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An Asymptotic Solution and the Green's Function for the Transverse Vibration of Beams with Variable Properties

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ABSTRACT: *An analytical solution procedure for the dynamic response of beams with variable properties is developed by using an asymptotic solution and the Green's function. This asymptotic closed form solution is derived for the transverse vibration of beams under the assumption of slowly varying properties, such as mass, cross-section, tension etc., along the beam length. However, this solution is still found to be very accurate even in the case of large variation, such as step change in cross-section, mass, and tension . Therefore, this derived asymptotic closed form solution and the Green's function can be easily applied to find dynamic responses for various kind of beam vibration problems.*

1. Introduction

Most of structures are subject to dynamic forces, such as ocean wave load, wind load, forces from rotaing machineries, moving harmonic forces etc. As the modern engineering technology develops, the size of the structure becomes huge and complex. Also, the modern structures consist of many beams with variable properties for different purposes.

Many research works have been done for the dynamic responses of beams subjected to general loading. Most analytical works are restricted to simple shaped beams, such as constant cross-section, mass, tension etc. In analytical method for variable properties Goel (1976) studied free transverse vibration of tapered beams and Naguleswaran (2004) studied free vibration for Step change in cross-section and linearly varying axial tension. Lee et al. (1990) studied on the analysis of non-unifirm beam and Firouz et al. (2007) recently published a paper on the asympotic solution to the transverse free vibration of variable-section beams. Most of published papers deal with free vibration problems including natural frequencies and mode shapes. Also, many numerical methods, such as Finite element method (FEM), are used to obtain solutions for more complex structures and beams with variable properties.

Main purpose of this paper is to derive analytical solution procedure for beams with variable properties along the beam length, which has not been available in the literature except for constant cross-section. The analytic solutions for transverse vibration of the Euler-Bernoulli Beam with constant properties are well known in the literature. However, the closed form solutions for the variable properties along it's length, such as variable cross-section, tension, mass etc., have not been available until the author (Kim, 1983) found asymptotic closed form solutions by WKB (Wentzel, Kramers and Brillouin) method and published it in the papers (Kim and Triantafyllou, 1984). These solutions were found to be very accurate though the variation of tension was large (Kim, 1988) and even for the step-change in cross-section (Nam and Kim, 2004). In the limit when the variation goes to zero, this asymptotic solution becomes exact solution. Therefore, this derived asymptic closed from solutions and the Green's function can be easily applied to many engineering problems, such as long offshore drilling pipes subject to varying tension due to gravity force and the dynamic responses of bridges with variable cross-section and bending rigidity subject to moving loads.

2. Equation of Motion

The equation of motion for the beam with variable properties subjected to dynamic forces can be described as follows:

$$\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 W}{\partial x^2}] - \frac{\partial}{\partial x}[T(x)\frac{\partial W}{\partial x}] + M(x)\frac{\partial^2 W}{\partial t^2} = F(x,t) \tag{1}$$

where EI is bending rigidity, T is axial tension, M is mass per unit length, W is transverse displacement, F is a dynamic force.

The typical boundary conditions and initial conditions can

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be stated as follows:

(1) Simply Supported Beam

$$W(0,t) = W''(0,t) = W(L,t) = W''(L,t) = 0 \tag{2}$$

where a prime denotes derivative with respect to x.

(2) Fixed-Fixed beam

$$W(0,t) = W'(0,t) = W(L,t) = W'(L,t) = 0 \tag{3}$$

(3) Free-Free Beam

$$W''(0,t) = W'''(0,t) = W''(L,t) = W'''(L,t) = 0 \tag{4}$$

Also, the two initial conditions are imposed.

$$W(x,0) \quad \text{and} \quad \dot{W}(x,0) \tag{5}$$

where a dot denotes derivative with respect to time.

