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A VERY SINGULAR SOLUTION OF A DOUBLY DEGENERATE PARABOLIC EQUATION WITH NONLINEAR CONVECTION

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Abstract. We here investigate an existence and uniqueness of the nontrivial, nonnegative solution of a nonlinear ordinary differential equation:

$$[|(w^m)'|^{p-2}(w^m)']' + \beta r w' + \alpha w + (w^q)' = 0$$

satisfying a specific decay rate: $\lim_{r\to\infty} r^{\alpha/\beta} w(r) = 0$ with $\alpha := (p-1)/[pq-(m+1)(p-1)]$ and $\beta := [q-m(p-1)]/[pq-(m+1)(p-1)]$. Here $m(p-1) > 1$ and $m(p-1) < q < (m+1)(p-1)$. Such a solution arises naturally when we study a very singular solution for a doubly degenerate equation with nonlinear convection:

$$u_t = [|(u^m)_x|^{p-2}(u^m)_x]_x + (u^q)_x$$

defined on the half line.

1. Introduction

In this paper, we consider a one dimensional doubly degenerate equation with nonlinear convection term

$$u_t = [|(u^m)_x|^{p-2}(u^m)_x]_x + (u^q)_x, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \tag{1.1}$$

with *Neumann* boundary condition

$$u_x(0,t) = 0, \tag{1.2}$$

where $m(p-1) > 1$, $q > m(p-1)$.

Equation (1.1) (sometimes called the non-*Newtonian* filtration equation) arises in the study a compressible fluid flows in a homogeneous isotropic rigid porous medium, flows of polytrophic gas and has various other applications, see, [23], [5]. From a mathematical point of view, we note that (1.1) is a quasilinear equation which is nonuniform parabolic and it is *doubly degenerate* on

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the sets $\{u_x = 0\}$ and $\{u = 0\}$ (if $q = 1$, equation (1.1) reduces to the standard doubly degenerate equation by an easy change of variables). The case $m(p-1) > 1$ occurs in slow diffusion process and $0 < m(p-1) < 1$ in fast diffusion process (see [19] and [22] for examples).

We are mostly interested in nonnegative solutions of (1.1) having the form

$$u(x,t) = t^{-\alpha} w(xt^{-\beta}) := t^{-\alpha} w(r), \tag{1.3}$$

where α, β are positive numbers. We substitute (1.3) into (1.1) to find

$$\alpha := (p-1)/[pq-(m+1)(p-1)], \quad \beta := [q-m(p-1)]/[pq-(m+1)(p-1)] \tag{1.4}$$

and w, as a function of $r = xt^{-\beta}$, solves an ordinary differential equation:

$$[|(w^m)'|^{p-2}(w^m)']' + \beta r w' + \alpha w + (w^q)' = 0. \tag{1.5}$$

We observe that if $u(x,t)$ solves (1.1), then the *rescaled functions*

$$u_\rho(x,t) = \rho^{(p-1)/[q-m(p-1)]} u(\rho x, \rho^{[pq-(m+1)(p-1)]/[q-m(p-1)]} t), \quad \rho > 0 \tag{1.6}$$

define a one parameter family of solutions to (1.1). A solution $u(x,t)$ is said to be *self-similar* when $u_\rho(x,t) = u(x,t)$ for every $\rho > 0$. It can be easily verified that $u(x,t)$ is a self-similar solution to (1.1) if and only if u has the form (1.3). We also remark that the self-similar solutions play an important role in the study of large time behaviors of general solutions (see [16, 18] and [24]).

Every nonnegative, bounded solution of (1.5) has exactly one critical point and since we here apply the shooting method, led to solve a more general initial value problem

$$[|(w^m)'|^{p-2}(w^m)']' + \beta r w' + \alpha w + (w^q)' = 0$$

for $r \geq 0$ with initial conditions

$$w'(0) = 0, \quad w(0) = \mu, \tag{1.7}$$

where μ may be any positive number.

Using the *Shauder's* fixed point theorem (or *Banach* contraction theorem), we find that initial value problem has an unique solution which we denote by $w(r;\mu)$. In many cases, it turns out that the limit

$$L(\lambda) = \lim_{r\to\infty} r^{\alpha/\beta} w(r) \tag{1.8}$$

exists and we distinguish between fast and slow orbits according to whether $L(\lambda) = 0$ or not respectively. The fast orbit will bring out a very singular solution of (1.1). The very singular solution has a stronger singularity at the origin than the singular solution of that equation. By a *singular solution* we mean a nonnegative and nontrivial solution which satisfies the equation and vanishes outside any open neighborhood of the origin as $t \to 0$. A singular solution is called a *very singular solution* if the integral of $u(x,t)$ over any open neighborhood of the origin becomes unbounded as $t \to 0$, which is equivalent to, if u is given by (1.3),

$$\lim_{r\to\infty} r^{\alpha/\beta} w(r) = 0. \tag{1.9}$$

Furthermore, if $0 < \beta < \alpha$ and a solution f of (1.5) satisfies (1.9), then $u(x,t)$ given explicitly by (1.3) becomes a very singular self-similar solution of (1.1).

Our goal is to find the relation of values m, p, q and initial data μ which insure that $w(\cdot,\mu)$ is a fast decaying solution and to give an exact asymptotic behavior of solutions at near infinity. More precisely, our main results include the followings;

- If $\alpha \leq \beta$ (i.e., $q \geq (m+1)(p-1)$), then there not exists any fast orbit (very singular solution) and indeed, only exists slow orbits for any $\mu > 0$.
- If $\alpha > \beta$ (i.e., $m(p-1) < q < (m+1)(p-1)$), then there exists μ_1 such that

  (i) $w(r;\mu)$ is changes sign with $w^m(R^-;\mu) < 0$ for $\mu \in (0,\mu_1)$.

