

QUASI-STATIC THERMOELASTIC PROBLEM OF AN INFINITELY LONG CIRCULAR CYLINDER

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ABSTRACT. The aim of this work is to determine the quasi-static thermal stresses of an infinitely long circular cylinder having constant initial temperature under steady-state field. The arbitrary heat flux is applied on the lower surface and the upper surface of the cylinder is at initial temperature. The fixed circular edge is thermally insulated. The results are obtained in series form in terms of Bessel's functions. These have been computed numerically and illustrated graphically.

1. INTRODUCTION

As known, thermal behaviors of structures must be considered in many situations. Study of thermal effect on deformations and stresses of a plate, especially a thick plate is increasingly important. Firstly, the problems of thick plates are more complicated and thus more attractive to many scientists. Secondly, there are practical requirements for thick plates in various modern projects, such as high building, raceway, high-way, container wharf, and so on.

Atsumi et al [9] determined the linear thermoelastic problem of an infinitely long circular cylinder with a circumferential edge crack is solved. The thermal stresses are caused by a uniform heat flow disturbed by the presence of the crack. The crack surfaces and the cylindrical surface are assumed to be insulated. Noda et al [7] has determined the two-dimensional problem of an infinitely long circular cylinder whose lateral surface is traction-free and subjected to an asymmetrical heating is considered within the context of the theory of generalized thermoelasticity with one relaxation time. Thomas et al [6] studied analysis is presented of the thermal stresses encountered during cooling of a solid circular cylinder initially heated from uniform temperature by Newtonian convection, followed by sudden cooling prior to reaching thermal

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equilibrium during the heating phase of the cycle. Sinha et al [5] has studied the transient heat conduction problem of a thick annular disk with transversely anisotropic coefficients of thermal conductivity is solved by using the Rayleigh-Ritz. Hany H. Sherief et al [4] determined the analysis of axisymmetric mechanical and thermal stresses for a long hollow cylinder made up of functionally graded material, as functions of radial and longitudinal directions is developed. Doo-Sung Lee [3] determined the three dimensional analysis of the stress distribution in a long circular cylinder containing a concentric very thin spherical cap cavity. The central plane of the cavity is perpendicular to the axis of the cylinder, and the cylinder is subjected to bending. Also Kulkarni et al [1] studied quasi-static thermal stresses in a thick circular plate subjected to arbitrary initial temperature on the upper surface with lower surface at zero temperature and the fixed circular edge thermally insulated.

In this work we study the quasi-static thermoelastic problem of an infinitely long circular cylinder having constant initial temperature (T_i) under steady-state field. The arbitrary heat flux is applied on the lower surface ($z = 0$) and the upper surface of the cylinder is at initial temperature (T_i). The fixed circular edge ($r = a$) is thermally insulated. The results are obtained in series form in terms of Bessel's functions. These have been computed numerically and illustrated graphically. But as yet nobody has studied the such type of problem.

In this work, we present some new interesting results of quasi-static thermoelastic problem of an infinitely long circular cylinder under steady-state field. The results presented here will be more useful in engineering problem particularly in the determination of the state of strain in a long circular cylinder constituting foundations of containers for hot gases or liquids, in the foundations for furnaces etc.

2. FORMULATION OF THE PROBLEM

Consider an infinitely long solid circular cylinder of radius a subjected to steady-state temperature field. Let the initial temperature of the circular cylinder is given by a constant temperature T_i . The heat flux $\left(\frac{-Q_0 f(r)}{\lambda}\right)$ is applied on the lower surface of the cylinder and the upper surface of the cylinder is at initial temperature T_i . The fixed circular edges ($r = a$) is thermally insulated. Assume that the boundary surface of a circular cylinder is free from traction. Under these more realistic prescribed conditions, the quasi-static thermal stresses need to be determined.

The differential equation governing the displacement potential function $\phi(r, z)$ is given in [2] as,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = K\tau \quad (2.1)$$

where K is the restraint coefficient and temperature change $\tau = T - T_i$, where T_i is initial temperature. The displacement function ϕ is known as Goodier's thermoelastic displacement potential.

