J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 17, Number 1 (February 2010), Pages 107–113

A NOTE ON THE RANK 2 SYMMETRIC HYPERBOLIC KAC-MOODY ALGEBRAS

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ABSTRACT. In this paper, we study the root system of rank 2 symmetric hyperbolic Kac-Moody algebras. We give the sufficient conditions for existence of imaginary roots of square length $-2k\,(k \in \mathbb{Z}_{>0})$. We also give several relations between the roots on $\mathfrak{g}(A)$.

1. INTRODUCTION

Let A be a symmetric Cartan matrix $A = \left(\begin{smallmatrix} 2 & -a \\ -a & 2 \end{smallmatrix}\right)$ with $a \geq 3$ and $\mathfrak{g} = \mathfrak{g}(A)$ denote the associated symmetric rank 2 hyperbolic Kac-Moody Lie algebra over the field of complex numbers. Let $\Pi = \{\alpha_0, \alpha_1\}$ denote the set of simple roots. Let Δ be its root system, and W be its Weyl group. A root $\alpha \in \Delta$ is called a *real root* if there exists $w \in W$ such that $w(\alpha)$ is a *simple root*, and a root which is not real is called an *imaginary root*. We denote by Δ^{re}, Δ_+^{re}, Δ^{im}, and Δ_+^{im} the set of all real, positive real, imaginary and positive imaginary roots, respectively. Then $\Delta = \Delta^{re} \cup \Delta^{im}$, $\Delta^{re} = \Delta_+^{re} \cup (-\Delta_+^{re})$, $\Delta^{im} = \Delta_+^{im} \cup (-\Delta_+^{im})$, all of which are disjoint. Let $Q = \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1$ denote the root lattice. We also denote by $\Delta_{+,k}^{im}$ the set of all positive imaginary roots of the algebra $\mathfrak{g}(A)$ with square length $-2k$. In [2], A.J.Feingold show that the Fibonacci numbers are intimately related to the rank 2 hyperbolic GCM Lie algebras. In [5], S.J.Kang and D.J.Melville show that all the roots of a given length are Weyl conjugate to roots in a small region. These information help in determining the sufficient conditions for the existence of integral points on the hyperbola $\mathfrak{h}_k : x^2 - axy + y^2 = -k(k \in \mathbb{Z}_{>0})$.

In this paper, we prove the following theorem which determines the sufficient conditions for the existence of integral points on the hyperbola $\mathfrak{h}_k$.

Received by the editors January 7, 2010. Revised January 15, 2010. Accepted January 22, 2010.
2000 *Mathematics Subject Classification.* 17B67.
Key words and phrases. generalized Cartan matrix, Kac-Moody algebra, hyperbolic type.

Main Theorem. Let $x^2 - axy + y^2 = -k$ be the hyperbola and let $k = tm^2$ be any positive integer where t is a square free integer and $m \in \mathbb{Z}$. For some positive integer n, we have :

1 . If $(m, n) \in \Omega_k$, then $a - 2 \le t \le \frac{a^2-4}{4}$.

2 . $(m, n) \in \Omega_k$ if and only if $t = j(a-j) - 1$, with $j = 1, \ldots, [\frac{a}{2}]$, where $[x]$ denotes the smallest integer $\geq x$. Furthermore, $n = jm$ for each $j = 1, \ldots, [\frac{a}{2}]$.

In the following two sections, we prove Theorem 2.3 to express a sequence $\{A_n\}$ as a partial sum of some sequence. We prove our main Theorem in Theorem 2.5 and Theorem 2.6. The paper closes with several interesting relations between the integral points on the hyperbolas $\mathfrak{h}_k$.

