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# NUMERICAL DISCRETIZATION OF A POPULATION DIFFUSION EQUATION

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ABSTRACT. A numerical method is proposed and analyzed to approximate a mathematical model of age-dependent population dynamics with spatial diffusion. The model takes a form of nonlinear and nonlocal system of integro-differential equations. A finite difference method along the characteristic age-time direction is considered and primal mixed finite elements are used in the spatial variable. A priori error estimates are derived for the relevant variables.

## 1. INTRODUCTION

We are interested in the approximation of the solution of a mathematical model for an age-dependent single species population moving in a limited environment $\Omega$ in two space dimensions. The age-space structure of the population is described through the density distribution $u(\mathbf{x}, a, t)$ where $\mathbf{x}$ is spatial position, $a$ age, and $t$ time. We consider the following model to describe the dynamics of the population [20, 32]:

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$$u_t + u_a - \text{div}(\kappa(\mathbf{x},p)\nabla u) + \mu(\mathbf{x},a,p)u = 0, \quad \mathbf{x} \in \Omega, \quad a \geq 0, \quad t \geq 0, \tag{1.1}$$

$$p(\mathbf{x},t) = \int_0^\infty u(\mathbf{x},a,t)da, \quad \mathbf{x} \in \Omega, \quad t \geq 0, \tag{1.2}$$

$$u(\mathbf{x},0,t) = \int_0^\infty \beta(\mathbf{x},a,p)u(\mathbf{x},a,t)da, \quad \mathbf{x} \in \Omega, \quad t \geq 0, \tag{1.3}$$

$$u(\mathbf{x},a,t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad a \geq 0, \quad t \geq 0, \tag{1.4}$$

$$u(\mathbf{x},a,0) = u_0(\mathbf{x},a), \quad \mathbf{x} \in \Omega, \quad a \geq 0, \tag{1.5}$$

where $\kappa = \kappa(\mathbf{x},p)$ is the dispersal modulus, $\mu = \mu(\mathbf{x},a,p)$ the age-specific death rate, $\beta = \beta(\mathbf{x},a,p)$ the rate of fertility, and $p = p(\mathbf{x},t)$ the total size of the population. $\nabla$ denotes the gradient operator with respect to the space variable and the vector $\boldsymbol{\nu}$ denote outward unit normal vector to $\partial\Omega$. Homogeneous Dirichlet boundary condition (1.4) represents an extremely inhospitable habitat. The population flux in the model under consideration

$$\mathbf{q} = -\kappa(\mathbf{x},p)\nabla u, \tag{1.6}$$

describes random diffusion so that the local flow of population lies in the direction of decreasing density [20, 32]. Other fluxes have been considered such as $\mathbf{q} = -\kappa(p)u\nabla p$, directed dispersal in the direction of least crowding [20, 14], among others. Spatial diffusion in age-dependent population dynamics was first introduced by Gurtin [19]. Since then, the diffusion age-structured model have been extensively studied by many authors, along with many generalizations to the related problems such as two-sex population models, S-I-R models, and various other epidemic models. The effects of spatial diffusion on "infection-age" structured epidemic models were discussed in [18].

We refer the reader to, for example, [20, 32, 30, 31, 33, 8, 43] for existence and qualitative behavior of the solutions to (1.1)-(1.5). For the numerical study, Lopez and Trigiante [35] used finite difference methods to approximate solutions to the problem in the case of a linear flux $\mathbf{q} = \kappa(\mathbf{x})\nabla u$ and $\beta = \beta(a), \mu = \mu(a)$. Milner [36] studied the case of directed dispersal using finite element methods and derived some error estimates. An application of Galerkin's method to (1.1)-(1.5) was first given by Kim [22] and optimal order error estimates were derived. Since then, there are several works on the model under consideration, just to name a few [3, 4, 5, 17, 39]. For model *without* diffusion we refer to, for example, [28, 1, 26, 23, 24, 25] and the references contained therein.

A number of papers in recent years study primal mixed finite element methods for elliptic and parabolic problems [40, 13, 15, 6, 7, 38, 21]. This growing interest is driven by two important features of the primal mixed method. First, it conserves the mass locally at element level and conserves fluxes node-wise [13]. Nodal fluxes are of interest in some applications. For example, nodal fluxes are related with nodal forces in structural mechanics. Second, the primal mixed method provides a softening of the quadratic form associated with the principal part of model equations and it is a starting point of study of the method of enhanced assumed strains (EAS method), which is very popular in computational mechanics during recent years [41, 42, 9, 10, 11].