The steady-state temperature of the cylinder satisfies the heat condition equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (2.2)$$

with the boundary conditions,

$$\frac{\partial T}{\partial r} = 0 \quad \text{at } r = a \quad (2.3)$$

$$\lambda \frac{\partial T}{\partial z} = -Q_0 f(r) \quad \text{at } z = 0, 0 \leq r \leq a \quad (2.4)$$

$$T = T_i \quad \text{at } z \rightarrow \infty, 0 \leq r \leq a \quad (2.5)$$

The displacement function in the cylindrical coordinate system are represented by the Michell's function defined in [2] as,

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \quad (2.6)$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \quad (2.7)$$

The Michell's function M must satisfy

$$\nabla^2 \nabla^2 M = 0 \quad (2.8)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (2.9)$$

The components of the stresses are represented by the thermoelastic displacements potential ϕ and Michell's function M as

$$\sigma_{rr} = 2G \left[\frac{\partial^2 \phi}{\partial r^2} - K\tau + \frac{\partial}{\partial z} \left(\nu \nabla^2 M - \frac{\partial^2 M}{\partial r^2} \right) \right] \quad (2.10)$$

$$\sigma_{\theta\theta} = 2G \left[\frac{1}{r} \frac{\partial \phi}{\partial r} - K\tau + \frac{\partial}{\partial z} \left(\nu \nabla^2 M - \frac{1}{r} \frac{\partial M}{\partial r} \right) \right] \quad (2.11)$$

$$\sigma_{zz} = 2G \left[\frac{\partial^2 \phi}{\partial z^2} - K\tau + \frac{\partial}{\partial z} \left((2 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right] \quad (2.12)$$

and

$$\sigma_{rz} = 2G \left[\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left((1 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right] \quad (2.13)$$

where G and ν are the shear modulus and poisson's ratio respectively. The boundary conditions on the traction free surfaces of an circular plate are

$$\sigma_{rr} = \sigma_{rz} = 0 \quad \text{at } r = a \quad (2.14)$$

Equations (2.1) to (2.14) constitute the mathematical formulation of the problem under consideration.

3. SOLUTION OF THE PROBLEM

To obtain the expressions for temperature $T(r, z)$, we assume

$$T(r, z) = T_i + \sum_{n=1}^{\infty} A_n J_0(\xi_n r) e^{-\xi_n z}, \quad (3.1)$$

where ξ_1, ξ_2, \dots are the roots of the transcendental equation $J_1(\xi a) = 0$. $J_n(x)$ is Bessel function of the first kind of order n .

Using equations (2.4) and (3.1), one obtains

$$Q_0 f(r) = \lambda \xi_n A_n J_0(\xi_n r) \quad (3.2)$$

By theory of Bessel's function, one obtains

$$A_n = \frac{2Q_0 \bar{f}(\xi_n)}{a^2 \lambda \xi_n J_0^2(\xi_n r)}, \quad (3.3)$$

where $\bar{f}(\xi_n)$ is the Hankel transform of $f(r)$.

$$\bar{f}(\xi_n) = \int_0^a r J_0(\xi_n r) f(r) dr \quad (3.4)$$

The temperature change $\tau = T - T_i$ is

$$\tau = \sum_{n=1}^{\infty} A_n J_0(\xi_n r) e^{-\xi_n z}. \quad (3.5)$$

Now a suitable form of M satisfying Equation (2.8) is given by

$$M = \sum_{n=1}^{\infty} [B_n J_0(\xi_n r) + C_n \xi_n r J_1(\xi_n r)] e^{-\xi_n z}, \quad (3.6)$$

where B_n and C_n are arbitrary constants, which can be determined from the boundary condition (2.14).

The potential $\phi(r, z)$ is obtained from equations (2.1) and (3.5) as

$$\phi(r, z) = \left(\frac{K}{2}\right) \sum_{n=1}^{\infty} A_n \frac{r}{\xi_n} J_1(\xi_n r) e^{-\xi_n z} \quad (3.7)$$