  (ii) $w(r;\mu)$ is a slow orbit and having the behavior

$$w(r;\mu) \sim L(\mu) r^{-\alpha/\beta}$$

  at near of infinity for $\mu \in (\mu_1, +\infty)$, with $L(\mu) > 0$.

  (iii) $w(r;\mu_1)$ is the only fast orbit with $w^m(R^-;\mu) = 0$ and having the interface relation

$$\lim_{r\to R^-}(w^{[m(p-1)-1]/(p-1)})'(r) = -[m(p-1)-1]/[m(p-1)]\beta^{1/(p-1)}R^{1/(p-1)}$$

  for some $0 < R < \infty$.

There have been many works dealing with the existence, uniqueness and asymptotic behavior of self-similar solutions to a class of quasilinear parabolic equations with absorption (or source, convection) term. For instance, it is thoroughly treated on the P-Laplacian equation with absorption term;

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q \quad \text{in } \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^+$$

with $p > 1$, $q > 1$. For linear diffusion case($p = 2$), see [3], [6], [13] for slow diffusion case($p > 2$), see [22] and [1], [16] for fast diffusion case($1 < p < 2$).

Recently some authors studied (1.1) with $m(p-1) \geq 1$, $q > m(p-1)$. They derived some estimates, used a suitable scaling and convergence of re-scaled solutions to self-similar ones, and concluded that the asymptotic of general solutions is self-similar (see [16, 17, 18, 24]). Similar arguments have been used in the case of the multidimensional convection-diffusion equation (see [7, 8], for examples). In addition, very singular self-similar solutions are found for the linear diffusion equation with convection on half line under the homogeneous *Neumann* boundary condition which motivated our investigation (see [2], [11], and [20]).

Let $f = w^m$, $\lambda = \mu^m$. Then the initial value problem (1.5), (1.7) is replaced by the following problem with respect to f

$$\left\{\begin{array}{l}(|f'|^{p-2}f')' + \beta r(f^{1/m})' + \alpha f^{1/m} + (f^{q/m})' = 0 \quad \text{in} \quad r > 0 \\ f(r) > 0 \\ f'(0) = 0, \quad f(0) = \lambda > 0\end{array}\right. \tag{1.10}$$

and the condition (1.9) is replaced by

$$\lim_{r\to\infty} r^{\alpha/\beta} f^{1/m}(r;\lambda) = 0. \tag{1.11}$$

Indeed, all throughout this paper we consider the above problem with respect to f and always assume that $[0, R)$ is the maximal existence interval of nonnegative solution of w or f.

The plan of this paper is as follows. In Section 2, we study basic properties of f which will be useful in the proof of the main results. In Section 3, we study the nonexistence of the very singular solution(fast orbit) when $q \geq (m+1)(p-1)$. In Section 4, we study the existence changing sign solutions, fast(slow) decaying global solutions and finding the decay rates, the interface relation when $m(p-1) < q < (m+1)(p-1)$. In Section 5, we show that uniqueness of the very singular solution.

2. Preliminary results

In this section we shall derive some properties of f which will be useful in the proof of the main results.

We first show that the sign of f' are depending on the sign of α and f decreases as long as it is positive, and also give the behavior of f, f' at near of infinity.

Lemma 2.1. *Assume that $\alpha > 0$, $\beta > 0$ and $\lambda > 0$. Let f be a solutions to* (1.10) *such that $f > 0$ on $[0, R)$ with R possibly infinity. Then*

(i) $\lim_{r\to R^-} f(r) = 0$.

(ii) $f'(r) < 0$ *in* $(0, R)$.

(iii) $\lim_{r\to\infty} f'(r) = 0$ *when* $R = \infty$.

Proof. We first to show that (ii).

By (1.10) we obtain $(|f'|^{p-2}f')'(0) = -\alpha\lambda^{1/m} < 0$. Thus, the function is strictly decreasing for small r. Suppose that there exists first zero of f' is r_1 such that $f(r) > 0$ on $(0, r_1)$ and $f'(r_1) = 0$. From (1.10) one sees $(|f'|^{p-2}f')'(r_1) < 0$, which is impossible.

Since f is strictly decreasing and f is bounded below by 0, there exists

$$\lim_{r\to R^-} f(r) = l \in [0, \lambda). \tag{2.1}$$

We define the energy function $E(r) = (p-1)/p|f'|^p + m\alpha/(m+1)f^{(m+1)/m}$ and obtain

$$\frac{d}{dr}E(r) = -(f')^2/m(\beta r f^{(1-m)/m} + q f^{(q-m)/m}) < 0$$

for $r > 0$. Thus, $E(r)$ decreases monotonically to a limit and there also exists the limit

$$\lim_{r\to R^-} f'(r) = -l_1, \quad l_1 \in [0,\infty). \tag{2.2}$$

In particular l_1 must be zero so that f is positive for all positive r.

Next, we prove that $l = 0$. Suppose to the contrary $l > 0$. By (iii) we obtain

$$\liminf_{r\longrightarrow\infty} |f''(r)| = 0. \tag{2.3}$$

Moreover, we easy to see that

$$\lim_{r\longrightarrow\infty} r(f^{1/m})'(r) = -\alpha/\beta l^{1/m} < 0. \tag{2.4}$$

Indeed, the function $r(f^{1/m})'$ is either eventually oscillates or monotone. If monotone, clearly holds by (1.5), (2.3) and if oscillates, we choose the sequence r_j realizing the minima(or maxima) of the function $r(f^{1/m})'$, then remain holds above result (2.4).

