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REMOVAL OF HYPERSINGULARITY IN A DIRECT BEM FORMULATION

Lee, BongJu

Abstract. Using Green's theorem, elliptic boundary value problems can be converted to boundary integral equations. A numerical methods for boundary integral equations are boundary elementary method(BEM). BEM has advantages over finite element method(FEM) whenever the fundamental solutions are known. Helmholtz type equations arise naturally in many physical applications. In a boundary integral formulation for the exterior Neumann there occurs a hypersingular operator which exhibits a strong singularity like $\frac{1}{|x-y|^3}$ and hence is not an integrable function. In this paper we are going to remove this hypersingularity by reducing the regularity of test functions.

1. Introduction

The unique radial fundamental solution of the Helmholtz equation in $\mathbb{R}^3$

$$-\Delta E - k^2 E = \delta$$

which satisfies the Sommerfeld radiation condition(or outgoing wave condition)

$$\left|\frac{\partial u}{\partial r} - iku\right| \leq \frac{c}{r^2}$$

at infinity is

$$E(r) = \frac{e^{ikr}}{4\pi r}, \quad r = \sqrt{x^2 + y^2 + z^2}.$$

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The derivatives of E are

$$\begin{aligned}\frac{\partial E(r)}{\partial x} &= \frac{e^{ikr}}{4\pi r^2}\Big(ik-\frac{1}{r}\Big)x,\\ \nabla E &= \frac{e^{ikr}}{4\pi r^2}\Big(ik-\frac{1}{r}\Big)(x,y,z) = \frac{e^{ikr}}{4\pi r^2}\Big(ik-\frac{1}{r}\Big)\mathbf{x}\\ \frac{\partial E}{\partial \mathbf{n_x}} = \big(\nabla E\big)\cdot\mathbf{n} &= \frac{e^{ikr}}{4\pi r^2}\Big(ik-\frac{1}{r}\Big)\mathbf{x}\cdot\mathbf{n_x}\\ \frac{\partial E}{\partial \mathbf{n_x}}(\mathbf{x}-\mathbf{y}) &= \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|^2}\Big(ik-\frac{1}{|\mathbf{x}-\mathbf{y}|}\Big)(\mathbf{x}-\mathbf{y})\cdot\mathbf{n_x}\\ &= \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{1}{|\mathbf{x}-\mathbf{y}|}\,\Big(ik-\frac{1}{|\mathbf{x}-\mathbf{y}|}\Big)(\mathbf{x}-\mathbf{y})\cdot\mathbf{n_x}.\end{aligned}$$

We see that $\frac{\partial E}{\partial \mathbf{n_x}}(\mathbf{x}-\mathbf{y})$ blows up like $\frac{1}{|\mathbf{x}-\mathbf{y}|^2}$ in a neighborhood of $\mathbf{y}$ and hence it is an integrable function. The function $E(\mathbf{x}-\mathbf{y})$ may be interpreted as a composition of the following two functions:

$$\mathbb{R}^6_{(\mathbf{x},\mathbf{y})} \xrightarrow{(\mathbf{x},\mathbf{y})\mapsto\mathbf{x}-\mathbf{y}} R^3_{\mathbf{z}} \xrightarrow{\mathbf{z}\mapsto E(|\mathbf{z}|)} \mathbb{R}$$

With this in mind, we have

$$\frac{\partial\Big(E(\mathbf{x}-\mathbf{y})\Big)}{\partial \mathbf{n_y}} = -\big(\nabla_{\mathbf{z}}E\big)(\mathbf{x}-\mathbf{y})\cdot\mathbf{n_y} = -\frac{\partial E}{\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y})$$

$$\frac{\partial}{\partial \mathbf{n_y}}\Big(\frac{\partial E}{\partial \mathbf{n_x}}(\mathbf{x}-\mathbf{y})\Big) = -\frac{\partial^2 E}{\partial \mathbf{n_x}\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y}) = -\mathbf{n_x}^T\Big(\frac{\partial^2 E}{\partial z_i\partial z_j}(\mathbf{x}-\mathbf{y})\Big)\mathbf{n_y}.$$

Let u be a function satisfying outgoing radiation condition. Then, in [4] , it was shown that

$$u\in H=\{v \mid \frac{v}{(1+r^2)^{1/2}},\frac{\nabla v}{(1+r^2)^{(}1/2)},\frac{\partial v}{\partial r}-ikv\in L^2(\Omega_e)\}$$

$$\begin{cases}\Delta u+k^2u &= 0 \text{ in } \Omega_i,\\ \Delta u+k^2u &= 0 \text{ in } \Omega_e,\\ \big[u\big] &= u_{\text{int}}-u_{\text{ext}},\\ \big[\frac{\partial u}{\partial n}\big] &= \frac{\partial u}{\partial n}\big|_{\text{int}}-\frac{\partial u}{\partial n}\big|_{\text{ext}}.\end{cases}$$

Therefore

$$u(\mathbf{y}) = \int_\Gamma E(\mathbf{x}-\mathbf{y})\big[\tfrac{\partial u}{\partial n}\big](\mathbf{x})ds_{\mathbf{x}} - \int_\Gamma \tfrac{\partial E}{\partial \mathbf{n_x}}(\mathbf{x}-\mathbf{y})\big[u\big](\mathbf{x})ds_{\mathbf{x}},\quad \mathbf{y}\notin\Gamma=\partial\Omega$$

$$\frac{u_{\text{int}}(\mathbf{y})+u_{\text{ext}}(\mathbf{y})}{2}=\int_\Gamma E(\mathbf{x}-\mathbf{y})\big[\frac{\partial u}{\partial n}\big](\mathbf{x})ds_{\mathbf{x}}-\int_\Gamma \frac{\partial E}{\partial \mathbf{n}_{\mathbf{x}}}(\mathbf{x}-\mathbf{y})\big[u\big](\mathbf{x})ds_{\mathbf{x}},$$
$\mathbf{y}\in\Gamma=\partial\Omega.$

Let's consider $u(\mathbf{y})=\int_\Gamma E(\mathbf{x}-\mathbf{y})q(\mathbf{x})ds_{\mathbf{x}},\ \ q\in\mathcal{C}^0(\Gamma)$. Then u is continuous on $\mathbb{R}^3$. The normal derivative is discontinuous on Γ. In fact

$$\begin{aligned}\nabla u(\mathbf{y}) &= \int_\Gamma \nabla_{\mathbf{y}}\Big(E(\mathbf{x}-\mathbf{y})\Big)q(\mathbf{x})ds_{\mathbf{x}}\\ &= -\int_\Gamma (\nabla_{\mathbf{z}}E)(\mathbf{x}-\mathbf{y})q(\mathbf{x})ds_{\mathbf{x}}.\end{aligned}$$