In this work, we propose and analyze a primal mixed finite element method for the first time to approximate the population flux and density distribution of the system (1.1)-(1.5). An extensive study of mixed finite element methods can be found in [12, 40]. Observe that along the characteristics in the age-time direction, (1.1) can be viewed as a parabolic differential equation. Thus, parabolic mixed methods [27] can be applied to our problem. However, it has the peculiarity that these parabolic differential equations are coupled between them due to the dependence on the total population of the age-specific death rate and dispersal modulus and also due to the age-initial condition (1.3). Following some of the ideas of [22, 29], we lag the evaluation of coefficients of (1.1)-(1.5) and compute the age-initial value (1.3) with appropriate numerical quadratures, which allow us to obtain a linear scheme.

The rest of the paper is organized as follows. In the next section, we describe the numerical method which combines the method of characteristics and primal mixed finite elements. Then, in Section 3, we derive optimal order error estimates in $l^{\infty,\infty}(L^2)$-norm, i.e., the discrete maximum norms for the time and age variables and $L^2$-norm for the spatial variable.

## 2. A Primal Mixed Finite Element Approximation

We first define some notations to be used in the paper. For $1 \leq q \leq \infty$ and $m$ any nonnegative integer, let

$$W^{m,q}(\Omega) := \{f \in L^q(\Omega) | D^\alpha f \in L^q(\Omega) \text{ if } |\alpha| \leq m\}$$

denote the Sobolev space endowed with the norm

$$||f||_{m,q;\Omega} := \left( \sum_{|\alpha| \leq m} ||D^\alpha f||^q_{L^q(\Omega)} \right)^{1/q},$$

with the usual modification for $q = \infty$. Let $H^m(\Omega) := W^{m,2}(\Omega)$ with norm $|| \cdot ||_{m;\Omega} := || \cdot ||_{m,2;\Omega}$. $H^m_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in the norm $|| \cdot ||_{m;\Omega}$, where $\mathcal{D}(\Omega)$ is the set of infinitely differentiable functions with compact support in $\Omega$. For $0 \leq r < \infty$, let $W^{r,q}(\Omega)$ and $H^r(\Omega)$ denote the fractional order Sobolev spaces with norms $|| \cdot ||_{r,q;\Omega}$ and $|| \cdot ||_{r;\Omega}$ (cf., [34]). The notation $|| \cdot ||$ will mean $|| \cdot ||_{L^2(\Omega)}$. Let $L^p(S; H^r(\Omega))$ denote the usual set of functions with the norm

$$||\psi||_{L^p(S;H^r(\Omega))} = \left\{ \int_S ||\psi(\cdot, s)||^p_{r;\Omega} ds \right\}^{1/p},$$

where if $p = \infty$, the integral is replaced by the essential supremum. The subscript $\Omega$ will be omitted in the notation above when there is no ambiguity. To take into account the discretization of age and/or time, we shall also find useful the following notations:

$$
\begin{aligned}
||\chi^n||_{l^p(H^r)} &:= \left( \sum_{j \geq 1} ||\chi_j^n||_{r;\Omega}^p \Delta t \right)^{1/p}, \\
||\chi||_{l^{q,p}(H^r)} &:= \left( \sum_{n=0}^{N} ||\chi^n||_{l^p(H^r)}^q \Delta t \right)^{1/q},
\end{aligned}
$$

with the usual modification for $q = \infty$.

Next, we shall make some assumptions on the data and solutions of the problem(1.1)-(1.5). Observe that the initial age-space distribution $u_0$ given in (1.5) must be nonnegative and compactly supported in the age variable for biological reasons. Thus, we shall assume that $u_0 \in C^0([0, a_\dagger]; L^\infty(\Omega))$ is nonnegative and

$$
u_0(\mathbf{x}, a) = 0, \qquad \text{for } a \geq a_\dagger.
$$

Note that the solution $u$ is then compactly supported in the age variable for any time $t$ so that

$$
u(\mathbf{x}, a, t) = 0, \qquad \text{for } a \geq t + a_\dagger.
$$

This implies that all the integrals which appear in this paper are indeed over a finite interval. Now, let $T > 0$ be the final time and let $J = [0, a_\dagger + T] \times [0, T]$. Let $\Omega$ be a bounded domain with $C^2-$boundary $\partial\Omega$. We shall assume that the coefficient $\kappa : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable with bounded derivatives through second order and

$$
0 < \kappa_0 \leq \kappa(\mathbf{x}, p) \leq \kappa_1 < \infty, \quad \text{for some real numbers } \kappa_0, \kappa_1,
$$

