A DIFFERENCE SET METHOD FOR CIRCULANT DECOMPOSITIONS OF COMPLETE PARTITE GRAPHS INTO GREGARIOUS 4-CYCLES

Eun Kyung Kim, Young Min Cho, and Jung Rae Cho

ABSTRACT. The complete multipartite graph $K_{n(m)}$ with $n \geq 4$ partite sets of size m is shown to have a decomposition into 4-cycles in such a way that vertices of each cycle belong to distinct partite sets of $K_{n(m)}$, if 4 divides the number of edges. Such cycles are called *gregarious*, and were introduced by Billington and Hoffman ([2]) and redefined in [3]. We independently came up with the result of [3] by using a difference set method, and improved the result so that the composition is circulant, in the sense that it is invariant under the cyclic permutation of partite sets. The composition is then used to construct gregarious 4-cycle decompositions when one partite set of the graph has different cardinality than that of others. Some results on joins of decomposable complete multipartite graphs are also presented.

1. Introduction

Edge disjoint decompositions of graphs into cycles has been considered in a number of ways. Necessary and sufficient conditions for a complete graph of odd order or a complete graph of even order minus an 1-factor to have a decomposition into cycles of some fixed length are known (see [1], [9] and [10] as well as their references). Some authors have also considered cycle decompositions with special properties such as resolvable cycle decompositions (see [4], [8], [7]). Although much work has gone into the decomposition of complete graphs into cycles, less attention has been paid to the same problem for complete multipartite graphs. Sotteau [11] showed that the complete bipartite graph $K_{m,n}$ has a decomposition into $2k$-cycles if and only if m, n are even and $2k$ divides mn. Note that the two obvious conditions turned out to be sufficient conditions. This result has been used as a powerful tool by later authors for various decompositions of multipartite graphs into cycles, as was done in [5].

Billington and Hoffman [2] introduced the notion of a gregarious cycle, meaning a cycle having at least one vertex from each partite set, and then changed

the definition slightly in [3]. There, a *gregarious cycle* is a cycle having at most one vertex from any given partite set. They gave a necessary and sufficient condition for an equipartite graph to have a decomposition into gregarious 4-cycles. An equipartite graph is a complete multipartite graph all of whose partite sets have the same size. They also considered complete multipartite graphs with one partite set of different size.

Throughout the paper, $K(A_1, A_2, \ldots, A_n)$ will denote the complete multipartite graph with partite sets $A_1, A_2, \ldots, A_n$, which we sometimes denote by $K(m_1, m_2, \ldots, m_n)$ when $|A_i| = m_i$ for $i = 1, 2, \ldots, n$. When all m_i are equal to m, we use the notation $K_{n(m)}$. This notation is standard and can be found in [6] amongst other places.

In this paper, we use the new definition of gregarious cycles. The purpose of this paper is, using difference sets, to show that a complete graph K_n or an equipartite graph $K_{n(m)}$ has a decompositions into gregarious 4-cycles if n is at least 4 and the number of edges is divisible by 4. Note that the sufficient condition is an obvious necessary condition. Improving a result in [3], we show that the decomposition can be circulant if $n \geq 5$, in the sense that the decomposition is invariant under the cyclic permutations of the partite sets. The decompositions of graphs obtained so far by other authors heavily relied on successively taking *joins* of graphs, and so the decompositions were not circulant at all. Some of our results overlaps with some results in [3], but we want make it clear that the results are obtained independently and use a different method from theirs.

The following is informally mentioned in many papers, and formally proved in [5] when the graph is decomposable into cycles of a fixed even length. However, it also applies when graph is decomposable into cycles of arbitrary lengths.