Let's focus on a fixed point $\mathbf{y}_0\in\Gamma$. We introduce two symmetric points

$$\begin{cases}\mathbf{y}_+ &= \mathbf{y}_0+\rho\mathbf{n}_{\mathbf{y}_0},\\ \mathbf{y}_- &= \mathbf{y}_0-\rho\mathbf{n}_{\mathbf{y}_0}.\end{cases}$$

Then, with $\mathbf{n}_{\mathbf{y}_0}=\mathbf{n}_0$,

$$\begin{aligned}&\nabla u(\mathbf{y}_+)\cdot\mathbf{n}_0+\nabla u(\mathbf{y}_-)\cdot\mathbf{n}_0\\
&\quad=-\int_\Gamma\Big[\big(\nabla_{\mathbf{z}}E\big)(\mathbf{x}-\mathbf{y}_+)\cdot\mathbf{n}_0+\big(\nabla_{\mathbf{z}}E\big)(\mathbf{x}-\mathbf{y}_-)\cdot\mathbf{n}_0\Big]q(\mathbf{x})ds_{\mathbf{x}}\\
&\quad=\int_\Gamma\Big[\frac{e^{ik|\mathbf{x}-\mathbf{y}_+|}}{4\pi|\mathbf{x}-\mathbf{y}_+|^2}\Big(ik-\frac{1}{|\mathbf{x}-\mathbf{y}_+|}\Big)(\mathbf{y}_+-\mathbf{x})\cdot\mathbf{n}_0\,q(\mathbf{x})ds_{\mathbf{x}}\\
&\qquad+\int_\Gamma\Big[\frac{e^{ik|\mathbf{x}-\mathbf{y}_-|}}{4\pi|\mathbf{x}-\mathbf{y}_-|^2}\Big(ik-\frac{1}{|\mathbf{x}-\mathbf{y}_-|}\Big)(\mathbf{y}_--\mathbf{x})\cdot\mathbf{n}_0\,q(\mathbf{x})ds_{\mathbf{x}}\\
&\quad=\int_\Gamma\Big[\frac{e^{ik|\mathbf{x}-\mathbf{y}_+|}}{4\pi|\mathbf{x}-\mathbf{y}_+|^2}\Big(ik-\frac{1}{|\mathbf{x}-\mathbf{y}_+|}\Big)(\mathbf{y}_0-\mathbf{x})\cdot\mathbf{n}_0\,q(\mathbf{x})ds_{\mathbf{x}}\\
&\qquad+\int_\Gamma\Big[\frac{e^{ik|\mathbf{x}-\mathbf{y}_-|}}{4\pi|\mathbf{x}-\mathbf{y}_-|^2}\Big(ik-\frac{1}{|\mathbf{x}-\mathbf{y}_-|}\Big)(\mathbf{y}_0-\mathbf{x})\cdot\mathbf{n}_0\,q(\mathbf{x})ds_{\mathbf{x}}\\
&\qquad\quad+\int_\Gamma\Big[\frac{e^{ik|\mathbf{x}-\mathbf{y}_+|}}{4\pi|\mathbf{x}-\mathbf{y}_+|^2}\Big(ik-\frac{1}{|\mathbf{x}-\mathbf{y}_+|}\Big)\\
&\qquad\qquad-\frac{e^{ik|\mathbf{x}-\mathbf{y}_-|}}{4\pi|\mathbf{x}-\mathbf{y}_-|^2}\Big(ik-\frac{1}{|\mathbf{x}-\mathbf{y}_-|}\Big)\Big]\rho q(\mathbf{x})ds_{\mathbf{x}}\\
&\quad\to 2\int_\Gamma\Big[\frac{e^{ik|\mathbf{x}-\mathbf{y}_0|}}{4\pi|\mathbf{x}-\mathbf{y}_0|^2}\Big(ik-\frac{1}{|\mathbf{x}-\mathbf{y}_0|}\Big)(\mathbf{y}_0-\mathbf{x})\cdot\mathbf{n}_0\,q(\mathbf{x})ds_{\mathbf{x}}+0\\
&=\ \ -2\int_\Gamma\frac{\partial E}{\partial\mathbf{n}_0}(\mathbf{x}-\mathbf{y})q(\mathbf{x})ds_{\mathbf{x}}.\end{aligned}$$

Note also that

$$\frac{\partial E}{\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y}) = -\frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n_y}}.$$

On the other hand

$$\nabla u(\mathbf{y}_-)\cdot \mathbf{n}_0 - \nabla u(\mathbf{y}_+)\cdot \mathbf{n}_0 \to \left[\frac{\partial u}{\partial \mathbf{n}_0}\right] = q(\mathbf{y}_0).$$

Therefore we have a system of relations

$$\begin{cases} \nabla u(\mathbf{y}_+)\cdot \mathbf{n}_0 + \nabla u(\mathbf{y}_-)\cdot \mathbf{n}_0 & \to -2\int_\Gamma \frac{\partial E}{\partial \mathbf{n}_0}(\mathbf{x}-\mathbf{y})q(\mathbf{x})ds_{\mathbf{x}} \\ \nabla u(\mathbf{y}_-)\cdot \mathbf{n}_0 - \nabla u(\mathbf{y}_+)\cdot \mathbf{n}_0 & \to \left[\frac{\partial u}{\partial \mathbf{n}_0}\right] = q(\mathbf{y}_0) \end{cases}$$

which implies that

$$\begin{aligned} \frac{\partial u}{\partial \mathbf{n}}\Big|_{\text{int}}(\mathbf{y}) &= \frac{q(\mathbf{y})}{2} - \int_\Gamma \frac{\partial E}{\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y})q(\mathbf{x})ds_{\mathbf{x}},\ \mathbf{y}\in\Gamma \\ \frac{\partial u}{\partial \mathbf{n}}\Big|_{\text{ext}}(\mathbf{y}) &= -\frac{q(\mathbf{y})}{2} - \int_\Gamma \frac{\partial E}{\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y})q(\mathbf{x})ds_{\mathbf{x}},\ \mathbf{y}\in\Gamma. \end{aligned}$$