3. Asymptotic Solution for the Transverse Vibration of Beams

In general, the WKB method can be effectively used to find the solutions for the slowly varying coefficients in the differential equation. Kim (1983) obtained the asymptotic closed form solution by using coordinate transformation and WKB method and published in the paper (Kim, 1988).

By introducing non-dimensional quantities, which are defined in Appendix A, the non-dimensional equation for natural frequencies and mode shapes of the beam with variable properties are obtained as follows:

$$\frac{\partial^2}{\partial s^2}[P(s)\frac{\partial^2 Y}{\partial s^2}] - \frac{\partial}{\partial s}[Q(s)\frac{\partial Y}{\partial s}] + m(s)\frac{\partial^2 Y}{\partial \tau^2} = 0 \tag{6}$$

After the separation of variables $Y(s, \tau) = R(s)H(\tau)$, the governing equation can be separated as

$$\frac{d^2}{ds^2}\left[P(s)\frac{d^2R}{ds^2}\right] - \frac{d}{ds}\left[Q(s)\frac{dR}{ds}\right] - U(s)\Lambda^2 R = 0 \tag{7}$$

and

$$\ddot{H} + \Lambda^2 H = 0 \tag{8}$$

The asymptotic closed form solution for the above equation (7) was obtained as follows:

$$R(s) = T_2(s)\left[C_1 \sin\int_0^s h_2 d\xi + C_2 \cos\int_0^s h_2 d\xi\right] + T_1(s)\left[C_3 \sinh\int_0^s h_1 d\xi + C_4 \cosh\int_0^s h_1 d\xi\right] \tag{9}$$

where C_1, C_2, C_3, C_4 are constants and $R(s)$, s are non-dimensional displacement and axial coordinate, respectively, and $T_1(s)$, $T_2(s)$, $h_1(s)$, $h_2(s)$ are defined in the Appendix A.

4. Natural Frequencies and Mode Shapes

For a simply supported beam, the boundary condition is

$$R(0) = R''(0) = R(1) = R''(1) = 0, \tag{10}$$

where a prime denotes a derivative with respect to s.

By substituting equation (9) into equation (10), the following simple, asymptotic formulas to predict natural frequencies and mode shapes of the simply supported beam can be obtained:

$$\int_0^1 \sqrt{-\frac{1}{2}\left(\frac{Q}{P}\right) + \frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2 + 4\frac{U\Lambda_n^2}{P}}}\, d\xi = n\pi$$

$$n = 1,\ 2,\ \ldots \tag{11}$$

In dimensional form,

$$\int_0^L \sqrt{-\frac{1}{2}\left(\frac{T(x)}{EI(x)}\right) + \frac{1}{2}\sqrt{\left(\frac{T(x)}{EI(x)}\right)^2 + 4\frac{M(x)\omega_n^2}{EI(x)}}}\, dx = n\pi,$$

$$n = 1,\ 2,\ \ldots \tag{12}$$

Mode shapes are

$$R_n(s) = \sin\left\{\int_0^S \sqrt{-\frac{1}{2}\left(\frac{Q}{P}\right) + \frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2 + 4\frac{U\Lambda_n^2}{P}}}\, ds\right\}$$

$$n = 1,\ 2,\ \ldots \tag{13}$$

Also, the orthonormal characteristic functions becomes

$$\phi_n(S) = \frac{\sin\left\{\int_0^1 \sqrt{-\frac{1}{2}\left(\frac{\mathrm{Q}}{\mathrm{P}}\right) + \frac{1}{2}\sqrt{\left(\frac{\mathrm{Q}}{\mathrm{p}}\right)^2 + 4\frac{\mathrm{U}\Lambda_{\mathrm{n}}^2}{\mathrm{p}}}}\, \mathrm{ds}\right\}}{\int_0^1 U(s)\sin^2\left\{\int_0^S \sqrt{-\frac{1}{2}\left(\frac{\mathrm{Q}}{\mathrm{p}}\right) + \frac{1}{2}\sqrt{\left(\frac{\mathrm{Q}}{\mathrm{p}}\right)^2 + 4\frac{\mathrm{U}\Lambda_{\mathrm{n}}^2}{\mathrm{p}}}}\, \mathrm{ds}\right\}\mathrm{ds}}$$

$$n = 1,\ 2,\ \ldots \tag{14}$$

When the variation goes to zero, the above asymptotic solution becomes exact solution. The natural frequencies for other boundary conditions can be found in (Nam and Kim, 2004).