Now using Equations (3.5)-(3.7) in (2.6), (2.7) and (2.10)-(2.13), one obtains the expressions for displacement and thermal stresses as,

$$u_r = \sum_{n=1}^{\infty} \left\{ K \frac{A_n}{2} r J_0(\xi_n r) - B_n \xi_n^2 J_1(\xi_n r) + C_n \xi_n^2 [\xi_n r J_0(\xi_n r)] \right\} e^{-\xi_n z} \quad (3.8)$$

$$u_z = \sum_{n=1}^{\infty} \left\{ -K \frac{A_n}{2} r J_1(\xi_n r) - B_n \xi_n^2 J_0(\xi_n r) + C_n \xi_n^2 [4(1 - \nu) J_0(\xi_n r) - \xi_n r J_1(\xi_n r)] \right\} e^{-\xi_n z} \quad (3.9)$$

$$\begin{aligned} \sigma_{rr} = 2G \sum_{n=1}^{\infty} & \left\{ K \frac{A_n}{2} [J_0(\xi_n r) - \xi_n r J_1(\xi_n r)] - K A_n J_0(\xi_n r) \right. \\ & \left. - B_n \xi_n^2 \left[\xi_n J_0(\xi_n r) - \frac{J_1(\xi_n r)}{r} \right] \right. \\ & \left. + C_n \xi_n^3 [(1 - 2\nu) J_0(\xi_n r) - \xi_n r J_1(\xi_n r)] \right\} e^{-\xi_n z} \end{aligned} \quad (3.10)$$

$$\begin{aligned} \sigma_{\theta\theta} = 2G \sum_{n=1}^{\infty} & \left\{ K \frac{A_n}{2} J_0(\xi_n r) - K A_n J_0(\xi_n r) \right. \\ & \left. - B_n \xi_n^2 \left(\frac{J_1(\xi_n r)}{r} \right) + C_n \xi_n^3 (1 - 2\nu) J_0(\xi_n r) \right\} e^{-\xi_n z} \end{aligned} \quad (3.11)$$

$$\begin{aligned} \sigma_{zz} = 2G \sum_{n=1}^{\infty} & \left\{ K \frac{A_n}{2} \xi_n r J_1(\xi_n r) - K A_n J_0(\xi_n r) + B_n \xi_n^3 J_0(\xi_n r) \right. \\ & \left. - C_n \xi_n^3 [(4 - 2\nu) J_0(\xi_n r) - \xi_n r J_1(\xi_n r)] \right\} e^{-\xi_n z} \end{aligned} \quad (3.12)$$

$$\begin{aligned} \sigma_{rz} = 2G \sum_{n=1}^{\infty} & \left\{ -K \frac{A_n}{2} \xi_n r J_0(\xi_n r) + B_n \xi_n^3 J_1(\xi_n r) \right. \\ & \left. - C_n \xi_n^3 [2(1 - \nu) J_1(\xi_n r) + \xi_n r J_0(\xi_n r)] \right\} e^{-\xi_n z} \end{aligned} \quad (3.13)$$

Now in order to satisfy the boundary conditions given in the equation (2.14), we use equations (3.10) and (3.13) for B_n and C_n one obtains,

$$B_n = -(1 - \nu) \frac{K A_n}{\xi_n^3}$$

and

$$C_n = -\frac{K A_n}{2\xi_n^3}$$

Using these values of B_n and C_n in equations (3.8) to (3.13), one obtains the expressions for displacements and stresses as

$$u_r = K(1 - \nu) \sum_{n=1}^{\infty} \frac{A_n}{\xi_n} J_1(\xi_n r) e^{-\xi_n z} \quad (3.14)$$

$$u_z = -K(1 - \nu) \sum_{n=1}^{\infty} \frac{A_n}{\xi_n} J_0(\xi_n r) e^{-\xi_n z} \quad (3.15)$$

$$\sigma_{rr} = -2GK(1 - \nu) \sum_{n=1}^{\infty} \frac{A_n}{r\xi_n} J_1(\xi_n r) e^{-\xi_n z} \quad (3.16)$$

$$\sigma_{\theta\theta} = -2GK(1 - \nu) \sum_{n=1}^{\infty} A_n \left[J_0(\xi_n r) - \frac{J_1(\xi_n r)}{r\xi_n} \right] e^{-\xi_n z} \quad (3.17)$$

$$\sigma_{zz} = 0 \quad (3.18)$$

and

$$\sigma_{rz} = 0 \quad (3.19)$$

SPECIAL CASE

Setting

$$f(r) = (r^2 - a^2)^2 \quad (3.20)$$

We apply the finite Hankel transform defined as in [8] to get

$$\begin{aligned} \bar{f}(\xi_n) &= \int_0^a r(r^2 - a^2)^2 J_0(\xi_n r) dr, \\ \bar{f}(\xi_n) &= \frac{8a \{ (8 - a^2 \xi_n^2) J_1(\xi_n a) - 4a \xi_n J_0(\xi_n a) \}}{\xi_n^5}. \end{aligned}$$