2. THE ROOT SYSTEM OF THE ALGEBRA $\mathfrak{g}(A)$

In this section, we give the sufficient conditions for the existence of imaginary roots of square length $-2k\,(k \in \mathbb{Z} > 0)$. We recall that some properties of the root system of symmetric hyperbolic Lie algebra $\mathfrak{g}(A)$. We identify an element

(1) $$\alpha = x_0\alpha_0 + x_1\alpha_1 \in Q \text{ with an ordered pair } (x_0, x_1) \in \mathbb{Z} \times \mathbb{Z}.$$

We call root $\alpha \in \mathbb{Z} \times \mathbb{Z}$ a *positive integral point* if $x, y \in \mathbb{Z}_{\geq 0}$. Define a symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ by the following equation:

(2) $$(\alpha_0 \,|\, \alpha_0) = (\alpha_1 \,|\, \alpha_1) = 2, (\alpha_0|\alpha_1) = -a.$$

We define the sequence of integers $\{A_n\}_{n\geq 0}$ by the recurrence relations

(3) $$A_0 = 0,\ A_1 = 1,\ A_{n+2} = aA_{n+1} - A_n \text{ for } n \geq 0.$$

If $a = 3$, then $A_n = F_{2n}$ for $n \geq 1$, where $\{F_n\}$ is a Fibonacci sequence defined by:

(4) $$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_n + F_{n+1}.$$

Following Proposition is well known.

Proposition 2.1 ([3]). $\Delta_+^{re} = \{(A_n, A_{n+1}),\ (A_{n+1}, A_n) \mid n \geq 0\}$.
Furthermore, $\Delta_+^{re} = \{(F_{2j}, F_{2j+2}),\ (F_{2j+2}, F_{2j}) \,|\, j \in \mathbb{Z}_{\geq 0}\}$ *for* $a = 3$.

Proposition 2.2 ([5]). *Let* $\{A_n\}_{\geq 0}$ *be the sequence defined in* (3). *Then*

$$A_n = \frac{1 - \gamma^{2n}}{\gamma^{n-1}(1-\gamma^2)}\ (n \geq 0),$$

where $\gamma = \dfrac{a+\sqrt{a^2-4}}{2}$ *is a zero of* $1 - ax + x^2$.

Using Proposition 2.2, we obtain the following Theorem.

Theorem 2.3. *Let* $\{A_n\}$ *be the sequence as above. Then for all* $n \geq 1$,

$$A_n = \frac{1}{2^{n-1}} \sum_{k=1}^{[\frac{n+1}{2}]} {}_nC_{2k-1} a^{n-(2k-1)} (a^2-4)^{k-1},$$

where $[x]$ *denotes the smallest integer* $\geq x$.

Proof. Proposition 2.2 shows that

$$\begin{aligned} A_n &= \frac{1-\gamma^{2n}}{\gamma^{n-1}(1-\gamma^2)} \\ &= \frac{1}{\gamma^2-1}(\gamma^{n+1} - \frac{1}{\gamma^{n-1}}) \\ &= \frac{1}{\gamma - \frac{1}{\gamma}}(\gamma^n - \frac{1}{\gamma^n}) \\ &= \frac{1}{2^n\sqrt{a^2-4}}((a+\sqrt{a^2-4})^n - (a-\sqrt{a^2-4})^n). \end{aligned}$$

Thus we have

$$\begin{aligned} A_{2n} &= \frac{1}{2^{2n}\sqrt{a^2-4}}\{(a+\sqrt{a^2-4})^{2n} - (a-\sqrt{a^2-4})^{2n}\} \\ &= \frac{1}{2^{2n-1}} \sum_{k=1}^{n} {}_{2n}C_{2k-1} a^{n-(2k-1)} (a^2-4)^{k-1}. \end{aligned}$$

Similarly,

$$\begin{aligned} A_{2n+1} &= \frac{1}{2^{2n+1}\sqrt{a^2-4}}\{(a+\sqrt{a^2-4})^{2n+1} - (a-\sqrt{a^2-4})^{2n+1}\} \\ &= \frac{1}{2^{2n}} \sum_{k=1}^{n+1} {}_{2n+1}C_{2k-1} a^{n-(2k-1)} (a^2-4)^{k-1}. \end{aligned}$$

Combing above two equations, we get the desired result. □

The following Proposition gives a nice description of the set of positive imaginary roots of square length $-2k$.