By (2.4) yields there exists r_0 such that

$$(f^{1/m})' < -C/r \quad \text{for } r \geq r_0,$$

where $C > 0$, which implies that $f(r) \to -\infty$ as $r \to +\infty$, which leads to a contradiction. □

By Lemma 2.1 (ii), $f'(r) < 0$ in $(0, R)$ for any $\lambda > 0$ and we find that if $R < \infty$, then $f(R) = 0$ and $f'(R) \leq 0$. We next show that if $f'(R) = 0$, then f vanishes identically after R.

Lemma 2.2. *Assume that $\alpha > 0$ and $\lambda > 0$. Let f be any solution of* (1.10) *with $f(R) = f'(R) = 0$ for $R > 0$. Then $f = 0$ for all $r \geq R$.*

Proof. By convention, (1.10) is rewritten as

$$(|f'|^{p-2}f')' + \beta r(f^{1/m})' + \alpha f^{1/m} + (|f|^{(q-m)/m}f)' = 0. \tag{2.5}$$

Thus, without loss of generality, we may assume that $f(r) > 0$ and $f'(r) > 0$ for r near R with $r > R$. For such r, we find easily from (2.5) that $(|f'|^{p-2}f')'(r) < 0$. Integrating over (R, r), we see that $|f'|^{p-2}f'(r) < 0$, which contradict to the assumption. □

3. The case $\beta \geq \alpha$ ($q \geq 2(p-1)$)

In this section, we show that there does not exist any fast orbit for the problem (1.10) and thus no very singular solution for (1.1) when $0 < \alpha \leq \beta$.

Theorem 3.1. *Assume $\beta \geq \alpha$ ($q \geq (m+1)(p-1)$). For each $\lambda > 0$, let $f(r;\lambda)$ be the solution of* (1.10). *Then $R = \infty$ and $\liminf_{r\to\infty} r^{\alpha/\beta} f^{1/m}(r;\lambda) > 0$.*

Proof. We assume $R < \infty$ to the contrary and integrate (1.10) over $(0, R)$ to get

$$|f'|^{p-2}f'(R) + (\alpha - \beta)\int_0^R f^{1/m}(r)dr - \lambda^{q/m} = 0,$$

which is impossible. Thus f is positive for all $r \geq 0$ and $R = \infty$.

Moreover, we have, for $r > 0$,

$$\begin{aligned}&\{r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta}f^{1/m}\}'\\ &= r^{\alpha/\beta-1}\{(|f'|^{p-2}f')' + \frac{\alpha/\beta - 1}{r}|f'|^{p-2}f' + \alpha f^{1/m} + \beta r(f^{1/m})'\}.\end{aligned}$$

By (1.5), we get

$$\{r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta}f^{1/m}\}' = r^{\alpha/\beta-1}\{\frac{\alpha/\beta - 1}{r}|f'|^{p-2}f' - (f^{q/m})'\} > 0$$

by the condition $\beta \geq \alpha$ and $f' < 0$. If we define the function

$$F(r) := r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta}f^{1/m},$$

then we see that $F(0) = 0$ and $F(r)$ is strictly increasing for all $r > 0$. Since f is a decreasing function, one must have $\liminf_{r\to\infty} r^{\alpha/\beta}f^{1/m}(r; \lambda) > 0$. □

We will see later that the limit $\lim_{r\to\infty} r^{\alpha/\beta}f^{1/m}(r; \lambda)$ exists for each $\lambda > 0$. Thus we may conclude together with Theorem 3.1 that there exist slow orbits only.

4. The case $\alpha > \beta$ $(m(p-1) < q < (m+1)(p-1))$

In this section, we first show that the solution changes sign for small λ and we next show that the solution becomes a slow orbit for suitably large λ. We then find a fast orbit in-between. The slow orbits will be shown to be ordered and the minimal one becomes the fast orbit as we have seen in many cases, see [10], [13], [1], [19], [20] for examples.

Define the following three sets for any initial value $\lambda > 0$,

$$\begin{aligned}\mathcal{S}_1 &= \{\lambda > 0;\ R < \infty,\ f'(R^-, \lambda) < 0\},\\ \mathcal{S}_2 &= \{\lambda > 0;\ R < \infty,\ f'(R^-, \lambda) = 0\},\\ \mathcal{S}_3 &= \{\lambda > 0;\ \ R = \infty,\ f(r, \lambda) > 0\}.\end{aligned}$$

Obviously, there sets are disjoint and $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = (0, \infty)$.

We first show that the problem (1.10) has changes sign for "small" $\lambda > 0$.

Theorem 4.1. *The set $\mathcal{S}_1 \neq \emptyset$ and open.*

Proof. By integrating (1.10), one has

$$|f'|^{p-2}f' + \beta r f^{1/m} = \phi(r) := -(\alpha - \beta)\int_0^r f^{1/m}dr - f^{q/m} + \lambda^{q/m}. \tag{4.1}$$

One finds easily that $\phi(0) = 0$, $\phi'(r) = -(\alpha - \beta)f^{1/m} - q/mf^{(q-m)/m}f'$, and $\phi'(0) = -(\alpha - \beta)\lambda < 0$.