5. The Green's Function

In order to find the dynamic responses subjected to harmonic excitation, the Green's function approach is used.

$$\frac{d^2}{ds^2}\left[P(s)\frac{d^2R}{ds^2}\right] - \frac{d}{ds}\left[Q(s)\frac{dR}{ds}\right] - U(s)\Lambda^2 R = \delta(s-\eta)F$$

$$\text{for } 0 \le s, \eta \le 1 \tag{15}$$

where F is the non-dimensional magnitude of harmonically excited force at $s=\eta$ and Λ is non-dimensional exciting frequency.

In order to find the Green's function for a simply supported beam, let $F=1$ and divide the region into two parts, then the asymptotic solutions for each region become

$$R_1(s)=T_2(s)\left[C_1\sin\int_0^s h_2 d\xi+C_2\cos\int_0^s h_2 d\xi\right]+ \qquad (16)$$
$$T_1(s)\left[C_3\sinh\int_0^s h_1 d\xi+C_4\cosh\int_0^s h_1 d\xi\right]$$
$$\text{for } 0\le s<\eta$$

and

$$R_2(s)=T_2(s)\left[D_1\sin\int_0^s h_2 d\xi+D_2\cos\int_0^s h_2 d\xi\right]+$$
$$T_1(s)\left[D_3\sinh\int_0^s h_1 d\xi+D_4\cosh\int_0^s h_1 d\xi\right]$$
$$\text{for } \eta<s\le 1$$

The matching conditions at $s=\eta$ are

$$R_1(\eta)=R_2(\eta) \qquad (27)$$
$$R_1'(\eta)=R_2(\eta)$$
$$R_1''(\eta)=R_2''(\eta)$$

and additional condition can be found by cosidering the discontinuity in the shear force at $s=\eta$.

After integrating the equation (15) over the small region ($\eta-\sigma$, $\eta+\sigma$) and neglecting higher order terms, where σ is a small parameter, we obtain

$$R_2'''(\eta)-R_1'''(\eta)=\frac{1}{P(\eta)} \qquad (18)$$

With these solutions and matching conditions the Green's function can be obtained as follow:

$$G(s,\eta)=\begin{Bmatrix}G_1(s,\eta)\\ G_2(s,\eta)\end{Bmatrix}$$
$$=\frac{T_2(\eta)T_2(s)}{B_1}\begin{Bmatrix}\sin\int_0^\eta h_2(\xi)d\xi.\sin\int_\eta^1 h_2(\xi)d\xi\\ \sin\int_0^\eta h_2(\xi)d\xi.\sin\int_\eta^1 h_2(\xi)d\xi\end{Bmatrix}$$
$$-\frac{T_1(\eta)T_1(s)}{B_3}\begin{Bmatrix}\sinh\int_0^\eta h_1(\xi)d\xi.\sinh\int_\eta^1 h_1(\xi)d\xi & \dots 0\le s\le\eta\\ \sinh\int_0^\eta h_1(\xi)d\xi.\sinh\int_\eta^1 h_1(\xi)d\xi & \dots \eta<s<1\end{Bmatrix} \qquad (19)$$

Once the Green;s function is obtained, total solution can be easily found for various loading conditions.