Lemma 1.1. *If $K(m_1, m_2, \ldots, m_n)$ is decomposable into cycles, then all m_1, $m_2, \ldots, m_n$ have the same parity, and furthermore n must be odd if the parity of each m_i is odd.*

Proof. For any i, the degree of any vertex in A_i is $\sum_{k \neq i} m_k$, which must be even. Thus, for any pair i and j, $m_j - m_i = \sum_{k \neq i} m_k - \sum_{k \neq j} m_k$ is also even. Thus, all $m_1, m_2, \ldots, m_n$ have the same parity. If this parity is odd, since $\sum_{k=1}^{n-1} m_k$ is even, $n - 1$ must be even, i.e., n is odd. $\square$

For simplicity, we will call a gregarious 4-cycle a γ_4-cycle. A graph will be called γ_4-*decomposable* if it is decomposable into γ_4-cycles, and a decomposition into γ_4-cycles will be called a γ_4-*decomposition*.

The following lemma is proved in [5] for decompositions into arbitrary (non-necessarily gregarious) cycles, by the standard "expanding points method". However, exactly the same method can be applied for decompositions into gregarious cycles.

Lemma 1.2. *If $K(m_1, m_2, \ldots, m_n)$ is decomposable into gregarious k-cycles for an even integer k, then so is $K(m_1 t, m_2 t, \ldots, m_n t)$ for every integer $t \geq 1$.*

Put $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}\}$ if n is odd and $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm \frac{n-2}{2}, \frac{n}{2}\}$ if n is even. Then, $\mathcal{D}_n$ is a complete set of differences of pairs of distinct integers in $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$.

A sequence (r_1, r_2, r_3, r_4) of differences in $\mathcal{D}_n$ is called a *feasible sequence*, or an *f-sequence* for simplicity, if $r_1 + r_2 + r_3 + r_4 = 0$ and $r_i + r_{i+1} \neq 0$ for $i = 1, 2, 3$, where the arithmetic is done in $\mathbb{Z}_n$ The *sequence of initial sums*, or the *s-sequence* for short, corresponding to an *f*-sequence $\rho = (r_1, r_2, r_3, r_4)$ is the sequence $\sigma_\rho = \langle s_0, s_1, s_2, s_3 \rangle$ of elements in $\mathbb{Z}_n$ such that $s_0 = 0$ and $s_i = \sum_{j=1}^{i} r_j$ for $i = 1, 2, 3$. Note that, $s_i = s_{i-1} + r_i$ for each $i = 1, 2, 3$ and $s_3 + r_4 = s_0$. Here, all the arithmetic is done in $\mathbb{Z}_n$.

Intuitively, an *s*-sequence is the sequence of partite sets of a multipartite graph or the sequence of vertices of a complete graph by which a 4-cycle traverses, and the feasibility of the *f*-sequence guarantees that the 4-cycle is proper and gregarious.

In the next section, we give circulant γ_4-decompositions of equipartite graphs with odd partite sets. In Section 3, we show that $K_{n(2t)}$ has a γ_4-decomposition for every $n \geq 4$ and $t \geq 1$, which is circulant if $n \geq 5$. In Section 4, we study γ_4-decomposition of $K_{h,n(m)}$ when $h \neq m$. In Section 5, we consider joins of γ_4-decomposable graphs.

Before we continue to next section, we mention the following lemma.

Lemma 1.3. *If $K(m_1, m_2, m_3, m_4)$ is γ_4-decomposable, then $m_1 = m_2 = m_3 = m_4$ and this number is an even integer.*

Proof. Let A_i be the partite sets with $|A_i| = m_i$ for $i = 1, 2, 3, 4$. Without loss of generality, we may assume that $m_1 \geq m_2 \geq m_3 \geq m_4$. Suppose $\mathcal{C}$ is a γ_4-decomposition of $K(m_1, m_2, m_3, m_4)$. Let a, b, c be the numbers of gregarious 4-cycles in $\mathcal{C}$ traversing the partite sets by the orders (A_1, A_2, A_3, A_4), (A_1, A_2, A_4, A_3), and (A_1, A_4, A_2, A_3), respectively. Counting edges used by each type of gregarious 4-cycle, we have equations

$$m_1 m_2 = a + b = m_3 m_4, \quad m_1 m_3 = b + c = m_2 m_4, \quad m_1 m_4 = a + c = m_2 m_3.$$