Suppose that $\phi(r) < 0$ and thus

$$|f'|^{p-2}f' + \beta r f^{1/m} < 0, \qquad 0 < r < r_0 \tag{4.2}$$

for some r_0 to be determined later. An integration of (4.2) yields

$$f^{\frac{m(p-1)-1}{m(p-1)}}(r) < \lambda^{\frac{m(p-1)-1}{m(p-1)}} - \frac{m(p-1)^2\beta^{1/(p-1)}}{p[m(p-1)-1]} r^{p/(p-1)}.$$

Thus if $r_0 > R_0 := \left(\frac{p[m(p-1)-1]}{m(p-1)^2\beta^{1/(p-1)}}\lambda^{\frac{m(p-1)-1}{m(p-1)}}\right)^{(p-1)/p}$, then f must change sign and we are done. Otherwise, we may assume that $\phi(r_0) = 0$ for some $r_0 \le R_0$. From definition, we obtain $f'(r_0) = -\beta^{1/(p-1)} r_0^{1/(p-1)} f^{1/[m(p-1)]}(r_0)$ and

$$\phi'(r_0) = -(\alpha - \beta) f^{1/m}(r_0) - q/m f^{q/m-1} f'(r_0) \ge 0.$$

Combining these, we have

$$0 < \alpha - \beta \le q/m\beta^{1/(p-1)} r_0^{1/(p-1)} f^{q/m-[(m+1)(p-1)-1]/[m(p-1)]}.$$

Since f is a decreasing solution, we also have $f(r_0) \le \lambda$ and

$$\begin{aligned} \alpha - \beta \le q/m\beta^{1/(p-1)} &\left(\frac{p[m(p-1)-1]}{m(p-1)^2\beta^{1/(p-1)}}\right)^{1/p} \\ &\cdot \lambda^{\frac{m(p-1)-1}{mp} + q/m - [(m+1)(p-1)-1]/[m(p-1)]}. \end{aligned} \tag{4.3}$$

The inequality (4.3) does not hold for all sufficiently small λ, which proves the first part of theorem. The continuous dependence of solutions on the initial values implies that $\mathcal{S}_1$ is an open set. □

We next prove that the problem (1.10) has a global positive decaying solution for all suitably large λ.

Lemma 4.2. *Let $\alpha > \beta$. Then for any R_0 there exists λ_0 such that $f(r) = f(r, \lambda) > 0$ for $0 < r < R_0$ and $f(R_0) + |f'(R_0)|^{p-2} f'(R_0) > 0$ for all $\lambda \ge \lambda_0$.*

Proof. We define $f_\lambda(t) = \frac{1}{\lambda} f(r, \lambda)$, $t = r\lambda^\delta$ with $\delta = \frac{[q-m(p+1)]}{m(p-1)}$. Then f_λ satisfies $f_\lambda'(0) = 0$, $f_\lambda(0) = 1$ and the following equation

$$(|f_\lambda'|^{p-2} f_\lambda')' + \lambda^{-\frac{qp-(m+1)(p-1)}{m(p-1)}} [\beta t (f_\lambda^{1/m})' + \alpha f_\lambda^{1/m}] + (f_\lambda^{q/m})' = 0.$$

By integrating over $(0, t)$, we obtain

$$\begin{aligned} &|f_\lambda'|^{p-2} f_\lambda' + \lambda^{-\frac{qp-(m+1)(p-1)}{m(p-1)}} (\alpha - \beta) \\ &\cdot \int_0^t f_\lambda^{1/m} d\tau + \lambda^{-\frac{qp-(m+1)(p-1)}{m(p-1)}} \beta t f_\lambda^{1/m} + (f_\lambda^{q/m} - 1) = 0. \end{aligned}$$

Since f_λ is bounded by 1, for any $\epsilon > 0$ there is λ_0 such that whenever $\lambda \ge \lambda_0$,

$$1 - \epsilon < |f_\lambda'|^{p-2} f_\lambda' + f_\lambda^{q/m} < 1 + \epsilon$$

for $t \in [0, \frac{qp-(m+1)(p-1)}{m(p-1)} - \epsilon]$, which implies lemma. □

We also prove the next key-observation:

Proposition 4.3. *Assume that $\alpha > 0$, $\beta > 0$ and $\lambda > 0$. Let f be any solution to* (1.10). *Consider the function $E_c(r) := cf + rf'$ for $c > 0$. Then*
(i) *If $c > m\alpha/\beta$, then $E_c(r)$ is eventually positive.*
(ii) *If $c < m\alpha/\beta$, then $E_c(r)$ is eventually negative.*

Proof. By direct calculations and (1.10), we obtain

$$\begin{aligned}&(p-1)|f'|^{p-2}E_c'(r)\\&= (c+1)(p-1)|f'|^{p-2}f' - \beta r^2(f^{1/m})' - \alpha r f^{1/m} - q/mrf^{(q-m)/m}f'\end{aligned} \tag{4.4}$$

and at any $r = r_0$ for which $E_c(r_0) = 0$ we have

$$\begin{aligned}&(p-1)|f'|^{p-2}E_c'(r_0)\\&= -(c+1)(p-1)c^{p-1}(f/r_0)^{p-1} + (\beta/mc - \alpha)r_0 f^{1/m} + qc/mf^{q/m}.\end{aligned} \tag{4.5}$$

Since the middle term on the right hand side of (4.5) dominates the others for all sufficiently large r_0, the sign of $E_c'(r_0)$ is only decided by the sign of $\beta/mc - \alpha$ and thus $E_c(r)$ becomes of the same sign eventually.

In order prove (i), we suppose that there exists r_1 such that $E_c(r) < 0$ for all $r \geq r_1$. From equation (1.10) and Lemma 2.1 (ii) we deduce that

$$(|f'|^{p-2}f')' - (\beta/mc - \alpha)f^{1/m} = -\beta/mf^{1/m-1}E_c(r) - (f^{q/m})' > 0$$

for $r \geq r_1$. Multiplying the previous inequality by f' and integrating from r to τ with $r_1 \leq r \leq \tau$, we have

$$(p-1)/p|f'|^p(\tau) - c_1 f^{(m+1)/m}(\tau) \leq (p-1)/p|f'|^p(r) - c_1 f^{(m+1)/m}(r),$$

where $c_1 := (\beta c - m\alpha)/(m+1)$. Letting $\tau \to \infty$ and using Lemma 2.1(ii), (iii), we get the following inequality

$$-f'f^{-\frac{m+1}{mp}} \geq c_2 > 0, \qquad r \geq r_1.$$

Integrating the previous inequality from r_1 to $r \geq r_1$ we obtain

$$\begin{aligned}&mp/[m(p-1)-1]f^{[m(p-1)-1]/(mp)}(r_1)\\&- mp/[m(p-1)-1]f^{[m(p-1)-1]/(mp)}(r) \geq c_2(r - r_1).\end{aligned}$$

Letting $r \to \infty$ we get a contradiction.