If we take $u = 0$ on Ω_i, then

$$u(\mathbf{y}) = -\int_\Gamma E(\mathbf{x}-\mathbf{y})\frac{\partial u}{\partial \mathbf{n_x}}(\mathbf{x})ds_{\mathbf{x}} + \int_\Gamma \frac{\partial E}{\partial \mathbf{n_x}}(\mathbf{x}-\mathbf{y})u(\mathbf{x})ds_{\mathbf{x}},\ \mathbf{y}\in\Omega_e$$

$$\frac{u(\mathbf{y})}{2} = -\int_\Gamma E(\mathbf{x}-\mathbf{y})\frac{\partial u}{\partial \mathbf{n_x}}(\mathbf{x})ds_{\mathbf{x}} + \int_\Gamma \frac{\partial E}{\partial \mathbf{n_x}}(\mathbf{x}-\mathbf{y})u(\mathbf{x})ds_{\mathbf{x}},\ \mathbf{y}\in\Gamma=\partial\Omega$$

Now

$$\frac{\partial u}{\partial \mathbf{n_y}} = \int_\Gamma \frac{\partial E}{\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y})\frac{\partial u}{\partial \mathbf{n_x}}ds_{\mathbf{x}} - \int_\Gamma \frac{\partial^2 E}{\partial \mathbf{n_x}\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y})u(\mathbf{x})ds_{\mathbf{x}},\ \mathbf{y}\in\Omega_e.$$

Using the boundary condition

$$\frac{\partial u}{\partial \mathbf{n}_0}(\mathbf{y}) = q(\mathbf{y}) = \lim_{\mathbf{y}\in\Omega_e \mathbf{y}\to\mathbf{y}_0\in\Gamma} \nabla u(\mathbf{y})\cdot \mathbf{n}_0$$

we get

$$q(\mathbf{y}_0) = \frac{1}{2}q(\mathbf{y}_0) + \int_\Gamma \frac{\partial E}{\partial \mathbf{n}_0}(\mathbf{x}-\mathbf{y})q(\mathbf{x})ds_{\mathbf{x}} - \int_\Gamma \frac{\partial^2 E}{\partial \mathbf{n_x}\partial \mathbf{n}_0}(\mathbf{x}-\mathbf{y})u(\mathbf{x})ds_{\mathbf{x}}.$$

Therefore

$$\int_\Gamma \frac{\partial^2 E}{\partial \mathbf{n_x}\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y})u(\mathbf{x})ds_{\mathbf{x}} = -\frac{q(\mathbf{y})}{2} + \int_\Gamma \frac{\partial E}{\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y})q(\mathbf{x})ds_{\mathbf{x}}$$

$$u_h(\mathbf{x}) = \sum_{i=1}^{N} u_i \phi_i(\mathbf{x}), \quad q(\mathbf{x}) = \sum_{k=1}^{M} q_k \psi_k(\mathbf{x})$$

$$\sum_{i=1}^{N} \Big\{ \int_\Gamma \phi_j(\mathbf{y}) \int_\Gamma \frac{\partial^2 E}{\partial \mathbf{n_x} \partial \mathbf{n_y}} (\mathbf{x}-\mathbf{y}) \phi_i(\mathbf{x}) ds_{\mathbf{x}} ds_{\mathbf{y}} \Big\} u_i$$
$$= \sum_{k=1}^{M} \Big\{ \int_\Gamma \phi_j(\mathbf{y}) \Big(-\frac{1}{2}\psi_k(\mathbf{y}) + \int_\Gamma \frac{\partial E}{\partial \mathbf{n_y}} (\mathbf{x}-\mathbf{y}) \psi_k(\mathbf{x}) ds_{\mathbf{x}} \Big) ds_{\mathbf{y}} \Big\} q_k$$

or

$$\sum_{i=1}^{N} \Big\{ \int_\Gamma \phi_j(\mathbf{y}) \int_\Gamma \frac{\partial^2 E(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n_x} \partial \mathbf{n_y}} \phi_i(\mathbf{x}) ds_{\mathbf{x}} ds_{\mathbf{y}} \Big\} u_i$$
$$= \sum_{k=1}^{M} \Big\{ \int_\Gamma \phi_j(\mathbf{y}) \Big(\frac{1}{2}\psi_k(\mathbf{y}) + \int_\Gamma \frac{\partial E(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n_y}} \psi_k(\mathbf{x}) ds_{\mathbf{x}} \Big) ds_{\mathbf{y}} \Big\} q_k.$$

The distribution $\nabla^2 E$ is not an integrable function. Actually

$$\frac{\partial}{\partial \mathbf{n_y}} \Big(\frac{\partial E}{\partial \mathbf{n_x}} (\mathbf{x}-\mathbf{y}) \Big) = -\frac{\partial^2 E}{\partial \mathbf{n_x} \partial \mathbf{n_y}} (\mathbf{x}-\mathbf{y})$$

blows up like $\frac{1}{|\mathbf{x}-\mathbf{y}|^3}$, a hypersingularity. However the limit

$$\int_\Gamma \frac{\partial^2 E}{\partial \mathbf{n_x} \partial \mathbf{n_y}} (\mathbf{x}-\mathbf{y}) \phi(x) ds_x \quad \text{as } y \to \Gamma,\ \phi \in \mathcal{C}^0(\Gamma),\ \Gamma = \partial\Omega$$

exists.

2. The integral $\int_\Gamma \phi_j(\mathbf{y}) \int_\Gamma \frac{\partial^2 E}{\partial \mathbf{n_x} \partial \mathbf{n_y}} (\mathbf{x}-\mathbf{y}) \phi_i(\mathbf{x}) ds_{\mathbf{x}} ds_{\mathbf{y}}$

Singular integrals occur on the diagonal of the BEM influence matrices. The fundamental solution of 2D Laplace equation is proportional to $\ln r$ and this type of singularity is frequently to be integrated over $[0, 1]$. However singular integrals arising in BEM applications are not certainly limited to logarithmic singularity. Singularities of type $\frac{1}{r}$, $\frac{1}{r^2}$ also frequently arise [1], [2]. Hypersingularities of type $\frac{1}{r^3}$ appear in a dual boundary integral formulation([3], [4]).