However, since $m_1 \geq m_2 \geq m_3 \geq m_4$, the first equation implies $m_1 = m_2 = m_3 = m_4$. Lemma 1.1 says that this number must be even. $\square$

2. Equipartite graphs with partite sets of odd size

Let $\tau : \mathbb{Z}_n \to \mathbb{Z}_n$ be the mapping defined by $\tau(i) = i+1$ for all i in $\mathbb{Z}_n$, i.e., the circulant permutation $(0, 1, 2, \ldots, n-1)$ on $\mathbb{Z}_n$. We can extend τ to a mapping $\tau_* : \mathbb{Z}^4 \to \mathbb{Z}^4$ by defining $\tau_* \langle s_0, s_1, s_2, s_3 \rangle = \langle \tau(s_0), \tau(s_1), \tau(s_2), \tau(s_3) \rangle$. Then,

$\tau_*^j \langle s_0, s_1, s_2, s_3 \rangle = \langle s_0 + j, s_1 + j, s_2 + j, s_3 + j \rangle$ for each $j = 0, 1, \ldots, n-1$, and τ_*^n is the identity mapping. By convention, τ_*^0 is also the identity mapping.

Lemma 2.1. *(i) K_n is decomposable into 4-cycles if and only if $n \equiv 1 \pmod 8$.*
(ii) If m is odd and $K_{n(m)}$ is decomposable into 4-cycles, then $n \equiv 1$ (mod 8).

Proof. (i) is obtained from [1]. For (ii), note that n is odd by Lemma 1, and 4 must divide the number $\frac{n(n-1)m^2}{2}$ of edges in $K_{n(m)}$. $\qquad\square$

Note that cycle decompositions of K_n obtained by the method in [1] are not circulant at all. Here, we want to give a circulant decomposition.

Let $n \equiv 1 \pmod 8$, say $n = 8k+1$. Then $\mathcal{D}_n = \{\pm1, \pm2, \ldots, \pm4k\}$. We use $\mathbb{Z}_n$ as the vertex set of K_n as well. An edge ij of K_n will be called *an edge of distance d* if $|i - j| = d$ for $d = 1, 2, \ldots, 4k$.

Consider the f-sequences $\rho_i = (4i+1, -(4i+2), -(4i+3), 4i+4)$ for $i = 0, 1, \ldots, k-1$. For each i, the corresponding s-sequence is $\sigma_{\rho_i} = \langle 0, 4i+1, -1, -(4i+4) \rangle$, and the set

$C_i = \{\tau_*^j(\sigma_{\rho_i}) \mid 0 \le j \le n-1\} = \{\langle j, 4i+1+j, -1+j, -(4i+4)+j \rangle \mid 0 \le j \le n-1\}$

consists of disjoint 4-cycles and these cycles exhaust all and only edges of distance $4i+1$, $4i+2$, $4i+3$ or $4i+4$. Thus, the 4-cycles in the set $\bigcup_{i=0}^{k-1} C_i$ are disjoint and exhaust all edges of every distance in K_n. Furthermore, this set is clearly invariant under τ_* as each C_i is. By this, we have the following lemma, which extends Lemma 2.1(i) so that the decomposition is circulant.

Lemma 2.2. *Let $n \equiv 1 \pmod 8$, say $n = 8k + 1$. Then, K_n is decomposable into 4-cycles and the set*

$\mathcal{C} = \bigcup_{i=0}^{k-1} C_i = \{\langle j, 4i+1+j, -1+j, -(4i+4)+j \rangle \mid 0 \le i \le k-1,\ 0 \le j \le n-1\}$

is a circulant decomposition of K_n into 4-cycles, in the sense that $\mathcal{C}$ is invariant under τ_*.