We prove (ii) similarly. Suppose that there exists r_2 such that $E_c(r) > 0$ for all $r \geq r_2$. From (1.10) and assumption,

$$\begin{aligned}(|f'|^{p-2}f')' + \alpha f^{1/m} &= -\beta r(f^{1/m})' + \alpha f^{1/m} - q/mf^{q/m-1}f'\\&\leq \beta/mcf^{1/m} + q/mc/rf^{q/m}.\end{aligned}$$

Since f decrease, we may rewrite this as

$$(|f'|^{p-2}f')' \leq -c_2 f < c_2\frac{rf'}{c}, \tag{4.6}$$

where we define $c_2 = \alpha\lambda^{1/m-1} - c\beta/m\lambda^{1/m-1} + \frac{cq}{mr_2}\lambda^{q/m-1}$ and assume to be positive by retaking r_2. The inequality (4.6) is rewritten as $(p-1)/(p-2)(|f'|^{p-2}f')' \le -c_3 r$ for some positive constant c_3 and an integration from $r = r_2$ to $r = \infty$ yields a contradiction, which completes the proof. □

We rewrite the problem (1.10) as the following system;

$$\text{(4.7)} \qquad \begin{cases} f' = |h|^{-(p-2)/(p-1)}h \\ h' = -\beta/mrf^{1/m-1}|h|^{-(p-2)/(p-1)}h \\ \qquad -\alpha f^{1/m} - q/mf^{(q-m)/m}|h|^{-(p-2)/(p-1)}h. \end{cases}$$

Given any $\delta > 0$, we denote

$$\mathcal{L}_\delta := \{(f,h) : 0 < f \le 1,\ 0 > h > -\delta f\}$$

then we obtain the following lemma.

Lemma 4.4. *For given $\delta > 0$ there exists a $r_\delta := m[\delta + \alpha\delta^{-1/(p-1)}]/\beta$ such that $\mathcal{L}_\delta$ is positively invariant for $r > t_\delta$. That is $(f(r_\delta), h(r_\delta)) \in \mathcal{L}_\delta$ then the orbit $(f(r), h(r))$ of (4.7) remains in region $\mathcal{L}_\delta$ for all $r \ge r_\delta$.*

Proof. We shall show that given $\delta > 0$ there exists a $r_\delta > 0$ such that if $r > r_\delta$, then the vector field determined by (4.7) points into $\mathcal{L}_\delta$, except at the critical point (0. 0). It is easy to see this fact on the top $h = 0$ and the line $f = 1$ and it is enough to verify this only on the line $h = -\delta f$. By the system (4.7), we have

$$\begin{aligned} &\frac{h'}{f'} \\ &= \frac{-\beta/mrf^{1/m-1}|h|^{-(p-2)/(p-1)}h - \alpha f^{1/m} - q/mf^{(q-m)/m}|h|^{-(p-2)/(p-1)}h}{|h|^{-(p-2)/(p-1)}h} \\ &= -\beta/mrf^{1/m-1} + \alpha\delta^{-1/(p-1)}f^{[(p-1)-m]/[m(p-1)]} - q/mf^{(q-m)/m} \\ &< -\beta/mrf^{1/m-1} + \alpha\delta^{-1/(p-1)}f^{[(p-1)-m]/[m(p-1)]} \le -\delta \end{aligned}$$

if $r \ge r_\delta := m[\delta + \alpha\delta^{-1/(p-1)}]/\beta$. □

As a consequence, we can prove the existence of globally positive solutions.

Theorem 4.5. *The set $\mathcal{S}_3$ is nonempty and open.*

Proof. From Lemma 4.2, we can find r_0 such that $f > 0$ for $0 \le r \le r_0$ and $f(r_0) + |f'(r_0)|^{p-2}f'(r_0) > 0$ for all sufficiently large λ. Thus $(f(r_0), |f'|^{p-2}f'(r_0)) \in \mathcal{L}_1$ and by Lemma 4.4, f is positive for all $r > 0$, which proves the first part of theorem.

We next prove $\mathcal{S}_3$ is an open set. Set $\lambda_0 \in \mathcal{S}_3$ and then by Proposition 4.3, $E_1(r) = f + rf'$ becomes positive for all large r. Thus there exist sufficiently large r_0 such that $(f(r_0), |f'|^{p-2}f'(r_0)) \in \mathcal{L}_1$. Then by continuous dependence of solutions on the initial value there is a neighborhood N of λ_0 such that $f(r;\lambda) > 0$ and $(f(r_0;\lambda), |f'|^{p-2}f'(r_0;\lambda)) \in \mathcal{L}_1$ for any $(r,\lambda) \in [0, r_0] \times N$. By

Lemma 4.4, we deduce that the orbits remains in $\mathcal{L}_1$ for any $r > r_0$, which implies in particular that $f(r, \lambda) > 0$ for any $r > r_0$ and $\lambda \in N$. Therefore, $f(r; \lambda) > 0$ for any $r > 0$ and $\lambda \in N$ and $\mathcal{S}_3$ is open. □

We are now going to find exact decay-rates for globally positive solutions.

