

## NUMERICAL PROPERTIES OF GAUGE METHOD FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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**ABSTRACT.** The representative numerical algorithms to solve the time dependent Navier-Stokes equations are projection type methods. Lots of projection schemes have been developed to find more accurate solutions. But most of projection methods [4, 11] suffer from inconsistency and requesting unknown datum. E and Liu in [5] constructed the gauge method which splits the velocity  $\mathbf{u} = \mathbf{a} + \nabla\phi$  to make consistent and to replace requesting of the unknown values to known datum of non-physical variables  $\mathbf{a}$  and  $\phi$ . The errors are evaluated in [9]. But gauge method is not still obvious to find out suitable combination of discrete finite element spaces and to compute boundary derivative of the gauge variable  $\phi$ . In this paper, we define 4 gauge algorithms via combining both 2 decomposition operators and 2 boundary conditions. And we derive variational derivative on boundary and analyze numerical results of 4 gauge algorithms in various discrete spaces combinations to search right discrete space relation.

### 1. INTRODUCTION

Given an open bounded polygon  $\Omega$  in  $\mathbb{R}^d$  with  $d = 2$  or  $3$ , we consider the time dependent Navier-Stokes Equations [NSE]:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \mu\Delta\mathbf{u} &= \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, & \text{in } \Omega, \\ \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}^0, & \text{in } \Omega, \end{aligned} \tag{1.1}$$

with vanishing Dirichlet boundary condition  $\mathbf{u} = \mathbf{0}$  on  $\partial\Omega$  and pressure mean-value  $\int_{\Omega} p = 0$ . The unknowns are vector function  $\mathbf{u}$  (velocity) and the scalar function  $p$  (pressure). And  $\mu = Re^{-1}$  is the reciprocal of the Reynolds number.

Pressure  $p$  can be viewed in (1.1) as a Lagrange multiplier corresponding to the incompressibility condition  $\operatorname{div} \mathbf{u} = 0$ . This coupling is responsible for compatibility conditions between the spaces for  $\mathbf{u}$  and  $p$ , characterized by the celebrated inf-sup condition, and associated numerical difficulties [1, 7]. On the other hand, projection methods were introduced independently

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by Chorin [4] and Temam [15, 16] in the late 60's to decouple  $\mathbf{u}$  and  $p$  and thus reduce the computational cost. However, some projection methods impose artificial boundary and initial conditions on  $p$ , which leads to boundary layers and reduced convergence rates for  $p$  [6, 12]. We will introduce Chorin and Chorin-Uzawa method in §2 to discuss inconsistency and to compare with the gauge method which is studied in §3. Also we collect theoretical estimations which were proved in [2, 6, 14].

E and Liu in [5] introduced the gauge method which splits the velocity  $\mathbf{u} = \mathbf{a} + \nabla\phi$  in terms of non-physical variables  $\mathbf{a}$  and  $\phi$ . The gauge method impose initial and boundary condition on gauge variable  $\phi$  but pressure  $p$ . Moreover this scheme doesn't include inconsistency. So we can say it is more natural method than any other projection type method in PDE level. But troubles are to compute boundary derivation on the discrete space and to find out suitable combinations of discrete finite element space of each function. The former limits their application on 2d and the latter make calculation heavy. The goal of this paper is to implement the gauge method using finite element method without losing advantages of the method. In order to discuss about the drawback of the classical projection method, we introduce Chorin method and Chorin-Uzawa method in §2. And We will construct 4 time discrete gauge algorithms to solve the difficulties on boundary differentiation via using both stream line functions and boundary properties in §3. We construct variational approach of the boundary derivation in §4, and we compute error decay on various discrete space combinations to analyze stability condition among finite element spaces in §5.

In whole this paper,  $\boldsymbol{\nu}$  and  $\boldsymbol{\tau}$  are the unit vectors in normal and tangential direction, respectively. And  $\tau$  designates the time step. And we indicate with  $\|\cdot\|_s$  the norm in  $\mathbf{H}^s(\Omega)$ .

## 2. REVIEW OF PROJECTION METHODS

The main strategy of the projection type method is to find an artificial velocity  $\tilde{\mathbf{u}}$  via solving the momentum equation including transformed pressure term without divergence free constraint. And then we project  $\tilde{\mathbf{u}}$  to solenoidal space using the following Helmholtz decomposition lemma in [7]:

**Lemma 2.1** (Helmholtz decomposition theorem). *Let*

$$\mathbf{H} = \{\mathbf{v} \in \mathbf{L}^2(\Omega)^d : \nabla \cdot \mathbf{v} = 0 \text{ and } \mathbf{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}.$$

*Then we have the decomposition*

$$\mathbf{L}^2(\Omega)^d = \mathbf{H} \oplus \mathbf{H}^\perp,$$

*where  $\mathbf{H}^\perp$  is defined as*

$$\mathbf{H}^\perp = \{\nabla q \in L^2(\Omega) : q \in L^2(\Omega)\}.$$

Equivalently, all  $\tilde{\mathbf{u}} \in \mathbf{L}^2(\Omega)^d$  can be written by

$$\tilde{\mathbf{u}} = \mathbf{u} + \nabla q, \tag{2.1}$$

where  $\mathbf{u} \in \mathbf{H}$  and  $\nabla q \in \mathbf{H}^\perp$ . The classical projection method impose divergence operator in (2.1) to compute  $\mathbf{u}$  and  $q$ , which is

$$\begin{aligned} \Delta q &= \operatorname{div} \tilde{\mathbf{u}}, & \text{in } \Omega, \\ \partial_\nu q &= 0, & \text{on } \partial\Omega, \end{aligned} \quad (2.2)$$

and then we can obtain divergence free velocity via adding known 2 functions

$$\mathbf{u} = \tilde{\mathbf{u}} + \nabla q. \quad (2.3)$$

We now introduce a classical projection method by Chorin [4, 6, 11]:

**Algorithm 1** (Chorin method). *Start with  $\mathbf{u}^0 = \mathbf{u}(0)$ .*

**Step 1:** (*Momentum equation*) Find  $\tilde{\mathbf{u}}^{n+1}$  as the solution of

$$\begin{aligned} \frac{\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\tau} + (\mathbf{u}^n \cdot \nabla) \tilde{\mathbf{u}}^{n+1} - \mu \Delta \tilde{\mathbf{u}}^{n+1} &= \mathbf{f}(t_{n+1}), & \text{in } \Omega, \\ \tilde{\mathbf{u}}^{n+1} &= \mathbf{0}, & \text{on } \partial\Omega. \end{aligned} \quad (2.4)$$

**Step 2:** (*Projection step*)

$$\begin{aligned} \frac{\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1}}{\tau} + \nabla p^{n+1} &= \mathbf{0}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u}^{n+1} &= 0, & \text{in } \Omega, \\ \mathbf{u}^{n+1} \cdot \boldsymbol{\nu} &= 0, & \text{on } \partial\Omega. \end{aligned} \quad (2.5)$$

In virtue of (2.2) and (2.3),  $\tilde{\mathbf{u}}^{n+1}$  in (2.5) can be split into its solenoidal and irrotational parts by solving

$$\begin{aligned} \Delta p^{n+1} &= \frac{1}{\tau} \operatorname{div} \tilde{\mathbf{u}}^{n+1} & \text{in } \Omega, \\ \partial_\nu p^{n+1} &= 0, & \text{on } \partial\Omega, \end{aligned}$$

and then adding 2 functions

$$\mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1} - \tau \nabla p^{n+1}. \quad (2.6)$$

**Remark 2.2** (Artificial boundary condition). In the view of  $\mathbf{u}^{n+1} \cdot \boldsymbol{\nu} = 0$ , pressure  $p$  automatically satisfies the non-physical Neumann boundary condition  $\partial_\nu p^{n+1} = 0$  on  $\partial\Omega$ . This artificial boundary condition is responsible for a non-physical boundary layer for  $p$ .

**Remark 2.3** (Inconsistency). Upon plugging (2.6) into (2.4), we also discover an inconsistency in the momentum equation

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + (\mathbf{u}^n \cdot \nabla) \tilde{\mathbf{u}}^{n+1} - \mu \Delta \mathbf{u}^{n+1} + \nabla p^{n+1} - \mu \tau \Delta \nabla p^{n+1} = \mathbf{f}(t_{n+1}),$$

where  $-\mu \tau \Delta \nabla p^{n+1}$  is the inconsistent term.

There are several publications concerning error estimates for Chorin Algorithm 1. The most relevant for us is Prohl [11] who employs a variational approach with some reasonable assumptions. If  $\sigma(t) = \min\{t, 1\}$ , then

$$\begin{aligned} \|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_0 + \sigma(t^{n+1})\|p(t^{n+1}) - p^{n+1}\|_{-1} &\leq C\tau, \\ \|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_1 + \sqrt{\sigma(t^{n+1})}\|p(t^{n+1}) - p^{n+1}\|_0 &\leq C\sqrt{\tau}. \end{aligned}$$

The second paper of interest [6] is by E and Liu, who derive error estimates via an asymptotic expansion approach: If the exact solution  $(\mathbf{u}(t), p(t))$  of (1.1) is smooth enough, then

$$\|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_0 + \sqrt{\tau}\|p(t^{n+1}) - p^{n+1}\|_0 \leq C\tau.$$

This result requires regularity which is often not valid for realistic incompressible flows.

One of the famous projection method is the Chorin-Uzawa method which has been introduced by Prohl in [11] to get rid of the boundary layer and inconsistency of Chorin method:

**Algorithm 2** (Chorin-Uzawa method). *Start with given data  $(\mathbf{u}^0, p^0, \tilde{p}^0)$  such that*

$$\|\mathbf{u}(0) - \mathbf{u}^0\|_0 + \sqrt{\tau}\|p(0) - p^0\|_0 \leq C\tau, \quad \tilde{p}^0 = 0. \quad (2.7)$$

**Step 1:** (*Momentum equation*) Find  $\tilde{\mathbf{u}}^{n+1}$  as the solution of

$$\begin{aligned} \frac{\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\tau} + (\mathbf{u}^n \cdot \nabla)\tilde{\mathbf{u}}^{n+1} - \mu\Delta\tilde{\mathbf{u}}^{n+1} + \nabla(p^n - \tilde{p}^n) &= \mathbf{f}(t_{n+1}), \quad \text{in } \Omega, \\ \tilde{\mathbf{u}}^{n+1} &= \mathbf{0}, \quad \text{on } \partial\Omega. \end{aligned} \quad (2.8)$$

**Step 2:** (*Projection step*)

$$\begin{aligned} \frac{\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1}}{\tau} + \nabla\tilde{p}^{n+1} &= \mathbf{0}, \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u}^{n+1} &= 0, \quad \text{on } \Omega, \\ \mathbf{u}^{n+1} \cdot \boldsymbol{\nu} &= 0, \quad \text{on } \partial\Omega. \end{aligned} \quad (2.9)$$

**Step 3:** (*Pressure step*)

$$p^{n+1} = p^n - \alpha\mu\operatorname{div} \tilde{\mathbf{u}}^{n+1}, \quad 0 < \alpha < 1. \quad (2.10)$$

The Chorin-Uzawa method is a combination of Chorin Algorithm 1 and Uzawa Algorithm which is an iterative solver of the stationary Stokes equations [1, 7, 8]. The condition of relaxation parameter,  $0 < \alpha < 1$ , is necessitated to prove convergence of the Uzawa algorithm, but it is proved that the optimal  $\alpha$  is 1 and that its convergence range is  $0 < \alpha < 2$  in [8]. So  $\alpha$  can be chosen as 1 simply.