The double layer potential

$$u(\mathbf{y}) = \int_\Gamma \frac{\partial E}{\partial \mathbf{n_x}} (\mathbf{x}-\mathbf{y}) \phi(\mathbf{x}) ds_{\mathbf{x}},\ \phi \in \mathcal{C}^0(\Gamma)$$

solves

$$\begin{cases} \Delta u + k^2 u &= 0 \text{ in } \Omega_i \cup \Omega_e, \\ \big[u\big](\mathbf{y}) &= u_{\text{int}}(\mathbf{y}) - u_{\text{ext}}(\mathbf{y}) = -\varphi(\mathbf{y}), \\ \Big[\frac{\partial u}{\partial \mathbf{n}}\Big](\mathbf{y}) &= 0. \end{cases}$$

Consider a vector field

$$\mathbf{v} = \begin{cases} \nabla u \text{ on } \Omega_i, \\ \nabla u \text{ on } \Omega_e. \end{cases} \tag{1}$$

Then $\mathbf{v}$ satisfies the radiation condition at infinity. Note also that

$$\big[\nabla u \cdot \mathbf{n}\big]_\Gamma = 0.$$

LEMMA 2.1. *Let* $\mathbf{v}$ *be the vector field defined by (1), then*

$$div\, \mathbf{v} + k^2 u = 0.$$

Proof. Let Y be a domain with $\mathcal{C}^1$ boundary. Then

$$\partial_j \chi_Y = n_j dS$$

where χ_Y is the characteristic function for the set Y, $\mathbf{n}$ is the unit outward normal vector to the boundary of Y and dS is the surface measure of ∂Y. Hence

$$\partial_j(u\chi_Y) = \big(\partial_j u\big)\chi_Y + u n_j dS.$$

Remember that $\mathbf{v}$ is defined on $\mathbb{R}^3 \setminus \Gamma = \Omega_i \cup \Omega_e$. Thus

$$\mathbf{v} = \mathbf{v}\chi_{\Omega_i} + \mathbf{v}\chi_{\Omega_e}$$

as a distribution on $\mathbb{R}^3$. Let $v_j, j = 1, 2, 3$, be the j-the component of $\mathbf{v}$. Then

$$\begin{aligned} \partial_j v_j &= \partial_j\big(v_j\chi_{\Omega_i}\big) + \partial_j\big(v_j\chi_{\Omega_e}\big) \\ &= \Big(\partial_j v_j \chi_{\Omega_i} + v_j n_j d\Gamma\Big) + \Big(\partial_j v_j \chi_{\Omega_e} - v_j n_j d\Gamma\Big). \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{div}\mathbf{v} &= (\operatorname{div}\mathbf{v})\chi_{\Omega_i} + (\operatorname{div}\mathbf{v})\chi_{\Omega_e} + \big[\mathbf{v}\cdot\mathbf{n}\big] \\ &= (\operatorname{div}\mathbf{v})\chi_{\Omega_i} + (\operatorname{div}\mathbf{v})\chi_{\Omega_e}, \ \big[\mathbf{v}\cdot\mathbf{n}\big] = \big[\nabla u\cdot\mathbf{n}\big] = 0 \\ &= (-k^2 u)\chi_\Omega + (-k^2 u)\chi_\Omega \\ &= -k^2 u. \end{aligned}$$

This proves that the distribution $\operatorname{div}\mathbf{v}$ is in fact a function in $L^2(\mathbb{R}^3)$ and that in the sense of distribution in $\mathbb{R}^3$,

$$\operatorname{div}\mathbf{v} + k^2 u = 0.$$

□

Similarly the gradient of the distribution $u = u\chi_{\Omega_i} + u\chi_{\Omega_e}$ in $\mathbb{R}^3$ is

$$\begin{aligned}
\partial_j u &= \Big((\partial_j u)\chi_{\Omega_i} + u n_j d\Gamma\Big) + \Big((\partial_j u)\chi_{\Omega_e} + u(-n_j)d\Gamma\Big) \\
&= \Big((\partial_j u)\chi_{\Omega_i} + (\partial_j u)\chi_{\Omega_e}\Big) + \big(u_{\text{int}} - u_{\text{ext}}\big)d\Gamma \\
&= \Big((\partial_j u)\chi_{\Omega_i} + (\partial_j u)\chi_{\Omega_e}\Big) + \big[u\big]n_j d\Gamma
\end{aligned}$$

$$\begin{aligned}
\nabla u &= \big(\nabla u\big)\chi_{\Omega_i} + \big(\nabla u\big)\chi_{\Omega_e} + \big[u\big]\mathbf{n}d\Gamma \\
&= \big(\nabla u\big)\chi_{\Omega_i} + \big(\nabla u\big)\chi_{\Omega_e} - \varphi\mathbf{n}d\Gamma \\
&= \mathbf{v} - \varphi\mathbf{n}d\Gamma, \qquad \mathbf{v} = \big(\nabla u\big)\chi_{\Omega_i} + \big(\nabla u\big)\chi_{\Omega_e}.
\end{aligned}$$

We see that ∇u, as distribution on $\mathbb{R}^3$, is not a function. We finally have the following system of equations in $\mathbb{R}^3$:

$$\text{(2)} \qquad \begin{cases} \operatorname{div}\mathbf{v} + k^2 u &= 0, \\ \nabla u - \mathbf{v} &= -\varphi\mathbf{n}d\Gamma. \end{cases}$$

In matrix form

$$\begin{pmatrix}
k^2 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & k^2 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & 0 & k^2 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & 0 & 0 & -1 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & 0 & -1 & 0 \\
0 & 0 & \frac{\partial}{\partial z} & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix} u \\ u \\ u \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}
=
\begin{pmatrix} \operatorname{div}\mathbf{v} + k^2 u \\ \operatorname{div}\mathbf{v} + k^2 u \\ \operatorname{div}\mathbf{v} + k^2 u \\ \frac{\partial u}{\partial x} - v_1 \\ \frac{\partial u}{\partial y} - v_2 \\ \frac{\partial u}{\partial z} - v_3 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\varphi n_1 d\Gamma \\ -\varphi n_2 d\Gamma \\ -\varphi n_3 d\Gamma \end{pmatrix}.$$