Theorem 4.6. *For any given $\lambda > 0$, let f be any solution to* (1.10) *such that $f > 0$ for any $r > 0$. Then $\lim_{r\to\infty} r^{\alpha/\beta} f^{1/m}(r; \lambda) = \mathrm{L}(\lambda) > 0$ exists.*

Proof. Step 1: By Lemma 2.1 we know that $f'(r) < 0$ for $r > 0$ and $\lim_{r\to\infty} f(r) = 0$, $\lim_{r\to\infty} f'(r) = 0$. Moreover we have seen that if $c < m\alpha/\beta$, then $E_c(r) = cf + rf' < 0$ for all sufficiently large r, say, $r > r_0$. We easily find that

$$f(r) \le f(r_0) r^{-c}, \qquad r > r_0. \tag{4.8}$$

We also recall that if $d > m\alpha/\beta$, then $E_d(r) = df + rf' > 0$ and thus

$$-f'(r) < df(r)/r, \qquad r > r_1 \tag{4.9}$$

for some $r_1 > 0$.

Step 2: From (1.10), we get

$$\begin{aligned} &\{r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta} f^{1/m}\}' \\ &= r^{\alpha/\beta-1}\{\frac{\alpha/\beta-1}{r}|f'|^{p-2}f' - (f^{q/m})'\}, \end{aligned} \tag{4.10}$$

and integrating over $(0, r)$, we see that

$$\begin{aligned} &r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta} f^{1/m} \\ &= (\alpha/\beta - 1)\int_0^r |f'|^{p-2}f' s^{\alpha/\beta-2} ds + q/m \int_0^r f^{(q-m)/m}|f'| s^{\alpha/\beta-1} ds. \end{aligned} \tag{4.11}$$

Using (4.8) and (4.9), we find that two integrals of the right hand side of (4.11) converge and $\lim_{r\to\infty} r^{\alpha/\beta-1}|f'|^{p-2}f' = 0$. Therefore, the limit $\mathrm{L}(\lambda) = \lim_{r\to\infty} r^{\alpha/\beta} f^{1/m}(r; \lambda)$ exists and finite.

Step 3: We now show that $\mathrm{L}(\lambda) > 0$. Assume that $\mathrm{L}(\lambda) = 0$. Integrating (4.10) over (r, ∞), we have

$$\begin{aligned} &r^{\alpha/\beta-1}|f'|^{p-2}f' + \beta r^{\alpha/\beta} f^{1/m} \\ &= (1 - \alpha/\beta)\int_r^\infty |f'|^{p-2}f' s^{\alpha/\beta-2} ds - q/m \int_r^\infty f^{(q-m)/m}|f'| s^{\alpha/\beta-1} ds. \end{aligned}$$

Again using (4.9), we see that $\mathrm{L}(\lambda) = \lim_{r\to\infty} r^{\alpha/\beta} f^{1/m}(r; \lambda)$ exists and finite. On the other hand, by (4.9),

$$f(r) \ge f(r_1) r^{-d}, \qquad r > r_1.$$

These conflictions implies that $\mathrm{L}(\lambda) > 0$. □

Remark 4.7. Obviously, the limit value $\mathrm{L}(\lambda) = 0$ is achieved only when f has the compact support and Proposition 4.3 and Theorem 4.6 remain true for the case $\alpha \le \beta$.

We finally show that there exists a fast orbit.

Theorem 4.8. *The set $\mathcal{S}_2 \neq \emptyset$ and closed. Moreover, the interface relation holds*

$$\lim_{r\to R^-} (f^{[m(p-1)-1]/[m(p-1)]})'(r) = -[m(p-1)-1]/[m(p-1)]\beta^{1/(p-1)}R^{1/(p-1)}$$

for any $\lambda \in \mathcal{S}_2$.

Proof. By Theorems 4.1 and 4.5, we immediately see that $\mathcal{S}_2$ is nonempty and closed set. From Lemma 2.2, any solution $f = f(r,\lambda)$ with $\lambda \in \mathcal{S}_2$ has a compact support, say, $[0,R]$ and f satisfies condition $f(R) = 0$, $f'(R) = 0$. Integrating the equation (1.10) from r to R we get

$$|f'|^{p-2}f'(r) + \beta r f^{1/m}(r) = (\alpha-\beta)\int_r^R f^{1/m}(s)ds - f^{q/m}(r).$$

Dividing by f, we have

$$\begin{aligned} &f'|^{p-2}f'(r)/f^{1/m}(r) + \beta r \\ &= (\alpha-\beta)\int_r^R f^{1/m}(s)ds/f^{1/m}(r) - f^{(q-1)/m}(r). \end{aligned} \tag{4.12}$$

Since f is strictly decreasing, we find that

$$0 \le \int_r^R f^{1/m}(s)ds \le f^{1/m}(r)(R-r).$$

Hence

$$\lim_{r\to R^-}\int_r^R f^{1/m}(s)ds/f^{1/m}(r) = 0.$$

Letting $r \to R^-$ in (4.12) then we obtain

$$\lim_{r\to R^-} |f'|^{p-2}f'(r)/f^{1/m}(r) = -\beta R$$

and which is equivalent to the second result of theorem. □

In addition, we show the monotonicity of the solutions of the problem (1.10) with respect to λ in the sense that two positive orbits do not intersect each other.