and that $\mu \in C^1(J; C^0(\overline{\Omega}))$ is nonnegative, $\frac{\partial \beta}{\partial p}$ and $\frac{\partial \mu}{\partial p}$ are continuous and bounded. We shall assume that the initial-boundary value problem (1.1)-(1.5) has a unique solution $u \in C^2(J; H^{2+\epsilon}(\Omega))$ for some $\epsilon, 0 < \epsilon \ll 1$. We refer the reader to [32, 14, 33] for some explicit assumptions on the data for (1.1)-(1.5). Due to Sobolev's embedding theorem, our regularity assumption on the solution implies that $u(\cdot, a, t)$ belongs to $W^{1,\infty}(\Omega)$ and $p$ especially belongs to $C^1([0, T]; W^{1,\infty}(\Omega))$, which will be needed throughout the paper. In the rest of the paper, we shall omit the explicit dependence of any function on the arguments.

In order to formulate the weak form of (1.1)-(1.5) appropriate for the mixed method, we first consider the equations (1.1) and (1.4). Using (1.6) and letting $\alpha(p) = \kappa^{-1}(p)$, (1.1) and (1.4) can be written as the following nonlinear, nonlocal first-order system of equations for $\mathbf{q}$ and $u$:

$$
\begin{aligned}
\alpha(p)\mathbf{q} + \nabla u &= 0, \quad \text{in } \Omega \times J, \\
u_t + u_a + \text{div}\mathbf{q} + \mu(p)u &= 0, \quad \text{in } \Omega \times J, \\
u &= 0, \quad \text{on } \partial\Omega \times J.
\end{aligned} \tag{2.1}
$$

Here, the first equation of (2.1) is the constitutive relation and the second equation of (2.1) is the law of population balance [20].

Let $\mathbf{V} := [L^2(\Omega)]^2$ and $W := H_0^1(\Omega)$. The weak form of the equations associated with (2.1) is obtained by seeking a map $\mathbf{q}, u : J \to \mathbf{V} \times W$ such that

$$(\alpha(p)\mathbf{q}, \boldsymbol{\chi}) + (\boldsymbol{\chi}, \nabla u) = 0, \quad \boldsymbol{\chi} \in \mathbf{V}, \tag{2.2}$$

$$(u_t + u_a, w) - (\mathbf{q}, \nabla w) + (\mu(p)u, w) = 0, \quad w \in W, \tag{2.3}$$

where $(\cdot,\cdot)$ indicates an $L^2-$inner product in $\Omega$.

We shall discretize (2.2) using a finite difference method of characteristics in the age-time direction and a primal mixed finite element method for the spatial variable. Let $N$ be a fixed integer, and let $\Delta t = T/N$, $A = [a_\dagger/\Delta t + 1]$, and

$$t^n = n\Delta t, \quad 0 \le n \le N, a_j = j\Delta t, 0 \le j < N + A.$$

We refer to the age level $a_j$ by a subscript and the time level $t^n$ by a superscript. We define a directional derivative $D\chi$ of $\chi$ along the characteristic $t = a$, and a finite difference operator $\overline{D}$ as follows:

$$D\chi(a,t) := \lim_{\Delta t \to 0} \frac{\chi(a + \Delta t, t + \Delta t) - \chi(a,t)}{\Delta t},$$

and for $j \ge 1, n \ge 1$,

$$\overline{D}\chi_j^n := \frac{\chi_j^n - \chi_{j-1}^{n-1}}{\Delta t}.$$

We consider now a shape-regular family of decomposition of $\Omega$, $\mathfrak{T}_h$ , with boundary elements allowed to have one curved edge or side. Associated with it, we take finite dimensional subspaces $\mathbf{V}_h \subset \mathbf{V}$ and $W_h \subset W$ of index $k \ge 1$ given as follows:

For a triangle $T$,

$$\begin{aligned} \mathbf{V_h} :&= \{\mathbf{v} \in \mathbf{V} : \forall T \in \mathfrak{T}_h, \quad \mathbf{v} \in P_{k-1}(T)^2\}, \\ W_h :&= \{w \in W : \forall T \in \mathfrak{T}_h, \quad w \in P_k(T)\}. \end{aligned} \tag{2.4}$$

For a rectangle $T$,

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v} \in \mathbf{V} : \forall T \in \mathfrak{T}_h,\ \mathbf{v}|_T \in P_{k-1,k}(T) \times P_{k,k-1}(T)\}, \\ W_h &:= \{w \in W : \forall T \in \mathfrak{T}_h,\ q|_T \in P_{k,k}(T)\}, \end{aligned} \tag{2.5}$$

where $P_{l,m}(T)$ is the space of polynomials of degree$\le l$ in $x_1$ and $\le m$ in $x_2$.