If we regard K_n as $K(1, 1, \ldots, 1)$, then $K_{n(m)}$ has a γ_4-decomposition for any integers n with $n \equiv 1 \pmod 8$ and m by Lemmas 1.2 and 2.2. However, we want to be more elaborate with the "expanding points method" in [5] to see that the decomposition can be circulant.

Let $n \equiv 1 \pmod 8$ and m be a positive integer. Expand each vertex i of K_n to the set $A_i = \{i_1, i_2, \ldots, i_m\}$, and let the partite sets of $K_{n(m)}$ be A_i for $i = 0, 1, \ldots, n-1$. We extend the mapping τ to a mapping τ_{**} of 4-cycles of $K_{n(m)}$ by defining

$\tau_{**}\langle a_j, b_k, c_p, d_q \rangle = \langle \tau(a)_j, \tau(b)_k, \tau(c)_p, \tau(d)_q \rangle = \langle (a+1)_j, (b+1)_k, (c+1)_p, (d+1)_q \rangle$

for all j, k, p, q in $\{1, 2, \ldots, m\}$. Let $\mathcal{C}$ be the circulant decomposition of K_n in Lemma 2.2. Then, the set

$$\mathcal{C}^* = \{\langle a_j, b_k, c_j, d_k \rangle \mid \langle a, b, c, d \rangle \in \mathcal{C}, \ 1 \le j \le m, \ 1 \le k \le m\}.$$

is a γ_4-decomposition of $K_{n(m)}$. Since $\mathcal{C}$ is invariant under τ_*, if $\langle a_j, b_k, c_j, d_k \rangle$ is in $\mathcal{C}^*$, then $\tau_{**}\langle a_j, b_k, c_j, d_k \rangle = \langle \tau(a)_j, \tau(b)_k, \tau(c)_j, \tau(d)_k \rangle$ also belongs to $\mathcal{C}^*$. That is, $\mathcal{C}^*$ is invariant under τ_{**}.

Summarizing the above discussion, we have the following lemma.

Lemma 2.3. *Let $n \equiv 1 \pmod 8$ and m be any integer, then $K_{n(m)}$ has a circulant γ_4-decomposition, in the sense that it is invariant under τ_{**}.*

Combining the discussions in this section, we have the following theorem.

Theorem 2.4. *Let m be an odd integer. Then, $K_{n(m)}$ has a circulant γ_4-decomposition, in the sense that it is invariant under τ_{**} if and only if $n \equiv 1 \pmod 8$.*

We remark that Billington and Hoffman ([3]) independently showed Theorem 2.1 using a different method, except that their decomposition is not circulant.

3. Multipartite graphs with partite sets of even size

Let $A_i = \{i, \bar{i}\}$, $i = 0, 1, \ldots, n - 1$, be the partite sets of $K_{n(2)}$. Thus, the elements in $\mathbb{Z}_n$ are used as indices of the partite sets and as vertices of the graph as well. The next lemma, which appears in [4], also serves as an example. However, the decomposition is not circulant.

Lemma 3.1. *$K_{4(2)}$ is γ_4-decomposable.*

Proof. The six 4-cycles

$$\langle 0, 1, 2, 3 \rangle, \ \langle \bar{0}, \bar{1}, \bar{2}, \bar{3} \rangle, \ \langle 0, \bar{1}, \bar{3}, 2 \rangle, \ \langle \bar{0}, 1, 3, \bar{2} \rangle, \ \langle 0, \bar{3}, 1, \bar{2} \rangle, \ \langle \bar{0}, 3, \bar{1}, 2 \rangle$$

constitute a γ_4-decomposition of $K_{4(2)}$. $\qquad\square$

An edge joining a vertex in A_i to a vertex in A_j is called an *edge of distance d* if $i - j \equiv \pm d \pmod n$, where $\pm d$ is in $\mathcal{D}_n$. In particular, if n is even and $d = \frac{n}{2}$ then an edge of distance d is called a *diagonal edge*. For example, the edges $0\bar{4}$, 73, $\bar{7}\,\bar{2}$ and $\bar{8}3$ are all edges of distance 4 in $K_{9(2)}$, while the edges $4\bar{9}$ and $\bar{5}\,0$ are diagonal edges of $K_{10(2)}$. If $\langle a_1, a_2, a_3, a_4 \rangle$ is a 4-cycle, then the edges a_1a_2, a_2a_3, a_3a_4 and a_4a_1 will be called the first, second, third and fourth edges of the cycle, respectively.