In the projection step (2.9), we split  $\tilde{\mathbf{u}}^{n+1}$  into  $\mathbf{u}^{n+1}$  and  $\nabla\tilde{p}^{n+1}$  by the same manner with (2.2) and (2.3). Note the presence of the auxiliary pressure  $\tilde{p}^n$  with artificial boundary value  $\partial_{\boldsymbol{\nu}}\tilde{p}^n = 0$  in (2.9). No boundary condition is imposed on pressure  $p^n$  any longer. Regardless of this improvement, Chorin-Uzawa exhibits the following pitfalls:

**Remark 2.4** (Initial pressure). The initial value  $p^0$  can not be chosen arbitrarily, because of the initial condition  $\|p(0) - p^0\|$  in (2.7). So it requires estimating initial pressure or choosing small time distance  $\tau$  at initial steps.

**Remark 2.5** (Inconsistency). Upon plugging  $\tilde{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} + \tau \nabla \tilde{p}^{n+1}$  from (2.9) into (2.8), we see that

$$\begin{aligned} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + (\mathbf{u}^n \cdot \nabla) \tilde{\mathbf{u}}^{n+1} + \nabla (p^n - \alpha \mu \tau \Delta \tilde{p}^{n+1}) - \mu \Delta \mathbf{u}^{n+1} \\ + (\alpha - 1) \mu \tau \nabla \Delta \tilde{p}^{n+1} + \nabla (\tilde{p}^{n+1} - \tilde{p}^n) = \mathbf{f}(t_{n+1}). \end{aligned}$$

Since (2.9) and (2.10) imply  $p^{n+1} = p^n - \alpha \mu \tau \Delta \tilde{p}^{n+1}$ , we end up with

$$\begin{aligned} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + (\mathbf{u}^n \cdot \nabla) \tilde{\mathbf{u}}^{n+1} + \nabla p^{n+1} - \mu \Delta \mathbf{u}^{n+1} \\ + (\alpha - 1) \mu \tau \nabla \Delta \tilde{p}^{n+1} + \nabla (\tilde{p}^{n+1} - \tilde{p}^n) = \mathbf{f}(t_{n+1}). \end{aligned}$$

Here  $(\alpha - 1) \mu \tau \nabla \Delta \tilde{p}^{n+1} + \nabla (\tilde{p}^{n+1} - \tilde{p}^n)$  are the inconsistency terms. If we choose  $\alpha = 1$ , then the first term disappears but the second term still remained.

The following a priori error bound is stated by Prohl [11]:

$$\|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_1 + \sqrt{\tau} \|p(t^{n+1}) - p^{n+1}\|_0 \leq C\tau.$$

### 3. GAUGE METHODS

As we reviewed in §2, projection methods suffer from inconsistencies. E and Liu in [5] construct gauge method which is a consist projection type method hiring (2.2) and (2.3) which is called divergence operator:

**Algorithm 3** (Gauge method with div operator and Neumann boundary condition). Start with initial values  $\phi^0 = 0$  and  $\mathbf{a}^0 = \mathbf{u}^0 = \mathbf{u}(\mathbf{x}, 0)$ .

**Step 1:** Find  $\mathbf{a}^{n+1}$  as the solution of

$$\begin{aligned} \frac{\mathbf{a}^{n+1} - \mathbf{a}^n}{\tau} + (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n - \mu \Delta \mathbf{a}^{n+1} = \mathbf{f}(t_{n+1}), \quad \text{in } \Omega, \\ \mathbf{a}^{n+1} \cdot \boldsymbol{\nu} = 0, \quad \mathbf{a}^{n+1} \cdot \boldsymbol{\tau} = -\partial_{\boldsymbol{\tau}} \phi^n, \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

**Step 2:** Find  $\phi^{n+1}$  as the solution of

$$\begin{aligned} -\Delta \phi^{n+1} = \operatorname{div} \mathbf{a}^{n+1}, \quad \text{in } \Omega, \\ \partial_{\boldsymbol{\nu}} \phi^{n+1} = 0, \quad \text{on } \partial\Omega, \end{aligned}$$

**Step 3:** Find

$$\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla \phi^{n+1}, \quad \text{in } \Omega.$$

One may compute the pressure whenever necessary as

$$p^{n+1} = -\frac{\phi^{n+1} - \phi^n}{\tau} + \mu \Delta \phi^{n+1}. \quad (3.2)$$

We can check easily that Algorithm 3 consists to (1.1), and a priori bound for Algorithm 3 is proved in [9]:

$$\tau \sum_{n=0}^N \left( \|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_0^2 + \tau \|p(t^{n+1}) - p^{n+1}\| \right) \leq C\tau^2. \quad (3.3)$$

But one difficulty in implementation is to compute boundary differentiation in (3.1). We will discuss about the variational calculation on boundary in §4. On the other hand, we can avoid the difficult boundary differentiation, provided we know  $\nabla \phi^n$  in (3.1). So we can consider to use stream line function in [7] instead of using (2.2) and (2.3).