In view of (2.2), the primal mixed finite element discretization appropriate for (1.1)-(1.5) is to seek a pair $\{\boldsymbol{\Xi}_j^n, U_j^n\}_{n,j\ge 1}$ in $\mathbf{V_h} \times W_h \subset \mathbf{V} \times W$ such that for any $\boldsymbol{\chi} \in \mathbf{V_h}$ and for any $w \in W_h$,

$$\begin{array}{rcll} U_j^0 &=& u_j^0, & j \ge 0, \\ P^0 &=& \sum_{j\ge 1} U_j^0 \Delta t, & \\ (\alpha(P^{n-1})\boldsymbol{\Xi}_j^n, \boldsymbol{\chi}) + (\boldsymbol{\chi}, \nabla u_j^n) &=& 0, & n \ge 1, \quad j \ge 1, \\ (\overline{D}U_j^n, w) - (\boldsymbol{\Xi}_j^n, \nabla w) + (\mu_{j,h}^{n-1} U_j^n, w) &=& 0, & n \ge 1, \quad j \ge 1, \\ P^n &=& \sum_{j \ge 1} U_j^n \Delta t, & n \ge 1, \\ U_0^n &=& \sum_{j\ge 1} \beta_{j,h}^n U_j^n \Delta t, & n \ge 1, \end{array} \tag{2.6}$$

where $\mu_{j,h}^n := \mu(\cdot, a_j, P^n), \beta_{j,h}^n := \beta(\cdot, a_j, P^n)$.

Note that $U_0^n \in W$ is not necessarily in $W_h$ due to the factor $\beta_{j,h}^n$. We also note that the resulting scheme (2.6) is linear and uniqueness and existence of the discrete solution are essentially proved in [40, 21].

## 3. $L^2$-ERROR ESTIMATE

Let $\Pi_h^1 : H_0^1(\Omega) \to W_h$ be the orthogonal $L^2$ projection, and let $\Pi_h^0 : [L^2(\Omega)]^2 \to \mathbf{V_h}$ be the orthogonal $L^2$ projection which has the following useful commuting property:

$$\nabla \circ \Pi_h^1 = \Pi_h^0 \circ \nabla : H_0^1(\Omega) \to \mathbf{V}_h. \tag{3.1}$$

These projections have the following approximation properties [40, 38]:

$$\begin{aligned} ||w - \Pi_h^1 w||_{0,q} &\le Ch^r ||w||_{r,q} && 0 \le r \le k+1, \\ & && \text{if } w \in W \cap W^{r,q}(\Omega), \\ ||w - \Pi_h^1 w||_{0,\infty} &\le Ch^r (\log h^{-1})^{1/2} ||w||_{r+1}, && 0 \le r \le k+1, \\ & && \text{if } w \in H^{r+1}(\Omega), \\ ||\mathbf{v} - \Pi_h^0 \mathbf{v}||_{0,q} &\le Ch^r ||\mathbf{v}||_{r,q} && 0 \le r \le k, \\ & && \text{if } \mathbf{v} \in \mathbf{V} \cap [W^{r,q}(\Omega)]^2, \\ ||\mathbf{v} - \Pi_h^0 \mathbf{v}||_{0,\infty} &\le Ch^r (\log h^{-1})^{1/2} ||\mathbf{v}||_{r+1}, && 0 \le r \le k, \\ & && \text{if } \mathbf{v} \in \mathbf{V} \cap [H^{r+1}(\Omega)]^2, \end{aligned} \tag{3.2}$$

We now consider the primal mixed method elliptic projection

$$\begin{aligned} (\alpha(p)(\mathbf{q} - \widetilde{\mathbf{q}}, \boldsymbol{\chi}) + (\boldsymbol{\chi}, \nabla(u - \widetilde{u})) &= 0, && \boldsymbol{\chi} \in \mathbf{V}, \\ -(\mathbf{q} - \widetilde{\mathbf{q}}, \nabla w) + (\mu(p)(u - \widetilde{u}), w) &= 0, && w \in W. \end{aligned} \tag{3.3}$$

It is proved [40] in the case $\mu \equiv 0$ that the projection $\{\widetilde{\mathbf{q}}, \widetilde{u}\} \in \mathbf{V}_h \times W_h$ exists and has optimal convergence properties in the energy norm. The following error estimates can be easily proved for general death rate $\mu \ge 0$. In particular, the $L^2$ error estimate for $\widetilde{u}$ follows from a duality argument.