Theorem 4.9. *Assume that $\alpha > 0$, $\beta > 0$ and f_i are solutions of problem* (1.10) *on $[0,R_i)$ with initial data $f_i(0) = \lambda_i > 0$, $i = 1,2$, where $[0,R_i)$ denotes the maximal existence interval of f_i and the R_i are possibly infinity. Then*

$$\lambda_2 > \lambda_1 \Rightarrow f_2(r) > f_1(r) \quad \text{for all } 0 \le r \le R := \min\{R_1,R_2\}.$$

Proof. Suppose contrarily that there exists $R_0 \in [0, R]$ such that $f_1(r) < f_2(r)$ for $r \in [0, R_0)$ and $f_1(R_0) = f_2(R_0)$. We define

$$g_k(r) := k^{-mp/[m(p-1)-1]} f_1(kr), \quad r \in [0, R_1/k)$$

for $k > 0$ and then $g_k(r)$ solves

$$\begin{aligned} &(|g_k'|^{p-2} g_k')' + \beta r (g_k^{1/m})' \\ &+ \alpha g_k^{1/m} + k^{[pq-(m+1)(p-1)]/[m(p-1)-1]} (g_k^{q/m})' = 0. \end{aligned} \tag{4.13}$$

By Lemma 2,1 we know that f_1 is strictly decreasing on $[0, R_1)$ and so g_k is strictly decreasing with respect to k. In particular, $\lim_{k\to 0} g_k(r) = +\infty$ for any $r \in [0, R]$. Thus there exists a small $k_0 > 0$ such that

$$g_k(r) > f_2(r) \quad \text{for } r \in [0, R] \text{ and } k \in [0, k_0]$$

and the set

$$I := \{k \in (0, k_0); g_k(r) > f_2(r) \quad \text{for } r \in [0, R_0]\}$$

is nonempty and open. Setting $l := \sup I$, we see that $l < 1$, $l \notin I$ and there exists $r_0 \in [0, R_0]$ such that $g_l(r_0) = f_2(r_0)$.

If $r_0 = R_0$, then $g_l(R_0) = l^{-mp/[m(p-1)-1]} f_1(lR_0) = f_2(R_0)$. Since $f_1(R_0) = f_2(R_0)$ and g_l is strictly decreasing with respect to l, we conclude that $l = 1$ and which contradicts to the hypothesis. If $r_0 \in (0, R_0)$, then g_l much touch f_2 at $r = r_0$ from the above. But in this case we deduce from (1.10) that

$$\begin{aligned} &(|g_l'|^{p-2} g_l')''(r_0) - (|f_2'|^{p-2} f_2')''(r_0) \\ &= (1 - l^{[pq-(m+1)(p-1)]/[m(p-1)-1]})(f_2^{q/m})'(r_0) < 0, \end{aligned}$$

which obviously violates the strong maximum principle. Thus g_l much touch f_2 at $r = 0$ from the above. But also from (1.10), we find $(|f_2'|^{p-2} f_2')'(0) = -\alpha \lambda_2^{1/m}$ and $(|f_2'|^{p-2} f_2')''(0) = -(f_2^{q/m})''(0) = -q/m \lambda_2^{q/m-1} f_2''(0)$. Similarly for g_l and we obtain

$$\begin{aligned} &(|g_l'|^{p-2} g_l')''(0) - (|f_2'|^{p-2} f_2')''(0) \\ &= (l^{[pq-(m+1)(p-1)]/[m(p-1)-1]} - 1) q/m \lambda_2^{q/m-1} f_2''(0) < 0, \end{aligned}$$

which leads to another contradiction and completes all the proofs. □

5. Uniqueness

In this section, we show that there exists only one fast decaying solution for the problem (1.10).

Recall that such a solution has compact support $[0, R]$ and has an interface relation

$$\begin{aligned} &\lim_{r\to R^-} (f^{[m(p-1)-1]/[m(p-1)]})'(r) \\ &= -[m(p-1)-1]/[m(p-1)] \beta^{1/(p-1)} R^{1/(p-1)} \end{aligned} \tag{5.1}$$

by Theorem 4.8.

Theorem 5.1. *The set $\mathcal{S}_2$ consists only one element.*

Proof. Let F and f be any two fast orbits with compact supports $[0, R_i]$ for $i = 1, 2$ respectively and satisfy $F(0) > f(0)$. We define

$$f_k(r) = kf(k^{-\gamma}r), \quad \gamma = [m(p-1)-1]/(mp)$$

and then f_k will be larger than F on $[0, R_2]$ for sufficiently large k. We now define

$$\tau = \min\{k \geq 1; f_k(r) \geq F(r),\ 0 \leq r \leq R_2\}.$$

The uniqueness proof is now reduced to showing that τ is not greater than 1. Suppose that $\tau > 1$, to the contrary. We will show that there exists $\epsilon > 0$ such that $f_{\tau-\epsilon}(r) \geq F(r)$ for every $r \in [0, R_2]$. Indeed, we are going to show that $f_\tau(r)$ does not touch $F(r)$ in compact support $[0, R_2]$ by dividing into three cases;

(i) in the interior of the support,
(ii) at the origin,
(iii) at R_2.

In fact, $f_\tau(r)$ solves

$$\begin{aligned}&(|f_\tau'|^{p-2}f_\tau')' + \beta r(f_\tau^{1/m})' + \alpha f_\tau^{1/m} + (f_\tau^{q/m})' \\ &= -\tau(1-\tau^{q/m-\gamma-1})(f^{q/m})'.\end{aligned} \tag{5.2}$$

(i) If f_τ touches F at $r_0 \in (0, R_2)$, then $f_\tau(r_0) = F(r_0)$, $f_\tau'(r_0) = F'(r_0) < 0$ and

$$(|f_\tau'|^{p-2}f_\tau')'(r_0) < (|F'|^{p-2}F')'(r_0).$$

But $f_\tau(r) \geq F(r)$ near $r = r_0$, which obviously violates the strong maximum principle.