The right hand side of the above equation has compact support. Therefore if we know the fundamental solution of the above system, we can express the solution as a convolution of the fundamental solution with the right hand side. The fundamental solution of the above 6×6 system is a 6×6 matrix of distributions. And if we take into account the right hand side which has 0 in the first half part, we only have to decide the second half of the fundamental solution. However just for convenience we give the whole one:

$$F = \begin{pmatrix} -E & 0 & 0 & -\frac{\partial E}{\partial x} & -\frac{\partial E}{\partial y} & -\frac{\partial E}{\partial z} \\ 0 & -E & 0 & -\frac{\partial E}{\partial x} & -\frac{\partial E}{\partial y} & -\frac{\partial E}{\partial z} \\ 0 & 0 & -E & -\frac{\partial E}{\partial x} & -\frac{\partial E}{\partial y} & -\frac{\partial E}{\partial z} \\ -\frac{\partial E}{\partial x} & -\frac{\partial E}{\partial x} & -\frac{\partial E}{\partial x} & & & \\ -\frac{\partial E}{\partial y} & -\frac{\partial E}{\partial y} & -\frac{\partial E}{\partial y} & & \Sigma & \\ -\frac{\partial E}{\partial z} & -\frac{\partial E}{\partial z} & -\frac{\partial E}{\partial z} & & & \end{pmatrix}$$

where

$$\begin{aligned} E &= \frac{e^{ikr}}{4\pi r} \\ \Sigma &= -\operatorname{curl}\operatorname{curl}(EI_3) + k^2 EI_3. \end{aligned}$$

I_3 is the 3×3 identity matrix.

$$AF = \begin{pmatrix} \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \end{pmatrix} = \begin{pmatrix} \delta I_3 & O \\ O & \delta I_3 \end{pmatrix}$$

where

$$A = \begin{pmatrix} k^2 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & k^2 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & k^2 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & 0 & -1 \end{pmatrix}.$$

Let's take a look at the second part of F. If we adopt the notation $U = -\nabla E$, then

$$\operatorname{div}(EI_3) = \nabla E$$

and hence

$$\begin{aligned} \operatorname{div}\Sigma &= \operatorname{div}\Big(-\operatorname{curl}\operatorname{curl}(EI_3)\Big) + \operatorname{div}(k^2 EI_3) \\ &= k^2 \operatorname{div}(EI_3) = k^2 \nabla E = -k^2 U. \end{aligned}$$

Thus

$$\operatorname{div}\Sigma + k^2 U = O.$$

On the other hand, if we apply

$$\Delta = \nabla \operatorname{div} - \operatorname{curl} \operatorname{curl}$$

we have

$$\begin{aligned} \nabla U &= -\nabla \operatorname{div}(EI_3) \\ &= -\Big(\Delta + \operatorname{curl} \operatorname{curl}\Big)(EI_3) \\ &= -(\Delta E)I_3 + \big(- \operatorname{curl} \operatorname{curl}(EI_3) + k^2 EI_3\big) - k^2 EI_3 \\ &= \big(-\Delta E - k^2 E\big)I_3 + \Sigma \\ &= \delta I_3 + \Sigma \end{aligned}$$

$$\nabla U - \Sigma = \delta I_3.$$

3. Some differential geometry

Let Γ be a regular surface in $\mathbb{R}^3$. Its tubular neighborhood Γ_ϵ is defined by

$$\Gamma_\epsilon = \{\mathbf{y} \mid d(\mathbf{y}) < \epsilon\}$$

where

$$d(y) := \inf_{x \in \Gamma} \|y - x\|.$$

If Γ is a regular orientable surface and ϵ is sufficiently small, for each $\mathbf{y} \in \Gamma_\epsilon$ there is a unique $\mathcal{P}(\mathbf{y}) \in \Gamma$ such that

$$d(\mathbf{y}) = \|y - \mathcal{P}(\mathbf{y})\|.$$

Therefore every point $\mathbf{y} \in \Gamma_\epsilon$ has the following expression

$$\mathbf{y} = \mathcal{P}(\mathbf{y}) + s(\mathbf{y})\mathbf{n}\big(\mathcal{P}(\mathbf{y})\big).$$

Here $\mathbf{n}$ is unit normal vector field on Γ and

$$s(\mathbf{y}) = \begin{cases} d(\mathbf{y}), & \mathbf{y} \in \Omega_e \\ -d(\mathbf{y}), & \mathbf{y} \in \Omega_i. \end{cases}$$

Clearly $\mathbf{n}$ can be extended to a unit normal vector filed $\mathbf{n}$ on Γ_ϵ by

$$\mathbf{n}(\mathbf{y}) = \nabla s(\mathbf{y}),\ \mathbf{y} \in \Gamma_\epsilon.$$

We introduce a family of surfaces Γ_s:

$$\Gamma_s = \{\mathbf{y} \mid \mathbf{y} = \mathbf{x} + s\mathbf{n}(\mathbf{x}), \mathbf{x} \in \Gamma\}.$$

Note that $\mathbf{n}$ is still normal to the surface $\Gamma_s, -\epsilon < s < \epsilon$. Moreover a function u defined on Γ can be extended to a function on a tubular neighborhood Γ_ϵ:

$$\widetilde{u}(\mathbf{y}) = u\big(\mathcal{P}(\mathbf{y})\big).$$

By using this extension. we define a tangential gradient, tangential rotation of a function u defined on Γ as the following:

$$\begin{aligned}
\nabla_\Gamma u &:= \big(\nabla \widetilde{u}\big)\Big|_\Gamma \\
\overrightarrow{\operatorname{curl}}_\Gamma u &= \operatorname{curl}\big(\widetilde{u}\mathbf{n}\big)\Big|_\Gamma \\
&= \Big(\nabla \widetilde{u} \wedge \mathbf{n} + \widetilde{u}\operatorname{curl}(\mathbf{n})\Big)\Big|_\Gamma \\
&= \Big(\nabla \widetilde{u} \wedge \mathbf{n}\Big)\Big|_\Gamma, \quad \operatorname{curl}(\mathbf{n}) = \operatorname{curl}(\nabla s) = \mathbf{0} \\
&= \big(\nabla_\Gamma u\big) \wedge \mathbf{n}.
\end{aligned}$$