$$\begin{aligned} ||u - \widetilde{u}|| + h||\nabla u - \nabla \widetilde{u}|| &\le Ch^{r+1} ||u||_{r+1}, && 1 \le r \le k, \\ ||\mathbf{q} - \widetilde{\mathbf{q}}|| &\le Ch^r ||\mathbf{q}||_r, && 1 \le r \le k. \end{aligned} \tag{3.4}$$

It follows from the inverse estimate [16], (3.2), and (3.4) that

$$||\widetilde{\mathbf{q}}||_{0,\infty} + ||\widetilde{u}||_{0,\infty} \le Q, \tag{3.5}$$

where the constant $Q$ depends on $||\mathbf{q}||_{1+\epsilon}$ and $||u||_2$.

Since the elliptic projection commutes with the differential operator $D$, we can prove the following estimates using a duality argument (cf., [22, 27])

**Lemma 3.1.** *There is a constant C, independent h, such that*

$$||D(u - \widetilde{u})|| \le Ch^{r+1}(||u||_{r+1} + ||Du||_{r+1}), \qquad 1 \le r \le k. \tag{3.6}$$

Let $\boldsymbol{\zeta} := \boldsymbol{\Xi} - \widetilde{\mathbf{q}},\ \xi := U - \widetilde{u},\ \boldsymbol{\eta} := \mathbf{q} - \widetilde{\mathbf{q}},\ \rho := u - \widetilde{u}$. In order to prove our main Theorem 3.1 below, we need the following estimates (3.8)-(3.10). First, for $n \geq 1$,

$$||P^n - p^n|| = \left|\left|\sum_{j\geq 1} U_j^n \Delta t - \int_0^\infty u^n(a)da\right|\right| \leq C\{||\xi^n||_{l^1(L^2)} + ||\rho^n||_{l^1(L^2)} + \Delta t\}, \quad (3.7)$$

and

$$\begin{aligned} U_0^n - u_0^n &= \sum_{j\geq 1} \beta_{j,h}^n U_j^n \Delta t - \int_0^\infty \beta(a, p^n) u^n(a) da \\ &= \sum_{j\geq 1} \beta_{j,h}^n (U_j^n - u_j^n)\Delta t + \sum_{j\geq 1} (\beta_{j,h}^n - \beta_j^n) u_j^n \Delta t + O(\Delta t) \\ &= \sum_{j\geq 1} \beta_{j,h}^n (U_j^n - u_j^n)\Delta t + \sum_{j\geq 1} \hat{\beta}_p (P^n - p^n) u_j^n \Delta t + O(\Delta t), \end{aligned}$$

where the hat denotes evaluation at some intermediate value of the variables. Hence, we have, by (3.7),

$$||U_0^n - u_0^n|| \leq C(||\xi^n||_{l^1(L^2)} + ||\rho^n||_{l^1(L^2)} + \Delta t), \quad \text{for } n \geq 1.$$

Since $\xi_0^n = U_0^n - u_0^n + \rho_0^n$, we obtain that

$$||\xi_0^n|| \leq C(||\xi^n||_{l^1(L^2)} + ||\rho^n||_{l^1(L^2)} + ||\rho_0^n|| + \Delta t), \quad \text{for } n \geq 1. \quad (3.8)$$

We also need the estimate for the term pertaining to the finite difference operator $\overline{D}$. From the integral form of Taylor's formula, we have

$$||\overline{D}\rho_j^n||^2 \leq \frac{1}{\Delta t} \int_{(a_{j-1}, t^{n-1})}^{(a_j, t^n)} ||D\rho||^2 ds = \frac{1}{\Delta t} ||D\rho||^2_{L^2(\Lambda_j^n; L^2)}, \quad n \geq 1,\ j \geq 1,$$

where $\Lambda_j^n$ is the line segment joining $(a_{j-1}, t^{n-1})$ and $(a_j, t^n)$. Thus, we have

$$\sum_{n,j\geq 1} ||\overline{D}\rho_j^n||^2 \Delta t \leq ||D\rho||^2_{L^2(\Lambda; L^2)}, \quad (3.9)$$

where $\Lambda = \bigcup_{n,j\geq 1} \Lambda_j^n$. We also have

$$\left|\left|\overline{D}u_j^n - \left(\frac{\partial u_j^n}{\partial t} + \frac{\partial u_j^n}{\partial a}\right)\right|\right| \leq \frac{1}{2} ||D^2 u||_{L^\infty(\Lambda_j^n; L^2)} \Delta t. \quad (3.10)$$