**Lemma 3.1** (Stream line function). *A function  $\mathbf{v}$  is in 2-dimension  $\mathbf{H}$  if and only if there exists a stream function  $\psi \in H^1(\Omega)$  such that*

$$\mathbf{u} = \mathbf{curl} \psi. \quad (3.4)$$

Since  $\mathbf{u}$  is in  $\mathbf{H}$ , there exists a stream function (3.4). And the stream function hold Dirichlet boundary condition because of  $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ . Owing Lemma 2.1,  $\mathbf{a}$  can be rewritten by

$$\begin{aligned} \mathbf{a} &= \mathbf{u} - \nabla \phi \\ &= \mathbf{curl} \psi - \nabla \phi. \end{aligned} \quad (3.5)$$

If we impose  $\text{rot}$  in (3.5), then we arrive at

$$\begin{aligned} -\Delta \psi &= \text{rot} \mathbf{a}, & \text{in } \Omega, \\ \psi &= 0, & \text{on } \partial\Omega, \end{aligned}$$

We now have  $\psi$  and easily obtain  $\mathbf{u}$  by solving (3.4) and then get  $\nabla \phi$  by computing

$$\nabla \phi = \mathbf{u} - \mathbf{a}.$$

So we do not need to compute the boundary derivative in (3.1), because we know  $\nabla \phi$  already. Finally, we are ready to define the gauge method to using rotational operator.

**Algorithm 4** (Gauge method with  $\text{rot}$  operator and Neumann boundary condition).

*Start with initial values  $\phi^0 = 0$  and  $\mathbf{a}^0 = \mathbf{u}^0 = \mathbf{u}(\mathbf{x}, 0)$ .*

**Step 1:** Find  $\mathbf{a}^{n+1}$  as the solution of (3.1)

**Step 2:** Find  $\psi^{n+1}$  as the solution of

$$\begin{aligned} -\Delta \psi^{n+1} &= \text{rot} \mathbf{a}^{n+1}, & \text{in } \Omega, \\ \psi^{n+1} &= 0, & \text{on } \partial\Omega. \end{aligned}$$

**Step 3:** Find

$$\begin{aligned} \mathbf{u}^{n+1} &= \mathbf{curl} \psi^{n+1}, & \text{in } \Omega, \\ \nabla \phi^{n+1} &= \mathbf{u}^{n+1} - \mathbf{a}^{n+1}, & \text{in } \Omega. \end{aligned}$$

We can compute pressure via (3.2) whenever necessary. Also we can obtain a priori error bound (3.3) for Algorithm 4 because it is equivalent to Algorithm 3.

In the view of (3.5) and  $\mathbf{u} = \mathbf{0}$  on  $\partial\Omega$ , we can also take boundary condition  $\phi = 0$  and  $\mathbf{a} = -\partial_\nu\phi$ . Therefore we can define a gauge method via imposing div operator and Dirichlet boundary condition:

**Algorithm 5** (Gauge method with div operator and Dirichlet boundary condition). *Start with initial values  $\phi^0 = 0$  and  $\mathbf{a}^0 = \mathbf{u}^0 = \mathbf{u}(\mathbf{x}, 0)$ .*

**Step 1:** Find  $\mathbf{a}^{n+1}$  as the solution of

$$\begin{aligned} \frac{\mathbf{a}^{n+1} - \mathbf{a}^n}{\tau} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n - \mu\Delta\mathbf{a}^{n+1} &= \mathbf{f}(t_{n+1}), & \text{in } \Omega, \\ \mathbf{a}^{n+1} \cdot \nu &= -\partial_\nu\phi^n, \quad \mathbf{a}^{n+1} \cdot \tau = 0, & \text{on } \partial\Omega. \end{aligned} \quad (3.6)$$

**Step 2:** Find  $\phi^{n+1}$  as the solution of

$$\begin{aligned} -\Delta\phi^{n+1} &= \operatorname{div} \mathbf{a}^{n+1}, & \text{in } \Omega, \\ \phi^{n+1} &= 0, & \text{on } \partial\Omega. \end{aligned}$$

**Step 3:** Find

$$\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^{n+1}, \quad \text{in } \Omega.$$

The following a priori bound for Algorithm 5 which is imposed Dirichlet boundary condition is proved in [9]

$$\tau \sum_{n=0}^N \|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_0^2 \leq C\tau. \quad (3.7)$$

Also we define a gauge method with rotational operator and with Dirichlet boundary condition.

**Algorithm 6** (Gauge method with rot operator and Dirichlet boundary condition). *Start with initial values  $\phi^0 = 0$  and  $\mathbf{a}^0 = \mathbf{u}^0 = \mathbf{u}(\mathbf{x}, 0)$ .*

**Step 1:** Find  $\mathbf{a}^{n+1}$  as the solution of (3.6).

**Step 2:** Find  $\psi^{n+1}$  as the solution of

$$\begin{aligned} -\Delta\psi^{n+1} &= \operatorname{rot} \mathbf{a}^{n+1}, & \text{in } \Omega, \\ \partial_\nu\psi^{n+1} &= 0, & \text{on } \partial\Omega. \end{aligned}$$

**Step 3:** Find

$$\begin{aligned} \mathbf{u}^{n+1} &= \operatorname{curl} \psi^{n+1}, & \text{in } \Omega, \\ \nabla\phi^{n+1} &= \mathbf{u}^{n+1} - \mathbf{a}^{n+1}, & \text{in } \Omega. \end{aligned}$$

We can impose the error bound (3.7) for Algorithm 6 because of equivalent to Algorithm 5.

**Remark 3.2** (Consistency). *We can see consistency easily by inserting  $\mathbf{a}^{n+1} = \mathbf{u}^{n+1} - \nabla\phi^{n+1}$  and pressure equation (3.2) into the gauge discrete momentum equation (3.1) or (3.6).*

**Remark 3.3** (Compatibility condition). We uncover  $\phi^n$  for Algorithms 5-6 doesn't satisfy compatibility condition in [9]. Since pressure  $p^n$  and  $\phi^n$  are linked via (3.2), we cannot expect convergence of  $p^n$  to  $p(t^{n+1})$ .