The rate of change of unit normal vector in the direction $\mathbf{v} \in T_p\Gamma$ is denoted by

$$S(\mathbf{v}) := \nabla_{\mathbf{v}}\mathbf{n} = \frac{d}{dt}\mathbf{n}\big(c(t)\big)\Big|_{t=0}, \quad c'(0) = \mathbf{v}$$

c is a curve in Γ. Since $\mathbf{n} = \nabla s$,

$$S(\mathbf{v}) = \Big(\frac{\partial^2 s}{\partial x_i \partial x_j}\Big)\mathbf{v}$$

where $\Big(\frac{\partial^2 s}{\partial x_i \partial x_j}\Big)$ is the Hessian matrix of s. Note also that

$$\nabla_{\mathbf{v}}\mathbf{n} = (\mathbf{v} \cdot \nabla)\mathbf{n}.$$

Now surfacic divergence, surfacic rotational of a vector field $\mathbf{v}$ defined on Γ is defined by

$$\begin{aligned}
\operatorname{div}_\Gamma \mathbf{v} &:= \big(\operatorname{div}\widetilde{\mathbf{v}}\big)\Big|_\Gamma, \ \widetilde{\mathbf{v}}(\mathbf{y}) := \mathbf{v}(\mathbf{x}) - s(\mathbf{y})S\big(\mathbf{v}(\mathbf{x})\big) \\
\operatorname{curl}_\Gamma \mathbf{v} &= \big(\operatorname{curl}\widetilde{\mathbf{v}} \cdot \mathbf{n}\big)\Big|_\Gamma.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{div}_\Gamma \overrightarrow{\mathrm{curl}}_\Gamma u &= \Big(\mathrm{div}\left(\mathrm{curl}\,(\widetilde{u}\mathbf{n})\right)\Big)\Big|_\Gamma \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{curl}_\Gamma \nabla_\Gamma u &= \Big(\mathrm{curl}\left(\nabla\widetilde{u}\right)\cdot\mathbf{n}\Big)\Big|_\Gamma \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{div}_\Gamma\big(\mathbf{v}\wedge\mathbf{n}\big) &= \Big(\mathrm{div}\left(\widetilde{\mathbf{v}}\wedge\mathbf{n}\right)\Big)\Big|_\Gamma \\
&= \Big(\mathrm{curl}\,(\widetilde{\mathbf{v}})\cdot\mathbf{n} - \widetilde{\mathbf{v}}\cdot\,\mathrm{curl}\,(\mathbf{n})\Big)\Big|_\Gamma \\
&= \Big(\mathrm{curl}\,(\widetilde{\mathbf{v}})\cdot\mathbf{n}\Big)\Big|_\Gamma \\
&= \mathrm{curl}_\Gamma\mathbf{v}.
\end{aligned}$$

If u is a function defined of Γ_ϵ and $\mathbf{v}$ is a vector field on Γ_ϵ, then $\nabla u = \nabla_{\Gamma_s} u + \frac{\partial u}{\partial s}\mathbf{n}$ and $\mathbf{v} = \mathbf{v}_{\Gamma_s} + (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$, where $\mathbf{v}_{\Gamma_s}$ is the tangential component of $\mathbf{v}$ to the surface Γ_s. Thus

$$\begin{aligned}
\mathbf{v}_{\Gamma_s} &= \mathbf{n}\wedge\big(\mathbf{v}\wedge\mathbf{n}\big) \\
\mathrm{div}\,\mathbf{v} &= \mathrm{div}\Big(\mathbf{v}_{\Gamma_s} + (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\Big) \\
&= \mathrm{div}\,\mathbf{v}_{\Gamma_s} + \mathrm{div}\,\big((\mathbf{v}\cdot\mathbf{n})\mathbf{n}\big).
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathrm{div}\,\big((\mathbf{v}\cdot\mathbf{n})\mathbf{n}\big) &= \nabla(\mathbf{v}\cdot\mathbf{n})\cdot\mathbf{n} + (\mathbf{v}\cdot\mathbf{n})\,\mathrm{div}\,\mathbf{n} \\
&= \frac{\partial(\mathbf{v}\cdot\mathbf{n})}{\partial s} + (\mathbf{v}\cdot\mathbf{n})\,\mathrm{Tr}\,(S) \\
&= \frac{\partial(\mathbf{v}\cdot\mathbf{n})}{\partial s} + 2(\mathbf{v}\cdot\mathbf{n})H(s)
\end{aligned}$$

where $H(s) = \frac{1}{2}TrS$ is the mean curvature of Γ_s.

LEMMA 3.1. *The restriction of surfacic divergence to Γ_s is the surfacic divergence on Γ_s of the restriction of the vector field to Γ_s.*

$$\Big(div\,\mathbf{v}_{\Gamma_s}\Big)\Big|_{\Gamma_s} = div_{\Gamma_s}\mathbf{v}_{\Gamma_s}.$$

Proof. It suffices to prove the case $s = 0$. Thus we have only to prove

$$\operatorname{div} \mathbf{v}_{\Gamma_s} = \operatorname{div}\left(\widetilde{\mathbf{v}_\Gamma}\right) \quad \text{when } s = 0.$$

For $\mathcal{C}^1$ vector field $\mathbf{v}$, $\mathbf{v}_{\Gamma_s}(\mathbf{y}) - \mathbf{v}_\Gamma(\mathbf{x})$ vanishes like s of Γ. More precisely

$$\mathbf{v}_{\Gamma_s}(\mathbf{y}) - \mathbf{v}_\Gamma(\mathbf{x}) = s(\mathbf{y})\mathbf{w}(s, \mathbf{y}), \;\; \mathbf{w}(s, \mathbf{y}) \in T\Gamma_s.$$

Now

$$\begin{aligned}
\mathbf{v}_{\Gamma_s}(\mathbf{y}) - \widetilde{\mathbf{v}_\Gamma}(\mathbf{y}) &= \mathbf{v}_{\Gamma_s}(\mathbf{y}) - \Big(\mathbf{v}_\Gamma(\mathbf{x}) - s(\mathbf{y})S(\mathbf{y})\mathbf{v}_\Gamma(\mathbf{x})\Big) \\
&= \Big(\mathbf{v}_{\Gamma_s}(\mathbf{y}) - \mathbf{v}_\Gamma(\mathbf{x})\Big) + s(\mathbf{y})S(\mathbf{y})\mathbf{v}_\Gamma(\mathbf{x}) \\
&= s(\mathbf{y})\Big(\mathbf{w}(s, \mathbf{y}) + S(\mathbf{y})\mathbf{v}_\Gamma(\mathbf{x})\Big) \\
&= s(\mathbf{y})\mathbf{t}(s, \mathbf{y}), \;\; \mathbf{t} \in T\Gamma_s.
\end{aligned}$$