We are now ready to prove the main theorem. Note that we do not require stability condition (CFL condition) relating the time and space step. For the sake of brevity, we introduce the following:

$$\begin{aligned} \varepsilon(u) := \quad & \Delta t + h^{r+1}\big(||u_0||_{l^\infty(H^{r+1})} + ||u|_{a=0}||_{l^2(H^{r+1})} + ||u||_{l^{2,2}(H^{r+1})}\big) \\ & + h^{r+1}\sqrt{\Delta t}\big(||u||_{L^2(\Lambda; H^{r+1})} + ||Du||_{L^2(\Lambda; H^{r+1})}\big) \end{aligned}$$

and

$$\varepsilon(\mathbf{q}) := \Delta t + h^r\big(||u_0||_{l^\infty(H^{r+1})} + ||u|_{a=0}||_{l^2(H^{r+1})} + ||u||_{l^{2,2}(H^{r+1})}\big) \\ +h^r\sqrt{\Delta t}\big(||u||_{L^2(\Lambda;H^{r+1})} + ||Du||_{L^2(\Lambda;H^{r+1})}\big)$$

**Theorem 3.1.** *Let $\varepsilon(u)$ and $\varepsilon(\mathbf{q})$ be defined as before. There is a constant C independent of $h$ and $\Delta t$ such that if $u$ is sufficiently smooth, then for $1 \le r \le k$ and $\Delta t$ sufficiently small, we have the following estimates:*

$$(i) \quad ||u - U||_{l^{\infty,\infty}(L^2)} \le C\varepsilon(u),$$
$$(ii) \quad ||\mathbf{q} - \mathbf{\Xi}||_{l^{2,2}(L^2)} \le C\varepsilon(\mathbf{q}),$$

*where the constant C depends on $||D^2u||_{l^{\infty,\infty}(L^2)}$ and $||\mathbf{q}||_{1+\epsilon}$.*

*Proof.* From (2.2) and (2.6), we have the following equation: for any $\boldsymbol{\chi} \in \mathbf{V_h}$ and $w \in W_h$,

$$(\alpha(P^{n-1})\mathbf{\Xi}_j^n - \alpha(p^n)\mathbf{q}_j^n, \boldsymbol{\chi}) + (\boldsymbol{\chi}, \nabla(U_j^n - u_j^n)) = 0, \quad (3.11)$$
$$(\overline{D}U_j^n - Du_j^n, w) - ((\mathbf{\Xi}_j^n - \mathbf{q_j^n}), \nabla w) + (\mu_{j,h}^{n-1}U_j^n - \mu_j^n u_j^n, w) = 0, \quad (3.12)$$

Using (3.3), we rewrite (3.11),(3.12) as follows: for any $\boldsymbol{\chi} \in \mathbf{V_h}$ and $w \in W_h$,

$$\begin{aligned} (\alpha(P^{n-1})\boldsymbol{\zeta}_j^n, \boldsymbol{\chi}) + (\boldsymbol{\chi}, \nabla\xi_j^n) &= ([\alpha(p^n) - \alpha(P^{n-1})]\widetilde{\mathbf{q}}_j^n, \boldsymbol{\chi}), \\ (\overline{D}\xi_j^n, w) - (\boldsymbol{\zeta}_j^n, \nabla w) + (\mu_{j,h}^{n-1}\xi_j^n, w) &= (Du_j^n - \overline{D}u_j^n, w) + (\overline{D}\rho_j^n, w) \\ &\quad +([\mu_j^n - \mu_{j,h}^{n-1}]\widetilde{u}_j^n, w). \end{aligned} \quad (3.13)$$

We now choose $\boldsymbol{\chi} = \boldsymbol{\zeta}_j^n, w = \xi_j^n$ in (3.13), and add the resulting equations to arrive at the following equations:

$$\begin{aligned} (\overline{D}\xi_j^n, \xi_j^n) + (\alpha(P^{n-1})\boldsymbol{\zeta}_j^n, \boldsymbol{\zeta}_j^n) + (\mu_{j,h}^{n-1}\xi_j^n, \xi_j^n) &= (\overline{D}\rho_j^n, \xi_j^n) + (Du_j^n - \overline{D}u_j^n, \xi_j^n) \\ &\quad +([\alpha(p^n) - \alpha(P^{n-1})]\widetilde{\mathbf{q}}_j^n, \boldsymbol{\zeta}_j^n) + ([\mu_j^n - \mu_{j,h}^{n-1}]\widetilde{u}_j^n, \xi_j^n) \\ &= I + II + III + IV. \end{aligned}$$