Let ϕ^+ and ϕ^- be the mappings of $\mathbb{Z}_n$ into $\bigcup_{i=0}^{n-1} A_i$ defined by $\phi^+(i) = i$ and $\phi^-(i) = \bar{i}$ for all i in $\mathbb{Z}_n$. A *flag* is a sequence $\phi^* = (\phi_0, \phi_1, \phi_2, \phi_3)$ each of whose entries is ϕ^+ or ϕ^-. Given such a flag ϕ^*, we also use the same notation

ϕ^* to denote the mapping defined by $\phi^*\langle a, b, c, d\rangle = \langle \phi_0(a), \phi_1(b), \phi_2(c), \phi_3(d)\rangle$ for every sequence $\langle a, b, c, d\rangle$ of numbers in $\mathbb{Z}_n$. So, $\phi^*\langle a, b, c, d\rangle$ is a γ_4-cycle of $K_{n(2)}$ if a, b, c, d are distinct, and every γ_4-cycle of $K_{n(2)}$ is in this form.

Recall that τ is the cyclic permutation $(0, 1, \ldots, n-1)$ on $\mathbb{Z}_n$. We extend τ to a mapping τ_* on the γ_4-cycles of $K_{n(2)}$ by defining

$$\tau_*(\phi^*\langle a, b, c, d\rangle) = \langle \phi_0(\tau(a)), \phi_1(\tau(b)), \phi_2(\tau(c)), \phi_3(\tau(d))\rangle$$
$$= \langle \phi_0(a+1), \phi_1(b+1), \phi_2(c+1), \phi_3(d+1)\rangle.$$

Suppose we are given an f-sequence $\rho = (r_1, r_2, r_3, r_4)$ and a flag $\phi^* = (\phi_0, \phi_1, \phi_2, \phi_3)$. With the s-sequence $\sigma_\rho = \langle s_0, s_1, s_2, s_3\rangle$ corresponding to ρ, we can generate a class $\{\tau_*^j(\phi^*(\sigma_\rho)) \mid 0 \le j \le n-1\}$ of γ_4-cycles of $K_{n(2)}$. This class is called the *n-class generated by* $\phi^*(\sigma_\rho)$, and $\phi^*(\sigma_\rho)$ is called the *starter cycle* of the class. It is clear that such an n-class is invariant under τ_*. For example, if $(\phi_0, \phi_1, \phi_2, \phi_3) = (\phi^+, \phi^-, \phi^-, \phi^+)$, the γ_4-cycles in the class are as below:

$$
\begin{array}{rcl}
\tau_*^0(\phi^*(\sigma_\rho)) & = & \langle \quad 0, \quad \overline{s_1}, \quad \overline{s_2}, \quad s_3 \ \rangle, \\
\tau_*^1(\phi^*(\sigma_\rho)) & = & \langle \quad 1, \quad \overline{s_1+1}, \overline{s_2+1}, s_3+1 \rangle, \\
& \vdots & \\
\tau_*^j(\phi^*(\sigma_\rho)) & = & \langle \quad j, \quad \overline{s_1+j}, \overline{s_2+j}, s_3+j \rangle, \\
& \vdots & \\
\tau_*^{n-1}(\phi^*(\sigma_\rho)) & = & \langle n-1, \overline{s_1-1}, \overline{s_2-1}, s_3-1 \rangle.
\end{array}
$$

Note that, every column of vertices above has one vertex from each partite set. Thus, the edge $p\,\overline{q}$ appears as the first edge of a γ_4-cycle above if $q - p = s_1 = r_1$. The edge $\overline{p}\,\overline{q}$ appears as the second edge of a γ_4-cycle above if $q - p = s_2 - s_1 = r_2$. Similarly, the edge $\overline{p}\,q$ with $q - p = r_3$ and the edge $p\,q$ with $q - p = r_4$ appear in the above γ_4-cycles.