Proposition 2.4 ([5]).

$$\Delta^{im}_{+,k} = \{(m,n),\ (n,m),\ (mA_{j+1} - nA_j, mA_{j+2} - nA_{j+1}),$$
$$(mA_{j+2} - nA_{j+1}, mA_{j+1} - nA_j),\ (nA_{j+1} - mA_j, nA_{j+2} - mA_{j+1}),$$

$$(nA_{j+2} - mA_{j+1}, nA_{j+1} - mA_j) \,|\, (m,n) \in \Omega_k\}.$$

where

$$\Omega_k = \left\{(m,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} | \frac{2\sqrt{k}}{\sqrt{a^2-4}} \leq m \leq \sqrt{\frac{k}{a-2}}, n = \frac{am - \sqrt{(a^2-4)m^2 - 4k}}{2}\right\}.$$

We know any positive integer k can be represented as a product of two integers t and m^2 ($m \in \mathbb{Z}$), where t is a square free integer. We have the following theorems.

Theorem 2.5. *Let $x^2 - axy + y^2 = -k$ be the hyperbola and let $k = tm^2$ be any positive integer where t is a square free integer and $m \in \mathbb{Z}_{>0}$. If $(m,n) \in \Omega_k$ for some positive integer n, then $a - 2 \leq t \leq \frac{a^2-4}{4}$ for $a \geq 3$.*

Proof. If $(m,n) \in \Omega_k$ for some positive integer n, then $\frac{2\sqrt{k}}{\sqrt{a^2-4}} \leq m \leq \sqrt{\frac{k}{a-2}}$. Since $k = tm^2$, we get $\frac{2\sqrt{t}}{\sqrt{a^2-4}} \leq 1$ and $\sqrt{\frac{t}{a-2}} \geq 1$, we have the desired result. □

Theorem 2.6. *Let $k = tm^2$ be any positive integer, where t is any square free integer. Then $(m,n) \in \Omega_k$ for some positive integer n if and only if $t = j(a-j) - 1$, with $j = 1, \ldots, [\frac{a}{2}]$, where $[x]$ denotes the smallest integer $\geq x$. Furthermore,*

$$n = jm \quad \textit{for each } j = 1, \ldots, \left[\frac{a}{2}\right].$$

Proof. Theorem 2.5 shows that $a^2 - 4 - 4t \geq 0$ and $a - 2 \leq t$. Hence

$$\begin{aligned}
\sqrt{(a^2-4)m^2 - 4k} &= \sqrt{(a^2-4)m^2 - 4tm^2} \\
&= \sqrt{a^2 - 4 - 4t}\, m \\
&\leq \sqrt{(a^2-4) - 4(a-2)}\, m \\
&= (a-2)m.
\end{aligned}$$

On the other hand, for $a \geq 3$,

$$\sqrt{(a^2-4) - 4t} \in \mathbb{Z} \text{ if and only if } \sqrt{(a^2-4) - 4t} = a - 2j$$

for $j = 1, \ldots, [\frac{a}{2}]$. Thus

$$t = j(a-j) - 1, \quad \text{where } j = 1, \ldots, [\frac{a}{2}].$$

Therefore,

$$n = \left(\frac{a - \sqrt{a^2-4-4t}}{2}\right)m = \left(\frac{a - (a-2j)}{2}\right)m = jm,$$

we are done. □

Example 2.7. Let $x^2 - 7xy + y^2 = -25t$ be the hyperbola where t is a square free integer. In case of $t = 5, 10$ and 11, we have $(5,5)$, $(5,10)$, and $(5,15)$ are in Ω_{25t}, respectively. In this case, There are infinitely many integral point on the hyperbola $x^2 - 7xy + y^2 = -25t$. In particular, $\{w(5,5) \mid w \in W\}$ is the set of all integral points on the hyperbola $x^2 - 7xy + y^2 = -75$.

Theorem 2.8. *If* $\sqrt{\frac{k}{a-2}}$ $(k \in \mathbb{Z} \geq 0)$ *is a positive integer, then there are infinitely many integral points on the hyperbola* $x^2 - axy + y^2 = -k$. *Equivalently, there are infinitely many imaginary roots of square length* $-2k$.

Proof. Let $\sqrt{\frac{k}{a-2}} = n \in \mathbb{Z}_{>0}$. Clearly, the hyperbola $x^2 - axy + y^2 = -k$ meets with the line $y = x$ at only one point (n, n). If n is a positive integer, then $k = (a-2)n^2$ and hence (n, n) is an integral point on the hyperbola $x^2 - axy + y^2 = -k$. This implies that $\alpha = n\alpha_0 + n\alpha_1$ is an imaginary root with square length $-2k$. Since $\Delta^{im}_{+,k}$ is W-invariant, $\{w\alpha \mid w \in W\} \subset \Delta^{im}_{+,k}$ and hence $\Delta^{im}_{+,k}$ is infinite. □

Example 2.9. There are infinitely many integral points on the hyperbola $x^2 - 4xy + y^2 = -8$. Clearly, $\{w(2,2) \mid w \in W\}$ is a set of integral points on that hyperbola.