**Remark 3.4** (Application in 3 dimension). Since the rot operator can be defined on only 2 dimension, Algorithms 4-6 are not applicable on 3 dimension. So the only Algorithm 3 is applicable to compute pressure on 3 dimension in conjunction with above Remark 3.3.

To apply gauge algorithms on finite element method, we consider stability relation between space of each variable.

**Remark 3.5** (Finite element space stability). Since the gradient of  $\phi$  is the addition of  $\mathbf{a}^{n+1}$  and  $\mathbf{u}^{n+1}$ , we can expect that the gauge variable  $\phi$  is necessary in one higher degree space than those of  $\mathbf{a}^{n+1}$  and  $\mathbf{u}^{n+1}$ . For example, if we consider Taylor-Hood family which is degree 2 for velocity and degree 1 for pressure, then degree 3 is required for  $\phi$ . But this combination requests too heavy computation by hiring high resolution for non-concerning variable  $\phi$ . It will be examined in §5 by comparing numerical results on several combinations.

#### 4. VARIATIONAL APPROACH TO COMPUTE DIFFERENTIATION ON BOUNDARY

A key difficulty in actual computations with gauge methods is to provide accurate approximation of boundary derivatives  $\partial_{\boldsymbol{\nu}}\phi^{n+1}$  or  $\partial_{\boldsymbol{\tau}}\phi^{n+1}$  on  $\partial\Omega$ . We recall now a variational approximation of boundary derivatives. First we consider the Laplace equation

$$\begin{aligned} -\Delta\phi &= f, & \text{in } \Omega, \\ \phi &= 0, & \text{on } \partial\Omega, \end{aligned} \quad (4.1)$$

and approximation of the normal derivative  $\partial_{\boldsymbol{\nu}}\phi$ . Integrating (4.1) by parts against  $\psi \in H^1(\Omega)$ , we find the variational expression

$$-\int_{\Omega} \Delta\phi\psi d\mathbf{x} = -\int_{\partial\Omega} \partial_{\boldsymbol{\nu}}\phi\psi d\Gamma + \int_{\Omega} \nabla\phi\nabla\psi d\mathbf{x}$$

or

$$\int_{\partial\Omega} \partial_{\boldsymbol{\nu}}\phi\psi d\Gamma = -\int_{\Omega} f\psi d\mathbf{x} + \int_{\Omega} \nabla\phi\nabla\psi d\mathbf{x}. \quad (4.2)$$

where the unit normal  $\boldsymbol{\nu}$  is well defined except at corners. Equality (4.2) defines  $\partial_{\boldsymbol{\nu}}\phi \in H^{-\frac{1}{2}}(\partial\Omega)$  uniquely as a linear functional in  $H^{\frac{1}{2}}(\partial\Omega)$  (Trace space of  $H^1(\Omega)$ ). One goal is to use a similar expression to defined the discrete counterpart. To this end, we follow Pehlivanov et al [3, 10]. The first issue is the concept of *normal derivative*, at a corner. Since  $\phi = 0$  on  $\partial\Omega$ , the tangential derivatives vanish, and so does  $\nabla\phi$ , at a corner (see (a) in Figure 1). We thus impose

$$\partial_{\boldsymbol{\nu}}\phi = 0 \quad \text{at corners of } \partial\Omega.$$

Let  $\mathfrak{T} = K$  be a shape-regular quasi-uniform partition of  $\Omega$ . Let  $\mathbb{B}_h$  be a conforming finite element space containing piecewise linear and let  $\mathbb{B}_h^b$  be the *boundary* finite element space

$$\mathbb{B}_h^b = \{w_h \in \mathbb{B}_h : w_h = 0 \text{ at the interior and corner nodes of } \Omega\}.$$



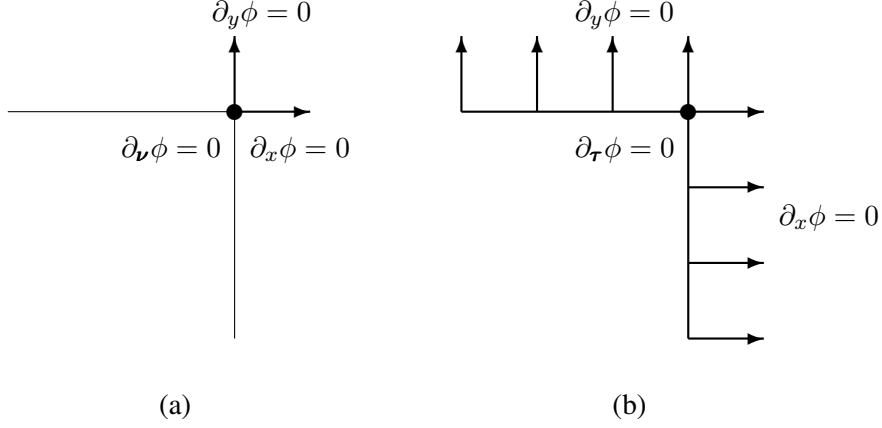


FIGURE 1. (a). deriving of  $\partial_\nu \phi = 0$  at each corner under the condition  $\phi = 0$ .  
 (b). deriving of  $\partial_\tau \phi = 0$  at each corner under the condition  $\partial_\nu \phi = 0$ .