Finally,

$$\begin{aligned}
\operatorname{div}\left(s(\mathbf{y})\mathbf{t}\right) &= \nabla s \cdot \mathbf{t} + s \operatorname{div} t \\
&= s \operatorname{div} t \\
&= 0, \;\; \text{when } s = 0.
\end{aligned}$$

Therefore

$$\operatorname{div} \mathbf{v}_{\Gamma_s} = \operatorname{div}\left(\widetilde{\mathbf{v}_\Gamma}\right), \quad \text{when } s = 0.$$

□

Let's consider the expression of the $\operatorname{curl} \mathbf{v}$. Since $\mathbf{v} = \mathbf{v}_{\Gamma_s} + (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$, we have

$$\begin{aligned}
\operatorname{curl} \mathbf{v} &= \operatorname{curl}\left(\mathbf{v}_{\Gamma_s}\right) + \operatorname{curl}\big((\mathbf{v} \cdot \mathbf{n})\mathbf{n}\big) \\
&= \operatorname{curl}\big(\mathbf{n} \wedge (\mathbf{v} \wedge \mathbf{n})\big) + \operatorname{curl}\big((\mathbf{v} \cdot \mathbf{n})\mathbf{n}\big).
\end{aligned}$$

Now

$$\begin{aligned}
\operatorname{curl}\big((\mathbf{v} \cdot \mathbf{n})\mathbf{n}\big) &= \nabla(\mathbf{v} \cdot n) \wedge \mathbf{n} + (\mathbf{v} \cdot \mathbf{n}) \operatorname{curl} \mathbf{n} \\
&= \nabla(\mathbf{v} \cdot n) \wedge \mathbf{n} + \mathbf{0} \\
&= \overrightarrow{\operatorname{curl}}_{\Gamma_s}(\mathbf{v} \cdot \mathbf{n}).
\end{aligned}$$

On the other hand, upon using the identity,

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \operatorname{div} \mathbf{b} - \mathbf{b} \operatorname{div} \mathbf{a} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b},$$

we have

$$
\begin{aligned}
&\operatorname{curl}\big(\mathbf{n}\wedge(\mathbf{v}\wedge\mathbf{n})\big)\\
&= \operatorname{div}(\mathbf{v}\wedge\mathbf{n})\mathbf{n} - (\operatorname{div} n)(\mathbf{v}\wedge\mathbf{n}) + \Big((\mathbf{v}\wedge\mathbf{n})\cdot\nabla\Big)\mathbf{n} - \Big(\mathbf{n}\cdot\nabla\Big)(\mathbf{v}\wedge\mathbf{n})\\
&= \operatorname{div}(\mathbf{v}_{\Gamma_s}\wedge\mathbf{n})\mathbf{n} - 2H(s)(\mathbf{v}\wedge\mathbf{n}) + S(\mathbf{v}\wedge\mathbf{n}) - \frac{\partial(\mathbf{v}\wedge\mathbf{n})}{\partial s}\\
&= \big(\operatorname{curl}_{\Gamma_s}\mathbf{v}_{\Gamma_s}\big)\mathbf{n} - 2H(s)(\mathbf{v}\wedge\mathbf{n}) + S(\mathbf{v}\wedge\mathbf{n}) - \frac{\partial(\mathbf{v}\wedge\mathbf{n})}{\partial s}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\operatorname{curl}\mathbf{v} &= \big(\operatorname{curl}_{\Gamma_s}\mathbf{v}_{\Gamma_s}\big)\mathbf{n} - 2H(s)(\mathbf{v}\wedge\mathbf{n}) + S(\mathbf{v}\wedge\mathbf{n})\\
&\qquad - \frac{\partial(\mathbf{v}\wedge\mathbf{n})}{\partial s} + \overrightarrow{\operatorname{curl}}_{\Gamma_s}(\mathbf{v}\cdot\mathbf{n})\\
&= \big(\operatorname{curl}_{\Gamma_s}\mathbf{v}_{\Gamma_s}\big)\mathbf{n} + \text{ tangential vector to } \Gamma_s.
\end{aligned}
$$

4. Removal of hypersingularity

The solution $\big(u,\mathbf{v}\big)$ of equation (2) on page 431 can be expressed as a convolution of the right hand side with the fundamental solution of the equation.

$$
\begin{aligned}
u(\mathbf{y}) &= U * \big(-\varphi\mathbf{n}\big)d\Gamma\\
&= \int_\Gamma \frac{\partial E}{\partial \mathbf{n_x}}(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})ds_{\mathbf{x}}\\
\mathbf{v} &= \Sigma * \big(-\varphi\mathbf{n}\big)d\Gamma\\
&= \operatorname{curl}\operatorname{curl}(EI_3) * (\varphi\mathbf{n})d\Gamma - k^2\int_\Gamma E(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\mathbf{n_x}ds_{\mathbf{x}}\\
&= \operatorname{curl}(EI_3) * \operatorname{curl}(\varphi\mathbf{n})d\Gamma - k^2\int_\Gamma E(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\mathbf{n_x}ds_{\mathbf{x}}
\end{aligned}
$$

$\operatorname{curl}\big(\varphi\mathbf{n}d\Gamma\big) = \big(\overrightarrow{\operatorname{curl}}_\Gamma\varphi\big)d\Gamma$. In fact

$$
\begin{aligned}
\langle \operatorname{curl}\big(\varphi\mathbf{n}d\Gamma\big),\mathbf{w}\rangle &= \langle\varphi\mathbf{d}\Gamma,\ \operatorname{curl}\mathbf{w}\rangle\\
&= \int_\Gamma \varphi\big(\mathbf{n}\cdot\ \operatorname{curl}\mathbf{w}\big)ds_{\mathbf{x}}.
\end{aligned}
$$

Note that

$$\begin{aligned}&\operatorname{curl}\mathbf{w}\\&=\big(\operatorname{curl}_\Gamma\mathbf{w}_\Gamma\big)\mathbf{n}+\overrightarrow{\operatorname{curl}}_\Gamma(\mathbf{w}\cdot\mathbf{n})+R_s(\mathbf{w}\wedge\mathbf{n})\\&\qquad\qquad\qquad\qquad-2H_s(\mathbf{w}\wedge\mathbf{n})-\frac{\partial}{\partial s}(\mathbf{w}\wedge\mathbf{n})\\&=\big(\operatorname{curl}_\Gamma\mathbf{w}_\Gamma\big)\mathbf{n}+\text{tangential component.}\end{aligned}$$