If $\phi^*(\sigma_\rho)$ has a diagonal edge of the form pq or $\overline{p}\,\overline{q}$ and we generate an n-class with it, the same diagonal edge would appear twice in the n-class. For example, if $n = 10$ and $r_2 = 5$, then the diagonal edge $\overline{1}\,\overline{6}$ appears twice, once in the form $\overline{1}\,\overline{6}$ and once in the form $\overline{6}\,\overline{1}$. Thus, to avoid such double appearances of the same 4-cycle, we need to generate a class $\{\tau_*^j(\phi^*(\sigma_\rho)) \mid j = 0, 1, \ldots, \frac{n-1}{2}\}$ with only $\frac{n}{2}$ γ_4-cycles. Such a class is called an $\frac{n}{2}$-class. When, we need an $\frac{n}{2}$-class, we will find one which is invariant under τ_*.

The above procedure is the basic method we adopt to produce a circulant γ_4-decomposition of $K_{n(2)}$. We note that each of the above classes is invariant under τ_*. As one may noticed already, the main problem then is how to choose f-sequences and flags so that, in the γ_4-cycles produced by them, each edge of distance d appears exactly once for every possible distance d of the graph. We do not have a method to produce all such f-sequences and flags. However, we can present at least one such a pair for every $n \ge 5$.

Since the number of edges of $K_{n(2)}$ is $2n(n-1)$, we need to produce $\frac{n(n-1)}{2}$ disjoint γ_4-cycles. We will produce $\frac{n-1}{2}$ n-classes if n is odd, and one $\frac{n}{2}$-class and $\frac{n-2}{2}$ n-classes if n is even.

Let $\Phi_1^* = (\phi^+, \phi^-, \phi^-, \phi^+)$ and $\Phi_2^* = (\phi^-, \phi^+, \phi^+, \phi^+)$ be two special flags. Suppose i, j are differences in $\mathcal{D}_n$ such that $i \neq j$ and $i + j \not\equiv 0 \pmod{n}$, and put $\eta = (i, j, -i, -j)$. Then η is an f-sequence and $\sigma_\eta = (0, i, i+j, j)$. We generate the following two n-classes from $\Phi_1^*(\sigma_\eta)$ and $\Phi_2^*(\sigma_\eta)$, respectively.

$$
\begin{array}{llll}
\langle\ 0, & \bar{i}, & \overline{i+j}, & \bar{j}\ \rangle, \\
\langle\ 1, & \overline{i+1}, & \overline{i+j+1}, & \overline{j+1}\rangle, \\
\langle\ 2, & \overline{i+2}, & \overline{i+j+2}, & \overline{j+2}\rangle, \\
\multicolumn{4}{c}{\vdots} \\
\langle n-2, & \overline{i-2}, & \overline{i+j-1}, & \overline{j-2}\rangle, \\
\langle n-1, & \overline{i-1}, & \overline{i+j-1}, & \overline{j-1}\rangle.
\end{array}
\qquad
\begin{array}{llll}
\langle\ \bar{0}, & i, & i+j, & j\ \rangle, \\
\langle\ \bar{1}, & i+1, & i+j+1, & j+1\rangle, \\
\langle\ \bar{2}, & i+2, & i+j+2, & j+2\rangle, \\
\multicolumn{4}{c}{\vdots} \\
\langle \overline{n-2}, & i-2, & i+j-2, & j-2\rangle, \\
\langle \overline{n-1}, & i-1, & i+j-1, & j-1\rangle.
\end{array}
$$

Then, each edge of distance i or j appears exactly once in the cycles above. We call these classes the *standard n-classes* produced from η.