We also define

$$\mathbb{B}_h^0 = \{w_h : w_h \in H_0^1(\Omega)\}$$

Let  $\phi_h \in \mathbb{B}_h^0$  be the finite element solution of (4.1), namely,

$$\phi_h \in \mathbb{B}_h^0 : \int_{\Omega} \nabla \phi_h \nabla \psi_h dx = \int_{\Omega} f \psi_h dx, \quad \forall \psi_h \in \mathbb{B}_h^0.$$

In view of (4.2), we define the approximate normal derivative  $\partial_\nu \phi_h$  to be:

$$\phi_h \in \mathbb{B}_h^0 : \int_{\partial\Omega} \partial_\nu \phi_h \psi_h d\Gamma = - \int_{\Omega} f \psi_h dx + \int_{\Omega} \nabla \phi_h \nabla \psi_h dx, \quad \forall \psi_h \in \mathbb{B}_h^0.$$

The following lemma was proved in [10]

**Lemma 4.1.** *If  $f \in H^2(\Omega)$  and  $\phi \in H^3(\Omega)$ , then*

$$\|\partial_\nu \phi - \partial_\nu \phi_h\|_{0,\Gamma} \leq Ch^{\frac{3}{2}} (\|\phi\|_{3,\Omega} + \|f\|_{2,\Omega}).$$

Thus derivative  $\partial_\nu \phi_h^n$  can be calculated by the variational formula:

$$\partial_\nu \phi_h^n \in \mathbb{B}_0^b : \int_{\partial\Omega} \partial_\nu \phi_h^n \psi_h d\Gamma = - \int_{\Omega} \operatorname{div} \mathbf{a}_h^n \psi_h dx + \int_{\Omega} \nabla \phi_h^n \nabla \psi_h dx, \quad \forall \psi_h \in \mathbb{B}_h^b.$$

Now we consider the approximation of tangential derivative  $\partial_\tau \phi$  on  $\partial\Omega$  provided  $\phi$  does no longer vanish on  $\partial\Omega$ . Integration by parts of  $-\Delta \phi = f$  yields for all  $\psi \in H^1(\Omega)$

$$\int_{\partial\Omega} \partial_\tau \phi \psi d\Gamma = \int_{\Omega} \nabla \phi \operatorname{curl} \psi dx. \quad (4.3)$$

If  $\phi$  satisfies Neumann boundary condition, the tangential derivative on each corner is 0 by (b) in Figure 1. If  $\phi^n \in \mathbb{B}_h$  is the finite element approximation in gauge Algorithms 3-6, then  $\phi_{\boldsymbol{\nu}}^n = \phi_{\boldsymbol{\tau}}^n = 0$  and the discrete of (4.3) reads:

$$\partial_{\boldsymbol{\tau}} \phi_h^n \in \mathbb{B}_h^b : \int_{\partial\Omega} \partial_{\boldsymbol{\tau}} \phi_h^n \psi_h d\Gamma = \int_{\Omega} \nabla \phi_h^n \mathbf{curl} \psi_h d\mathbf{x}, \quad \forall \psi_h \in H^1(\Omega). \quad (4.4)$$

Formula (4.4) can be used to approximate  $\partial_{\boldsymbol{\tau}} \phi_h^n$  in 2d. However, we have 2 orthogonal tangential differentiations in 3d, and we can calculate  $\mathbf{a}^{n+1}$  on  $\partial\Omega$  by solving linear system of these 2 tangential boundary differentiations and  $\mathbf{a}^{n+1} \cdot \boldsymbol{\nu} = 0$ .

## 5. NUMERICAL EXPERIMENTS

We, in this section, analyze and compare numerical results of projection methods which are Chorin, Chorin-Uzawa, gauge, and Gauge-Uzawa methods, with both smooth and singular solutions. The first Experiment comes from multiplication time function  $\cos(t)$  and the example of Prohl in [11]: the computational domain is  $\Omega = [0, 1] \times [0, 1]$  and  $\mu$  is 1. We choose the following exact solution of (1.1) and determine the corresponding force term  $\mathbf{f}$

$$\begin{aligned} u(x, y, t) &= \cos(t)(x^2 - 2x^3 + x^4)(2y - 6y^2 + 4y^3) \\ v(x, y, t) &= -\cos(t)(y^2 - 2y^3 + y^4)(2x - 6x^2 + 4x^3) \\ p(x, y, t) &= \cos(t) \left( x^2 + y^2 - \frac{2}{3} \right). \end{aligned}$$

**Remark 5.1** (Distorted mesh). In order to avoid super convergence due to mesh uniformity and symmetry, we choose the distorted quasi-uniform mesh Figure 2. Mesh distortion is crucial to uncover numerical difficulties that may go unnoticed otherwise. For instance, Gauge method is insensitive to the discrete inf-sup condition for uniform mesh.

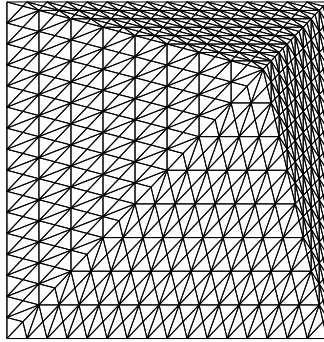


FIGURE 2. The Computational mesh for experiments.

In order to check the dependency of inf-sup condition and find out the stability condition of gauge variable  $\phi$  in the gauge methods, we make a mesh analysis in this section on the

following combinations of discretization parameters and polynomial degrees ( $K_v =$  polynomial degree of velocity,  $K_p =$  polynomial degree of pressure, and  $K_\phi =$  polynomial degree of  $\phi$ ):

- Combination 1:**  $K_v = 1, K_p = 1, K_\phi = 1.$
- Combination 2:**  $K_v = 2, K_p = 1, K_\phi = 1.$
- Combination 3:**  $K_v = 1, K_p = 1, K_\phi = 2.$
- Combination 4:**  $K_v = 2, K_p = 1, K_\phi = 2.$
- Combination 5:**  $K_v = 2, K_p = 1, K_\phi = 3.$

The gauge methods show different dependence on these combinations. As we know, the finite element spaces of Combinations 2 and 4-5 correspond to the Taylor-Hood family  $P_2 - P_1$  ( $K_v = 2, K_p = 1$ ) which satisfies the discrete inf-sup condition. In contrast, the finite element pairs  $P_1 - P_1$  ( $K_v = 1, K_p = 1$ ) of Combinations 1 and 3 do not satisfy the discrete inf-sup condition. Since gauge Algorithms 3-6 have convergence order 1, we compute with relation  $\tau = h^2$  to get same order 2 for space and time contributions. If  $\tau = h$  and the space errors are  $\mathcal{O}(h^{\kappa+1})$  with  $\kappa \geq 1$ , then the time error  $\mathcal{O}(\tau)$  dominates the calculation. All numerical results in this paper are computed by ALBERT which is a finite element toolbox [13].