If there are n-concentrated forces, F_k, and distributed harmonic force, $f(\eta)$, as shown in Fig. 1, the total solution can be obtained in the following form:

$$w^*(s,t)=R(s)e^{iwt} \qquad (20)$$

where w^* is non-dimensional response and

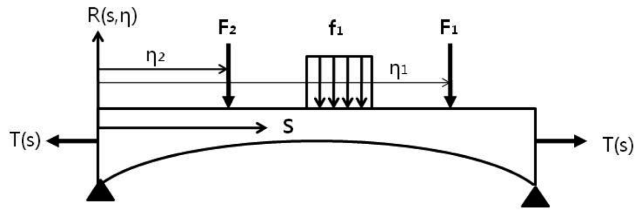

Fig. 1 A Beam with variable properties subjected to various kind of pulsating forces

$$R(s)=\int_0^1 G(s,\xi)f(\xi)d\xi+\sum_{k=1}^{n}F_kG(s,\eta_k) \qquad (21)$$
$$=\int_0^s G_2(s,\xi)f(\xi)d\xi+\int_s^1 G_1(s,\xi)f(\xi)d\xi+\sum_{k=1}^{n}F_kG(s,\eta_k)$$

In this formulation, the variation of tension is also taken into account along the beam length. In the case of long offsore drilling pipes, the tension is varying linearly and very large at the top due to its own weight.

For more general loadings including transient forced vibration, the total solution can be obtained by using the derived asymptotic solution and modal expansion method. Also, the dynamic response due to moving loads can be obtained by using the same solution procedure. Further more, this asymptotic closed form solution can be effectively used as base functions to find the dynamic response of beams for nonlinear systems and time varying boundary conditions (Kim and Triantafyllou, 1984).

6. Comparisions

In order to compare these solutions with the exact solution in the literature, let's consider the simple case, ie. beam with constant cross-section and no axial tension.

(1) Constant distributed force over the whole length

For comparison purpose, let $P=1$, $Q=0$. Then, the non-dimensional amplitude function, $R(s)$, is plotted in Fig. 2, 3, 4 for different values of Λ. Thse solution is compared with the