Example 2.10. There is no integral point on the hyperbola $x^2 - 3xy + y^2 = -2$.

3. Relations between the Integral Points on $\mathfrak{g}(A)$

In this section, we study the relation among the integral points on the hyperbolas $x^2 - axy + y^2 = 1$ and $x^2 - axy + y^2 = -k$. We give several relations between the integral points on $\mathfrak{g}(A)$.

Theorem 3.1. *Let* $\{A_n\}$ *be a sequence defined in* (3). *Then*

(a) $A_{j+k} = A_{j+1}A_k - A_jA_{k-1}$ *for* $j,\ k-1 \in \mathbb{Z}_{\geq 0}$.
(b) $((A_j, A_{j+1}) \mid (A_{j+k}, A_{j+k+1})) = A_{k+1} - A_{k-1}$ *for* $j,\ k-1 \in \mathbb{Z}_{\geq 0}$.

Proof. (a) Proceeding by induction on n, suppose that we have

$$A_{j+k} = A_{j+1}A_k - A_jA_{k-1} \text{ for } j,\ k-1 \in \mathbb{Z}_{\geq 0}.$$

Since,

$$A_k{}^2 - aA_{k-1}A_k + A_{k-1}{}^2 = 1 \quad \text{for } k \in \mathbb{Z}_{\geq 1},$$

we have

$$\begin{aligned} A_{j+k+1} &= aA_{j+k} - A_{j+k-1} \\ &= a(A_kA_{j+1} - A_{k-1}A_j) - (A_{k-1}A_{j+1} - A_{k-2}A_j) \\ &= (aA_k - A_{k-1})A_{j+1} - (aA_{k-1}A_{k-2})A_j \\ &= A_{k+1}A_{j+1} - A_kA_j. \end{aligned}$$

For (b),

$$\begin{aligned} &((A_j, A_{j+1}) \,|\, (A_{j+k}, A_{j+k+1})) \\ &= (-A_{j+1}^2 + A_j^2)aA_k + (2A_{j+1} - aA_jA_{j+1})A_{k+1} + (aA_jA_{j+1} - 2A_j^2)A_{k-1} \\ &= (-A_{j+1}^2 + A_j^2)(A_{k+1} + A_{k-1}) + (2A_{j+1} - aA_jA_{j+1})A_{k+1} + (aA_jA_{j+1} - 2A_j^2)A_{k-1} \\ &= (A_{j+1}^2 - aA_jA_{j+1} + A_j^2)A_{k+1} - (A_{j+1}^2 - aA_jA_{j+1} + A_j^2)A_{k-1} \\ &= A_{k+1} - A_{k-1}. \end{aligned}$$

□

Corollary 3.2. *Let* $\{A_n\}$ *be a sequence defined in* (4). *Then we have:*

$$((A_n, A_{n+1}) \,|\, (A_{n+1}, A_{n+2})) = a \text{ for all } j \in \mathbb{Z}_{\geq 0}.$$

Since $F_{2n} = A_n$, and $F_{2k+2} = F_{2k} + F_{2k+1}$, we have the following Corollary.

Corollary 3.3. *Let* $\{F_n\}$ *be a Fibonacci sequence defined in* (4). *Then*

(a) $F_{2j+k} = F_{2j}F_{k-1} + F_{2j+1}F_k$.

(b) $((F_{2j}, F_{2j+2}) \,|\, (F_{2j+2k}, F_{2j+2k+2})) = F_{2k+1} + F_{2k-1}$.

(c) $((F_{2j}, F_{2j+2}) \,|\, (F_{2j+2}, F_{2j+4})) = 3$ *for all* $j \in \mathbb{Z}_{\geq 0}$,

(d) $((F_{2j+1}, F_{2j+3}) \,|\, (F_{2j+3}, F_{2j+5})) = -3$ *for all* $j \in \mathbb{Z}_{\geq 0}$.

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