We note that Algorithms 5-6 do not necessary the error of pressure to decrease (see Remark 3.2). We use a pair  $(\alpha, \beta)$ , where  $\alpha$  and  $\beta$  are convergence orders in  $L^2(\Omega)$  and  $L^\infty(\Omega)$  spaces, respectively. We first consider Figure 3 which is error decay of Combination 1. As we know that Combination 1 doesn't satisfy inf-sup condition, so pressure  $p^{n+1}$  doesn't need to converge  $p(t^{n+1})$ . But the pressure in Algorithm 4 has (1.6, 1.0) convergence order and the velocity has

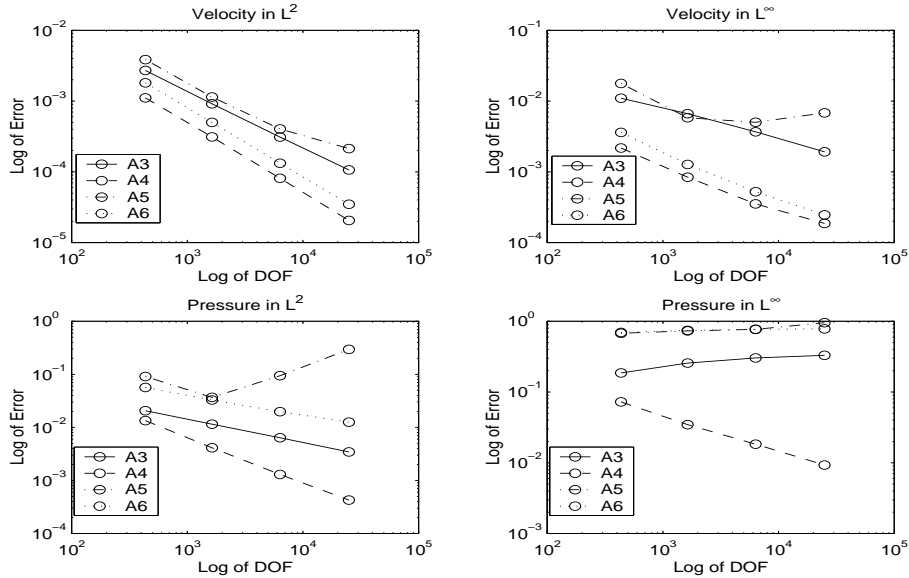


FIGURE 3. Error decay of gauge methods with  $\tau = h^2$  and  $P_1 - P_1 - P_1$  elements.

convergence order (2.0, 1.0). The reason of losing order in  $L^\infty(\Omega)$  space seems like due to the broken inf-sup condition.

Combinations 2 satisfies inf-sup condition, but pressure in  $L^\infty(\Omega)$  doesn't converge to exact solution for all gauge methods in Figure 4. Moreover velocity has only order (1.0, 1.0). So we conclude it is not stable combination. We now see Figures 5-7. The velocities of Algorithms 3-4 have convergence order (2.0, 2.0) which is optimal error decay. But pressures of both algorithms have order (1.5, 1.0) for Combination 3, (2.0, 1.8) for Combinations 4-5. The size of errors are also similar for both Combinations 4-5, but Combination 5 requests much higher computational cost than Combination 4. So we conclude, in this experiment, Combination 4 are the best family to imply Algorithms 3-4. Also we note Combination 3 is also acceptable family, if we have relatively big tolerance for pressure. In other word, Algorithms 5-6 have order (2.0, 1.0) for velocity and not computable for pressure on Combinations 3-5.

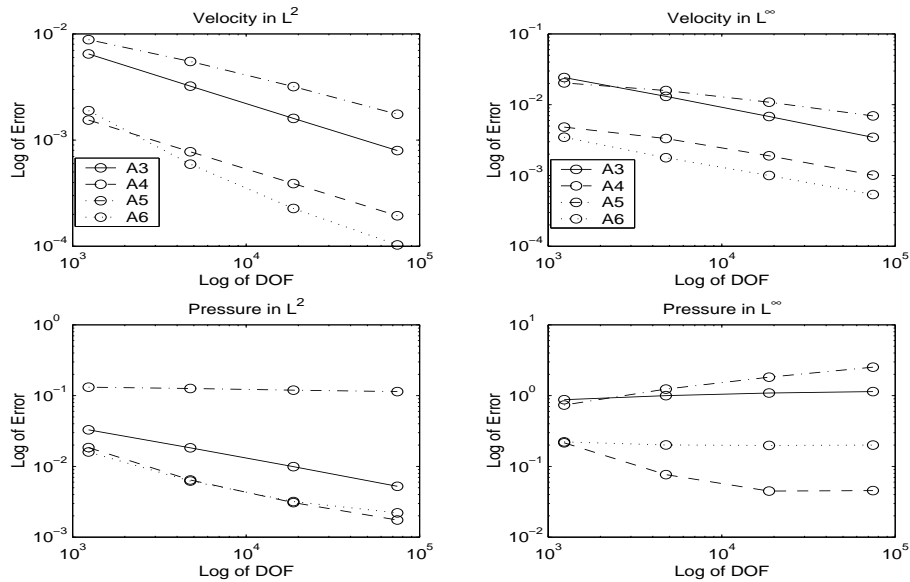


FIGURE 4. Error decay of gauge methods with  $\tau = h^2$  and  $P_2 - P_1 - P_1$  elements.