Note that the left-hand side is bounded below by

$$\frac{1}{2\Delta t}(||\xi_j^n||^2 - ||\xi_{j-1}^{n-1}||^2) + \alpha_0||\boldsymbol{\zeta}_j^n||^2,$$

where $\alpha_0 = 1/\kappa_1$. Using (3.5), (3.7) and (3.10), we have the following bounds:

$$
\begin{aligned}
I &\le \frac{1}{2}\big(||\overline{D}\rho_j^n||^2 + ||\xi_j^n||^2\big),\\
II &\le \frac{1}{2}||D^2u||_{L^\infty(\Lambda_j^n;L^2)}\Delta t||\xi_j^n||,\\
&\le \frac{1}{4}||D^2u||^2_{L^\infty(\Lambda_j^n;L^2)}(\Delta t)^2 + \frac{1}{4}||\xi_j^n||^2,\\
III &\le ||\widetilde{\mathbf{q}}_j^n||_{0,\infty}||\boldsymbol{\zeta}_j^n||||\alpha(p^n) - \alpha(P^{n-1})||\\
&\le ||\widetilde{\mathbf{q}}_j^n||_{0,\infty}||\boldsymbol{\zeta}_j^n||||\hat{\alpha_p}||_{0,\infty}||p^n - P^{n-1}||\\
&\le C||\boldsymbol{\zeta}_j^n||\big(||\xi^{n-1}||_{l^1(L^2)} + ||\rho^{n-1}||_{l^1(L^2)} + \Delta t\big),\\
&\le C_\epsilon\big(||\xi^{n-1}||_{l^1(L^2)} + ||\rho^{n-1}||_{l^1(L^2)} + \Delta t\big)^2 + \epsilon||\boldsymbol{\zeta}_j^n||^2,\\
IV &\le ||\widetilde{u}_j^n||_{0,\infty}||\mu_j^n - \mu_{j,h}^{n-1}||||\xi_j^n||\\
&\le C_1||\xi_j^n||\big(||\xi^{n-1}||_{l^1(L^2)} + ||\rho^{n-1}||_{l^1(L^2)} + \Delta t\big).
\end{aligned}
$$

Choosing $\epsilon = \alpha_0/2$ in the estimate of III and combining the estimates, we have

$$
\begin{aligned}
&\frac{1}{2\Delta t}\big(||\xi_j^n||^2 - ||\xi_{j-1}^{n-1}||^2\big) + \alpha_0||\boldsymbol{\zeta}_j^n||^2\\
&\quad\le C_2\big(||\xi^{n-1}||^2_{l^1(L^2)} + ||\rho^{n-1}||^2_{l^1(L^2)} + ||\xi_j^n||^2 + ||\overline{D}\rho_j^n||^2 + (\Delta t)^2\big) + \frac{\alpha_0}{2}||\boldsymbol{\zeta}_j^n||^2, \quad (3.14)
\end{aligned}
$$

or

$$
\begin{aligned}
&\big(||\xi_j^n||^2 - ||\xi_j^{n-1}||^2\big) + \big(||\xi_j^{n-1}||^2 - ||\xi_{j-1}^{n-1}||^2\big) + \alpha_0||\boldsymbol{\zeta}_j^n||^2\Delta t\\
&\qquad\le 2C_2\Delta t\big(||\xi_j^n||^2 + ||\xi^{n-1}||^2_{l^1(L^2)} + ||\rho^{n-1}||^2_{l^1(L^2)} + ||\overline{D}\rho_j^n||^2 + (\Delta t)^2\big).
\end{aligned}
$$

Summing over $j$, $1 \le j \le n + A$, and using Schwarz inequality, we see that

$$
\begin{aligned}
&\frac{1}{\Delta t}\big(||\xi^n||^2_{l^2(L^2)} - ||\xi^{n-1}||^2_{l^2(L^2)}\big) + ||\xi_{n+A}^{n-1}||^2 - ||\xi_0^{n-1}||^2 + \alpha_0||\boldsymbol{\zeta}^n||^2_{l^2(L^2)}\\
&\quad\le 2C_3\left(||\xi^n||^2_{l^2(L^2)} + ||\xi^{n-1}||^2_{l^2(L^2)} + ||\rho^{n-1}||^2_{l^2(L^2)} + \sum_{j\ge 1}||\overline{D}\rho_j^n||^2\Delta t + (\Delta t)^2\right),
\end{aligned}
$$

or

$$
\begin{aligned}
&||\xi^n||^2_{l^2(L^2)} - ||\xi^{n-1}||^2_{l^2(L^2)} + \Delta t\big(||\xi_{n+A}^{n-1}||^2 - ||\xi_0^{n-1}||^2\big) + \alpha_0||\boldsymbol{\zeta}^n||^2_{l^2(L^2)}\Delta t\\
&\le 2C_3\left[\left(||\xi^n||^2_{l^2(L^2)} + ||\xi^{n-1}||^2_{l^2(L^2)} + ||\rho^{n-1}||^2_{l^2(L^2)}\right)\Delta t + \Delta t\sum_{j\ge 1}||\overline{D}\rho_j^n||^2\Delta t + (\Delta t)^3\right].
\end{aligned}
$$

