\lambda^2}$ in (2.7) is not the best possible, then there exists a positive number k with $k \geq \frac{\pi^2}{6\lambda^2}$, such that (2.7) is still correct by changing $\frac{\pi^2}{6\lambda^2}$ to k, then, in particular, by (2.5), we have

$$\begin{aligned}\left(\frac{\pi^2}{6\lambda^2}+\widetilde{o}(1)\right)\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} &> J(\varepsilon) > k\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(n)]n^{p-1}\widetilde{a}_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q-1}\widetilde{b}_n^q\right\}^{1/q}\\ &= k\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}-\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^\lambda}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right\}^{1/q}\\ &= k\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{1-\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^\lambda}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}.\end{aligned}$$

It follows that

$$\frac{\pi^2}{6\lambda^2}+\widetilde{o}(1) > k\left\{1-\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^\lambda}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p},$$

and then $\frac{\pi^2}{6\lambda^2} \geq k(\varepsilon \to 0^+)$. Thus the constant factor $\frac{\pi^2}{6\lambda^2}$ in (2.7) is the best possible. The theorem is proved. □

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