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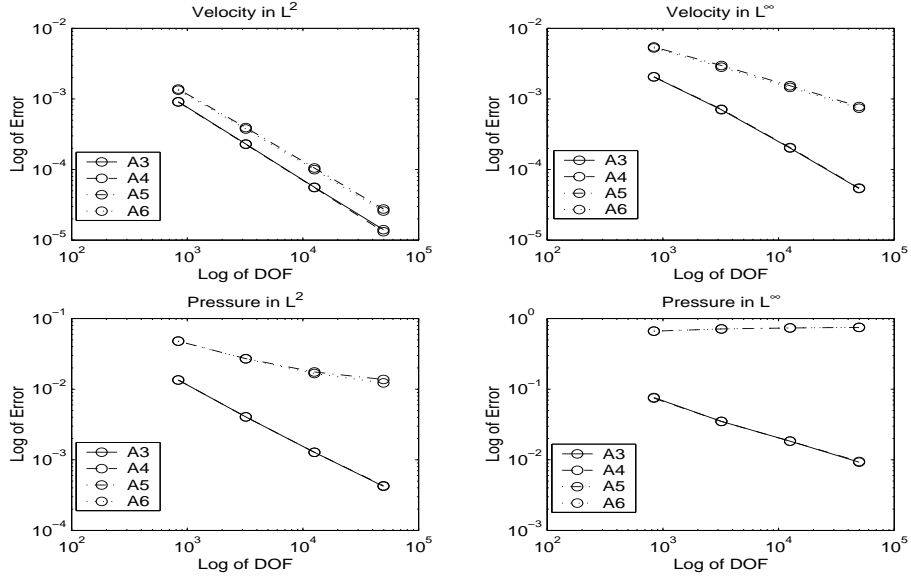


FIGURE 5. Error decay of gauge methods with  $\tau = h^2$  and  $P_1 - P_1 - P_2$  elements.

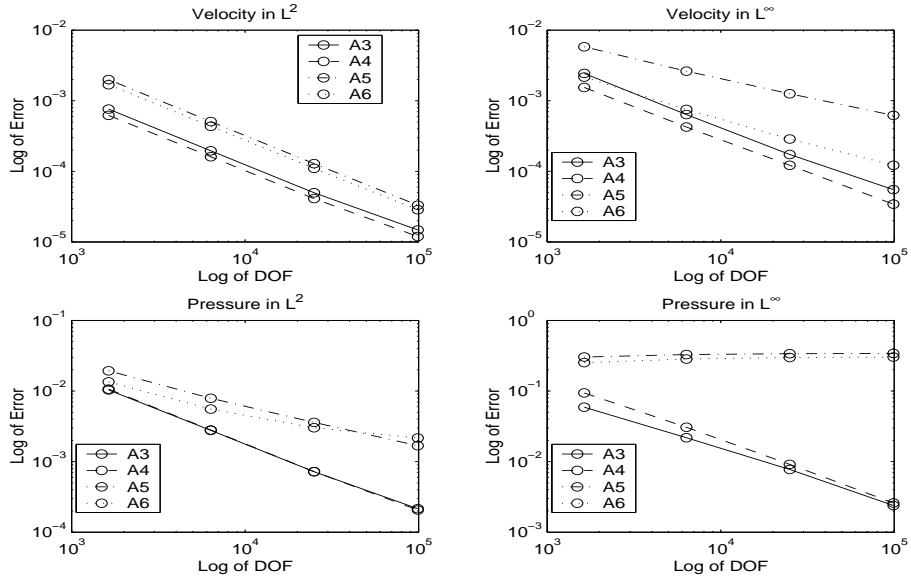


FIGURE 6. Error decay of gauge methods with  $\tau = h^2$  and  $P_2 - P_1 - P_2$  elements.

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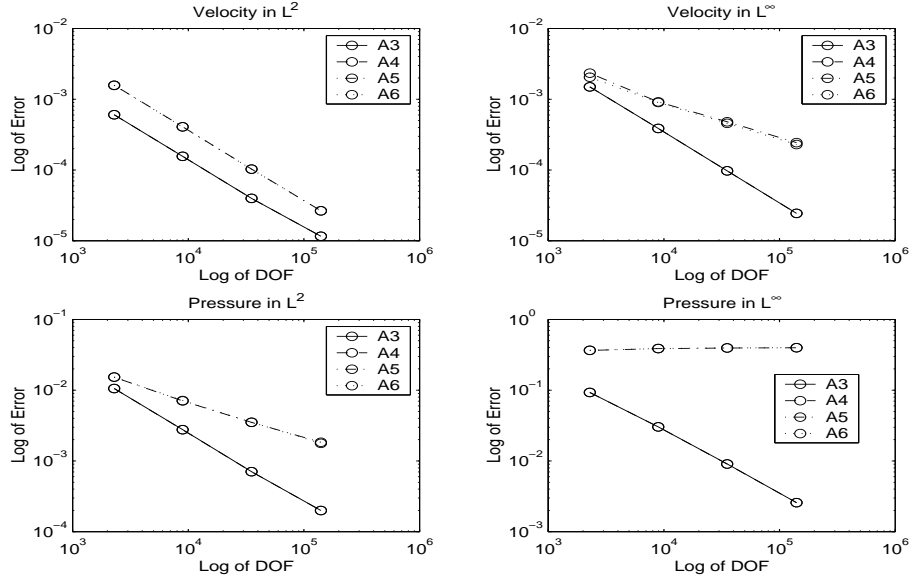


FIGURE 7. Error decay of gauge methods with  $\tau = h^2$  and  $P_2 - P_1 - P_3$  elements.

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