4. NUMERICAL CALCULATIONS

Numerical calculations have been carried out for a steel (*SN 50C*) plate with parameters chosen $a = 2m$, $z = 4m$. The thermal diffusivity is given by $k = 15.9 \times 10^6 (m^2 s^{-1})$ and the Poisson ratio by $\nu = 0.281$. The transcendental roots of $J_1(\xi_n a)$ as in [10] are $\xi_1 = 3.8317$, $\xi_2 = 7.0156$, $\xi_3 = 10.1735$, $\xi_4 = 13.3237$, $\xi_5 = 16.470$, $\xi_6 = 19.6159$, $\xi_7 = 22.7601$, $\xi_8 = 25.9037$, $\xi_9 = 29.0468$, $\xi_{10} = 32.18$.

For convenience we set

$$\alpha = \left(\frac{8K(1-\nu)}{a^2 \lambda} \right) \quad \beta = \left(\frac{16GK(1-\nu)}{a^2 \lambda} \right)$$

The numerical expressions for the displacement and stress components are obtained in equations (3.14) to (3.17).

The numerical calculation has been carried out with the help of computational mathematical software Mathcad-2007, and the graphs are plotted with the help of Excel(MS Office-2007).

5. CONCLUDING REMARKS

In this paper, we have discussed the long circular cylinder which is free from traction subjected to the arbitrary heat flux is applied on the lower surface and determine the expressions for the temperature, displacement and stress components.

As a special case a mathematical model is constructed for

$$f(r) = (r^2 - a^2)^2$$

and numerical calculations were performed. The thermoelastic behavior is examined such as temperature, displacement and stress components with the help of arbitrary heat flux is applied on the lower surface.

Figure 1, the radial displacement function u_r/α increases within the circular region $0 \leq r \leq 1$ and decreases within annular region $1 \leq r \leq 2$ in the radial direction.

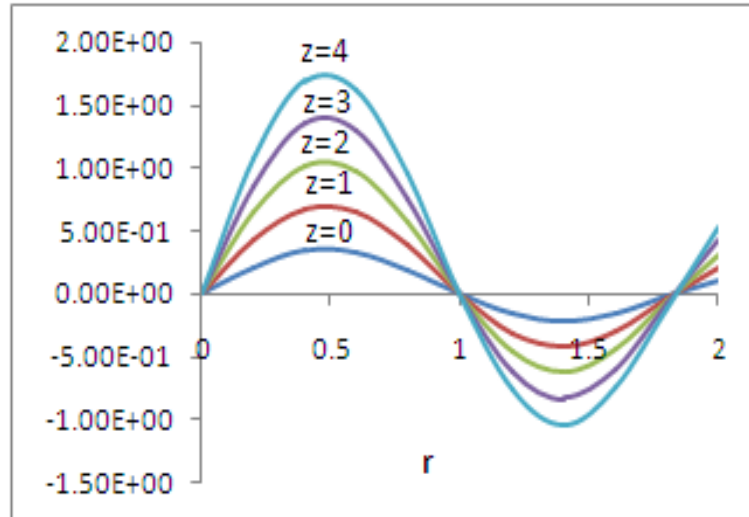


FIGURE 1. The radial displacement function u_r/α in radial direction.

Figure 2, the axial displacement function u_z/α oscillates in the radial direction .

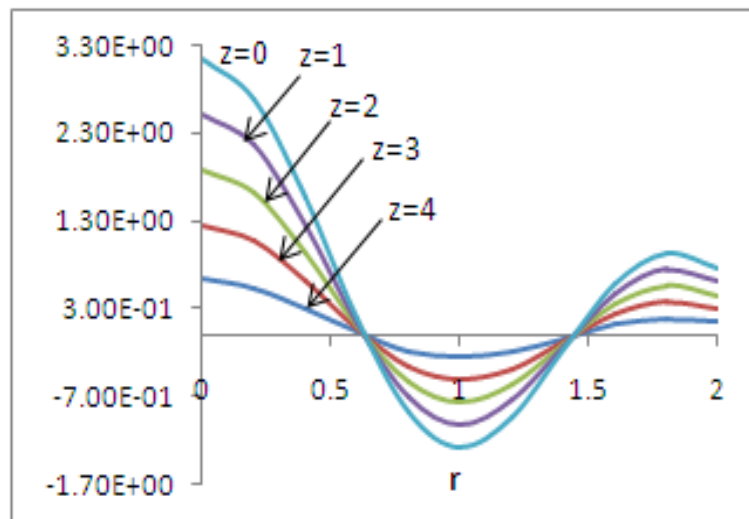


FIGURE 2. The axial displacement function u_z/α in radial direction.