Therefore

$$\mathbf{n}\cdot\mathbf{w}=\operatorname{curl}_\Gamma\mathbf{w}_\Gamma.$$

Hence

$$\begin{aligned}\langle\operatorname{curl}\big(\varphi\mathbf{n}d\Gamma\big),\mathbf{w}\rangle&=\langle\varphi\mathbf{d}\Gamma,\operatorname{curl}\mathbf{w}\rangle\\&=\int_\Gamma\varphi\big(\mathbf{n}\cdot\operatorname{curl}\mathbf{w}\big)ds_\mathbf{x}\\&=\int_\Gamma\varphi\big(\operatorname{curl}_\Gamma\mathbf{w}_\Gamma ds_\mathbf{x}\\&=\int_\Gamma\overrightarrow{\operatorname{curl}}_\Gamma\varphi\cdot\mathbf{w}_\Gamma ds_\mathbf{x}\\&=\int_\Gamma\overrightarrow{\operatorname{curl}}_\Gamma\varphi\cdot\mathbf{w}ds_\mathbf{x}\\&=\langle\overrightarrow{\operatorname{curl}}_\Gamma\varphi d\Gamma,\mathbf{w}\rangle.\end{aligned}$$

Let's consider the expression for

$$\mathbf{v}=\operatorname{curl}(EI_3)*\overrightarrow{\operatorname{curl}}_\Gamma\varphi d\Gamma-k^2\int_\Gamma E(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\mathbf{n}_\mathbf{x}ds_\mathbf{x}.$$

From the double layer potential

$$u(\mathbf{y})=\int_\Gamma\frac{\partial E}{\partial\mathbf{n}_\mathbf{x}}(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})ds_\mathbf{x},$$

we have

$$\begin{aligned}\big[u\big]&=u_{\text{int}}-u_{\text{ext}}=-\varphi,\\\nabla u(\mathbf{y})&=\int_\Gamma\nabla_\mathbf{y}\Big(\frac{\partial E}{\partial\mathbf{n}_\mathbf{x}}(\mathbf{x}-\mathbf{y})\Big)\varphi(\mathbf{x})ds_\mathbf{x},\ \mathbf{y}\in\Omega_e,\\\mathbf{v}\cdot\mathbf{n}&=\lim_{\mathbf{y}\to\Gamma}\nabla u(\mathbf{y})\cdot\mathbf{n}.\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_\Gamma (\mathbf{v}\cdot\mathbf{n})\psi(\mathbf{y})ds_{\mathbf{y}} \\
&= \langle\, \mathrm{curl}\,(EI_3) * \overrightarrow{curl_\Gamma}\varphi d\Gamma, \psi\mathbf{n}d\Gamma\rangle \\
&\qquad\qquad -k^2\int_\Gamma\int_\Gamma E(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\psi(\mathbf{y})\big(\mathbf{n}_{\mathbf{x}}\cdot\mathbf{n}_{\mathbf{y}}\big)ds_{\mathbf{x}}ds_{\mathbf{y}} \\
&= \langle EI_3 * \overrightarrow{curl_\Gamma}\varphi d\Gamma, \overrightarrow{\mathrm{curl}\,_\Gamma}\psi d\Gamma\rangle \\
&\qquad\qquad -k^2\int_\Gamma\int_\Gamma E(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\psi(\mathbf{y})\big(\mathbf{n}_{\mathbf{x}}\cdot\mathbf{n}_{\mathbf{y}}\big)ds_{\mathbf{x}}ds_{\mathbf{y}} \\
&= \int_\Gamma\int_\Gamma E(\mathbf{x}-\mathbf{y})\Big(\overrightarrow{\mathrm{curl}\,_\Gamma}\varphi(\mathbf{x})\cdot\overrightarrow{\mathrm{curl}\,_\Gamma}\psi(\mathbf{y})\Big)ds_{\mathbf{x}}ds_{\mathbf{y}} \\
&\qquad\qquad -k^2\int_\Gamma\int_\Gamma E(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\psi(\mathbf{y})\big(\mathbf{n}_{\mathbf{x}}\cdot\mathbf{n}_{\mathbf{y}}\big)ds_{\mathbf{x}}ds_{\mathbf{y}}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_\Gamma(\mathbf{v}\cdot\mathbf{n}_{\mathbf{y}})\psi(\mathbf{y})ds_{\mathbf{y}} &= \int_\Gamma\int_\Gamma \nabla_{\mathbf{y}}\Big(\frac{\partial E}{\partial\mathbf{n}_{\mathbf{x}}}(\mathbf{x}-\mathbf{y})\Big)\cdot\mathbf{n}\varphi(\mathbf{x})\psi(\mathbf{y})ds_{\mathbf{x}}ds_{\mathbf{y}} \\
&= -\int_\Gamma\int_\Gamma \frac{\partial^2 E}{\partial\mathbf{n}_{\mathbf{x}}\partial\mathbf{n}_{\mathbf{y}}}\big(\mathbf{x}-\mathbf{y}\big)\varphi(\mathbf{x})\psi(\mathbf{y})ds_{\mathbf{x}}ds_{\mathbf{y}}.
\end{aligned}$$

We have finally replaced the hyper-singular integral with a logarithmic sigularity.

$$\begin{aligned}
&\int_\Gamma\int_\Gamma \frac{\partial^2 E}{\partial\mathbf{n}_{\mathbf{x}}\partial\mathbf{n}_{\mathbf{y}}}\big(\mathbf{x}-\mathbf{y}\big)\varphi(\mathbf{x})\psi(\mathbf{y})ds_{\mathbf{x}}ds_{\mathbf{y}} \\
&\quad = k^2\int_\Gamma\int_\Gamma E(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\psi(\mathbf{y})\big(\mathbf{n}_{\mathbf{x}}\cdot\mathbf{n}_{\mathbf{y}}\big)ds_{\mathbf{x}}ds_{\mathbf{y}} \\
&\qquad -\int_\Gamma\int_\Gamma E(\mathbf{x}-\mathbf{y})\Big(\overrightarrow{\mathrm{curl}\,_\Gamma}\varphi(\mathbf{x})\cdot\overrightarrow{\mathrm{curl}\,_\Gamma}\psi(\mathbf{y})\Big)ds_{\mathbf{x}}ds_{\mathbf{y}}.
\end{aligned}$$

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Korea Institute for Curriculum and Evaluation
Seoul, Korea
E-mail: yibongju@kice.re.kr