(ii) If f_τ touches F at $r_0 = 0$, then $f_\tau(0) = F(0) > 0$, $f_\tau'(0) = F'(0) = 0$ and $(|f_\tau'|^{p-2}f_\tau')' = -\alpha f_\tau^{1/m}(0) = (|F'|^{p-2}F')'(0) < 0$. Differentiating the equation (1.10) and (5.2), we reduce that

$$(|f_\tau'|^{p-2}f_\tau')''(0) - (|F'|^{p-2}F')''(0) = -\tau(1-\tau^{q/m-\gamma-1})(f^{q/m})''(0) < 0.$$

Thus, we have

$$(|f_\tau'|^{p-2}f_\tau')''(r) - (|F'|^{p-2}F')''(r) \leq 0$$

near $r_0 = 0$, which leads to a contradiction.

(iii) For the final case, we define the functions u, U_τ corresponding to F and f_τ by

$$u(x,t) =: t^{-\alpha}F^{1/m}(r),$$
$$U_\tau(x,t) =: t^{-\alpha}f_\tau^{1/m}(r) =: \tau^{1/m}t^{-\alpha}f^{1/m}(\tau^{-\gamma}r),$$

where $\gamma = (p-2)/p$, $r = rt^{-\beta}$ as defined before. Then $u(x,t)$ is a solution of (1.1) and U_τ is a supersolution. Indeed, a straightforward computation shows that

$$U_t - (|(U^m)_x|^{p-2}(U^m)_x)_x - (U^q)_x = \tau^{1/m}(\tau^{q/m-\gamma-1}-1)|(f^{q/m})'| \geq 0 \ \text{ for } \tau > 1.$$

Following directly the proof of Lemma 10 in [12], we can show that for fixed $t > 0$ and all sufficiently small $\delta' > 0$, there exists $\theta = \theta(\delta') \in (0,1)$ such that $U_\tau(x,t) \le U_\tau(x,t+\delta')$ if x satisfies $\theta R_2 \le xt^{-\beta}\tau^{-\gamma} \le R_2$ and $\lim_{\delta'\downarrow 0}\theta(\delta') = \theta_0 \in (0,1)$. In the proof, we use the interface relation (5.1) crucially (see [12] for details). In particular, we have

$$U_\tau(x,1) \le U_\tau(x,1+\delta') \tag{5.3}$$

for $\theta R_2\tau^\gamma \le x < R_2\tau^\gamma(1+\delta')^\beta$. In other words, we found a separation near the right end $r = R_2$.

On the other hand, as previously proved, f_τ can not touches F at $r_0 \in [0,R_2)$, which implies for any $\epsilon_1 > 0$, there exists $\kappa = \kappa(\epsilon_1) \in (0,1)$ such that $F^{1/m}(x) \le \kappa f_\tau^{1/m}(x)$, that is

$$u(x,1) \le \kappa U_\tau(x,1). \tag{5.4}$$

We choose $\epsilon_1 > 0$ so that $0 < \epsilon_1 < 1-\theta_0$ and find $\delta_0 = \delta_0(\epsilon_1)$ such that

$$1-\epsilon_1 > \theta(\delta') \tag{5.5}$$

for $\delta' \in (0,\delta_0)$. By continuity of U_τ, there exists $\delta_1 = \delta_1(\epsilon_1) \in (0,\delta_0)$ such that

$$\kappa U_\tau(x,1) \le U_\tau(x,1+\delta') \tag{5.6}$$

for any $\delta' \in (0,\delta_1)$ and $0 \le x < (1-\epsilon_1)R_2\tau^\gamma$. Combining (5.3), (5.4), and (5.6) and using again the continuity of U_τ, we deduce that for $\delta \in (\delta',\delta_1)$, $\delta-\delta'$ small enough, we have

$$F^{1/m}(r) < U_\tau(x,1+\delta) = \tau^{1/m}(1+\delta)^{-\alpha}f^{1/m}(x(1+\delta)^{-\beta}\tau^{-\gamma})$$

for any $x \ge 0$. Furthermore, from the continuity with respect to τ, there exists $\tau_1 \in (0,\tau)$ such that

$$\begin{aligned} u(x,1) = F^{1/m}(r) &\le \tau_1^{1/m}(1+\delta)^{-\alpha}f^{1/m}(x(1+\delta)^{-\beta}\tau_1^{-\gamma}) \\ &= U_{\tau_1}(x,1+\delta) \end{aligned} \tag{5.7}$$

for any $x \ge 0$. By parabolic maximum principle, we have $u(x,t) \le U_{\tau_1}(x,t+\delta)$, that is,

$$t^{-\alpha}F^{1/m}(xt^{-\beta}) \le \tau_1^{1/m}(t+\delta)^{-\alpha}f^{1/m}(x(t+\delta)^{-\beta}\tau_1^{-\gamma}) \tag{5.8}$$

for any $t \ge 1$ and $x \ge 0$. Rewriting (5.8) of the form;

$$F^{1/m}(r) \le \tau_1^{1/m}[t/(t+\delta)]^{-\alpha}f^{1/m}(r[t/(t+\delta)]^{-\beta}\tau_1^{-\gamma})$$

and letting $t \longrightarrow \infty$, we find that

$$F^{1/m}(r) \le \tau_1^{1/m}f^{1/m}(r\tau_1^{-\gamma})$$

which contradicts the fact that τ is the smallest constant with that property. Thus f_τ does not meet at $r_0 = R_2$.

Hence we may find $\epsilon > 0$ so that

$$f_{\tau-\epsilon}(r) \ge F(r) \quad \text{for every } r \in [0,R_2]$$

which means that we can slightly reduce the factor τ. Hence we may conclude that $\tau = 1$ but it is obviously impossible. $\square$

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