It follows from (3.8) that

$$(1-2C_3\Delta t)||\xi^n||^2_{l^2(L^2)}+\alpha_0||\boldsymbol{\zeta}^n||^2_{l^2(L^2)}\Delta t \le (1+2C_3\Delta t)||\xi^{n-1}||^2_{l^2(L^2)}$$
$$+C_4\left(||\rho_0^{n-1}||^2\Delta t+||\rho^{n-1}||^2_{l^2(L^2)}\Delta t+\Delta t\sum_{j\ge1}||\overline{D}\rho_j^n||^2\Delta t+(\Delta t)^3\right).$$

Summing now over $n$, $1\le n\le N$, and using Gronwall's lemma, the Schwarz inequality, and (3.9), we see that for sufficiently small $\Delta t$ (e.g. $\Delta t\le 1/(4C_3)$),

$$||\xi^N||^2_{l^2(L^2)}+||\boldsymbol{\zeta}||^2_{l^{2,2}(L^2)}$$
$$\le C_5\big(||\xi^0||^2_{l^2(L^2)}+||\rho_0||^2_{l^2(L^2)}+||\rho||^2_{l^{2,2}(L^2)}+\Delta t||D\rho||^2_{L^2(\Lambda;L^2)}+(\Delta t)^2\big).$$

Hence, by (3.4) and (3.6),

$$||\xi||_{l^{\infty,2}(L^2)}+||\boldsymbol{\zeta}||_{l^{2,2}(L^2)}\le C_6\Big(||\xi^0||_{l^2(L^2)}+h^{r+1}||u|_{a=0}||_{l^2(H^{r+1})}+h^{r+1}||u||_{l^{2,2}(H^{r+1})}$$
$$+h^{r+1}\sqrt{\Delta t}\big(||u||_{L^2(\Lambda;H^{r+1})}+||Du||_{L^2(\Lambda;H^{r+1})}\big)+\Delta t\Big). \quad (3.15)$$

Since $\xi^0=U^0-u^0+\rho^0$, by our choice of initial approximation and (3.4), we have

$$||\xi^0||_{l^2(L^2)}\le Ch^{r+1}||u_0||_{l^\infty(H^{r+1})}. \quad (3.16)$$

Combining estimates (3.15),(3.16) we obtain

$$||\xi||_{l^{\infty,2}(L^2)}+||\boldsymbol{\zeta}||_{l^{2,2}(L^2)}\le C\varepsilon(u). \quad (3.17)$$

Substituting (3.17) into (3.14), we easily see that for sufficiently small $\Delta t$ (e.g., $\Delta t\le 1/(4C_3)$),

$$||\xi_j^n||^2\le\frac{1}{1-2C_3\Delta t}\big(||\xi_{j-1}^{n-1}||^2+C\Delta t\varepsilon(u)^2\big)\le(1+4C_3\Delta t)||\xi_{j-1}^{n-1}||^2+C\Delta t\varepsilon(u)^2.$$

Recursively using this inequality, we obtain that

$$||\xi_j^n||^2\le\begin{cases} e^{cj\Delta t}||\xi_0^{n-j}||^2+(e^{cj\Delta t}-1)C\varepsilon(u)^2, & \text{for } n>j,\\ e^{cn\Delta t}||\xi_{j-n}^0||^2+(e^{cn\Delta t}-1)C\varepsilon(u)^2, & \text{for } j\ge n.\end{cases} \quad (3.18)$$

Using (3.17) in (3.8), we see

$$||\xi_0^n||\le C\varepsilon(u),\ \text{for all } n\ge1. \quad (3.19)$$

As in (3.16), we find

$$||\xi_j^0||\le Ch^{r+1}||u_0||_{l^\infty(H^{r+1})},\ \text{for } j\ge0.$$

This together with (3.18) and (3.19) implies that

$$||\xi||_{l^{\infty,\infty}(L^2)}\le C\varepsilon(u),$$

which completes the proof in view of (3.17) and (3.4) and an application of the triangle inequality. □

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