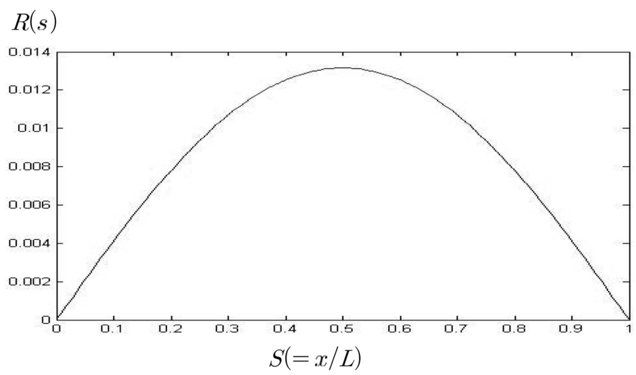

Fig. 2 Non-dimensional amplitude function for $\Lambda=1$

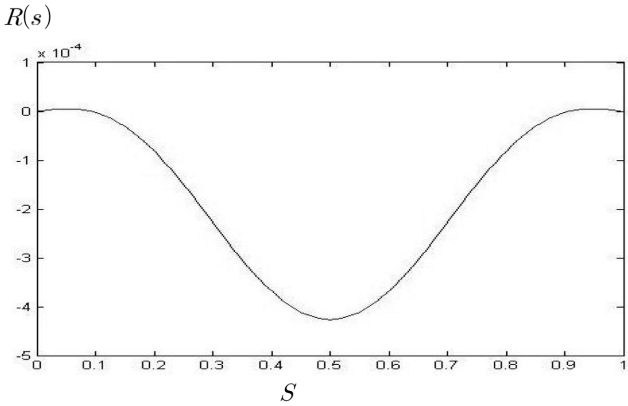

Fig. 3 Non-dimensional amplitude function for $\Lambda = 64$

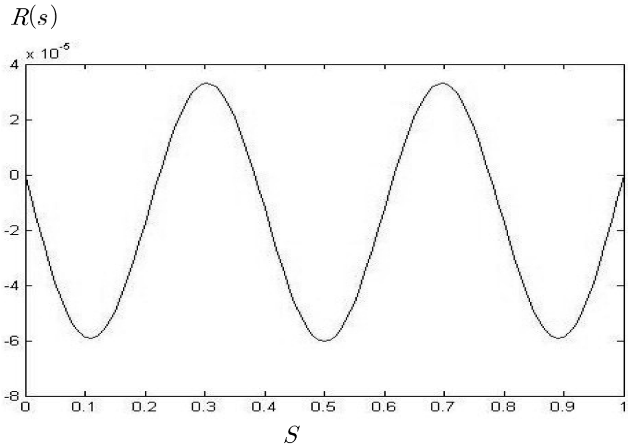

Fig. 4 Non-dimensional amplitude function for $\Lambda = 256$

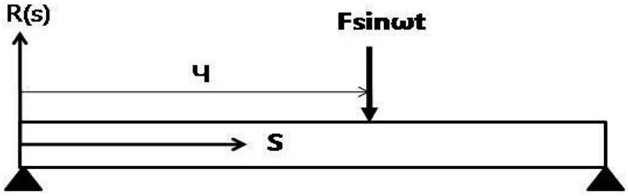

Fig. 5 A constant cross-section beam subjected to a single concentrated harmonic force

exact solution, which can be expressed as series expansion form (Lebedev et al., 1965), and the same result was obtained.

Fig. 2 shows the first mode excitation, Fig. 3. is for second and third mode, Fig. 4 is for high frequency excitation. Also, the amplitude becomes small as the exciting frequency increases.

(2) For a single concentrated harmonic force as shown in Fig. 5, the solution simply becomes

$$R(s) = FG(s,\ \eta) \tag{22}$$

The non-dimensional amplitude function is plotted in Fig. 6, 7, 8, 9 for different loading location.