Figure 3, the radial stress function σ_{rr}/β decreases within the circular region $0 \leq r \leq 1$ and increases within the annular region $1 \leq r \leq 2$ in the radial direction.

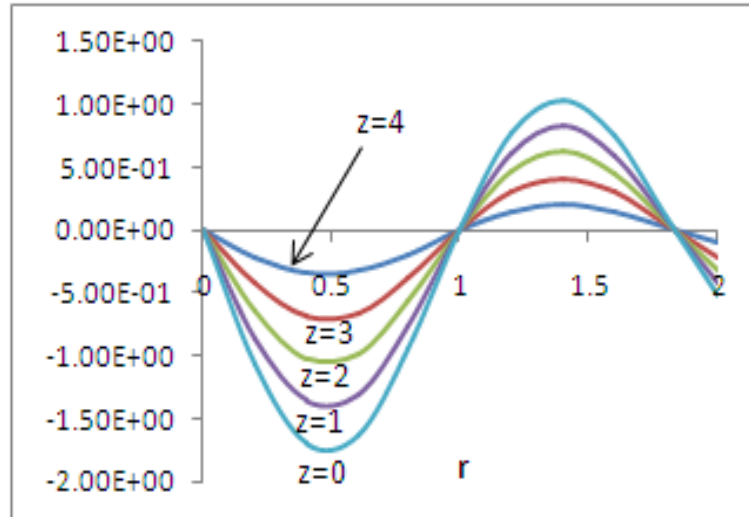


FIGURE 3. The radial stress function σ_{rr}/β in radial direction.

Figure 4, the angular stress function $\sigma_{\theta\theta}/\beta$ shows the oscillating behaviors in the radial direction.

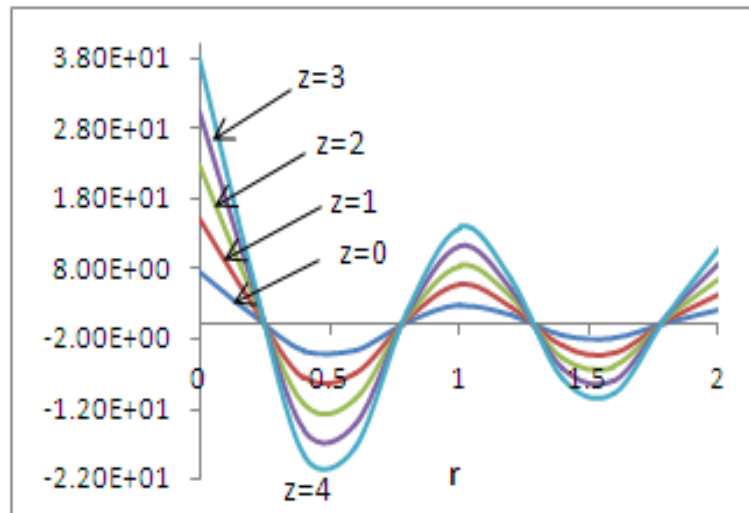


FIGURE 4. The angular stress function $\sigma_{\theta\theta}/\beta$ in radial direction.

It means we may find out the displacement and stress components occurs near heat source. With an increases in temperature the circular plate will tend to expand in the radial direction as well as in the axial direction. In the plane state of stress the stress components σ_{zz} and σ_{rz} are zero. Also from the figure of displacement it can observe that displacement occurs around the center towards downward direction. So it may conclude that due to arbitrary heat flux is applied on the lower surface of the circular cylinder expands in axial direction and bends concavely at the center. This expansion is inversely proportional to thickness of circular cylinder.

The results obtained here are more useful in engineering problems particularly in the determination of state of strain in a long circular cylinder. Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (3.14) to (3.17).

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