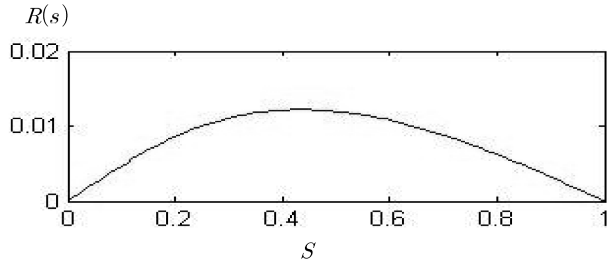

Fig. 6 Non-dimensional amplitude function at $\eta = 0.2$

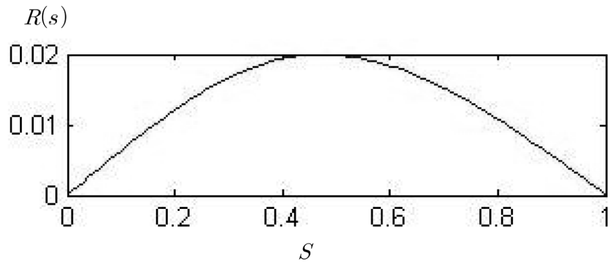

Fig. 7 Non-dimensioal amplitude function at $\eta = 0.4$

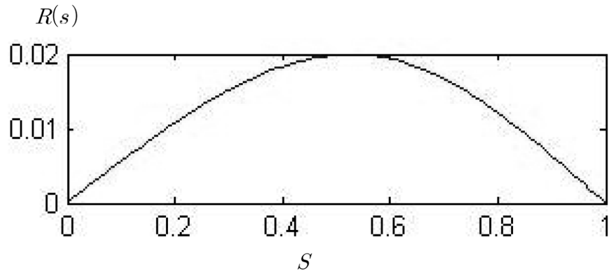

Fig. 8 Nondimensioal amplitude function at $\eta = 0.6$

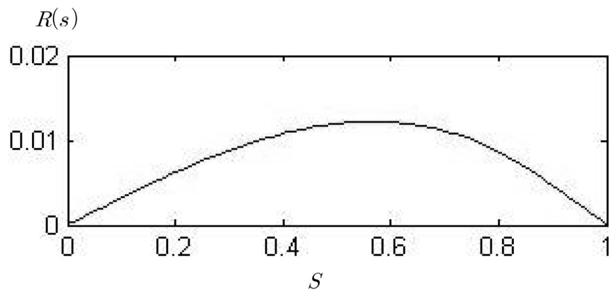

Fig. 9 Nondimensional amplitude function at $\eta = 0.8$

After comparisons, we can conclude that this asymptotic solution and the Green's function becomes exact when the variation goes to zero.

The non-dimensional magnitude of the dynamic responses at $\eta = 0.2$ and $\eta = 0.8$ are symmetrical about center of the beam. The peak value becomes large and moves to center of the beam as the loading point moves to center, as expected.

The effects of other boundary condtions, such as multi-supports and elastic foundations can be found in the paper

(Choi and Kim, 1991).

7. Conclusions

In order to find dynamic responses and stresses for the complex structures or beams with various properties subjected to dynamic forces, the typical numerical methods, such as Finite Element Method, can be employed. However, if the analytical solutions and procedure for the beams with variable properities are developed, it can be effectively used in the preliminary design stage.

In this paper, the analytical solution and procedure for the dynamic response of beams with variable properties is developed by using asymptotic closed form solution and Green's function. This asymptotic solution becomes exact when the variation in properties goes to zero. Further more, even in large variation, such as step change in cross-section, the solution is still accurate in predicting natural frequencies, mode shapes, and dynamic responses. Therefore, this asymptotic solution and the Green's function can be easily applied to obtain dynamic responses for various kind of beam vibration problems.

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Appendix A

The non-dimensional variables are defined as the followings:

$$s=\frac{x}{L},\quad \tau=w_o t\,,\quad w_o=\sqrt{\frac{EI_o}{M_o L^4}}\,,\quad Y=\frac{W}{D}$$

where subscript o denotes the reference section and D is the depth of the beam.

The non-dimensional parameters are given by

$$P(s)=\frac{EI(Ls)}{EI_o},\ Q(s)=\frac{T(Ls)L^2}{EI_o},\ U(s)=\frac{M(Ls)}{M_o},\ \Lambda=\frac{w}{w_o}$$

$$T_1(S)=\frac{1}{\sqrt{P}}\left[\frac{1}{2}\left(\frac{Q}{P}\right)^3+2\frac{QU\Lambda^2}{P^2}+\frac{1}{2}\left\{\left(\frac{Q}{P}\right)^2+4\frac{U\Lambda^2}{P}\right\}^{3/2}\right]^{-1/4}$$

$$T_2(S)=\frac{1}{\sqrt{P}}\left[-\frac{1}{2}\left(\frac{Q}{P}\right)^3-2\frac{QU\Lambda^2}{P^2}+\frac{1}{2}\left\{\left(\frac{Q}{P}\right)^2+4\frac{U\Lambda^2}{P}\right\}^{3/2}\right]^{-1/4}$$

$$h_1(S)=\sqrt{\frac{1}{2}\left(\frac{Q}{P}\right)+\frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2+4\frac{U\Lambda^2}{P}}}$$

$$h_2(S)=\sqrt{-\frac{1}{2}\left(\frac{Q}{P}\right)+\frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2+4\frac{U\Lambda^2}{P}}}$$

$$B_1=\sin\int_0^1 h_2(\xi)d\xi$$

$$B_2=\cos\int_0^1 h_2(\xi)d\xi$$

$$B_3=\sinh\int_0^1 h_1(\xi)d\xi$$

$$B_4=\cosh\int_0^1 h_1(\xi)d\xi$$

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