We assume $n \geq 5$, and divide the decomposition problem of $K_{n(2)}$ into four cases depending on n modulo 4.

Case (1). Suppose $n = 4k+1$ with $k \geq 1$. Then $\frac{n-1}{2} = 2k$ and $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm 2k\}$. Partition $\mathcal{D}_n$ into k subsets $\{\pm 1, \pm 2\}, \{\pm 3, \pm 4\}, \ldots,$ and $\{\pm(2k-1), \pm 2k\}$, and take f-sequences $\eta_1 = (1, 2, -1, -2)$, $\eta_2 = (3, 4, -3, -4)$, $\ldots$, and $\eta_k = (2k-1, 2k, -(2k-1), -2k)$ for the respective subsets of the partition. Now, we produce two standard n-classes from each η_i, and let $\mathcal{C}$ be the union of all these standard n-classes. Clearly, two distinct f-sequences generates disjoint classes, and so every edge of every distance appears exactly once in the γ_4-cycles of $\mathcal{C}$. Thus, $\mathcal{C}$ is a circulant γ_4-decomposition of $K_{n(2)}$, and is invariant under τ_* as is each n-class.

Note that if $n = 8k+1$ then Lemma 2.3 applies. However, The decomposition obtained here is different from the one obtained by the method in the preceding section.

Example 3.1 Let $n = 4k+1 = 9$. Then $k = 2$ and $D_9 = \{\pm 1, \pm 2, \pm 3, \pm 4\}$. Following the procedure in Case (1), we take $\eta_1 = (1, 2, -1, -2)$ and $\eta_2 = (3, 4, -3, -4)$. Then $\sigma_{\eta_1} = (0, 1, 3, 2)$ and $\sigma_{\eta_2} = (0, 3, 7, 4)$. The circulant γ_4-decomposition of $K_{9(2)}$ produced by the above method is as below:

$$
\begin{array}{llll}
\langle 0, \bar{1}, \bar{3}, \bar{2}\rangle, & \langle \bar{0}, 1, 3, 2\rangle, & \langle 0, \bar{3}, \bar{7}, \bar{4}\rangle, & \langle \bar{0}, 3, 7, 4\rangle, \\
\langle 1, \bar{2}, \bar{4}, \bar{3}\rangle, & \langle \bar{1}, 2, 4, 3\rangle, & \langle 1, \bar{4}, \bar{8}, \bar{5}\rangle, & \langle \bar{1}, 4, 8, 5\rangle, \\
\langle 2, \bar{3}, \bar{5}, \bar{4}\rangle, & \langle \bar{2}, 3, 5, 4\rangle, & \langle 2, \bar{5}, \bar{0}, \bar{6}\rangle, & \langle \bar{2}, 5, 0, 6\rangle, \\
\langle 3, \bar{4}, \bar{6}, \bar{5}\rangle, & \langle \bar{3}, 4, 6, 5\rangle, & \langle 3, \bar{6}, \bar{1}, \bar{7}\rangle, & \langle \bar{3}, 6, 1, 7\rangle, \\
\langle 4, \bar{5}, \bar{7}, \bar{6}\rangle, & \langle \bar{4}, 5, 7, 6\rangle, & \langle 4, \bar{7}, \bar{2}, \bar{8}\rangle, & \langle \bar{4}, 7, 2, 8\rangle, \\
\langle 5, \bar{6}, \bar{8}, \bar{7}\rangle, & \langle \bar{5}, 6, 8, 7\rangle, & \langle 5, \bar{8}, \bar{3}, \bar{0}\rangle, & \langle \bar{5}, 8, 3, 0\rangle, \\
\langle 6, \bar{7}, \bar{0}, \bar{8}\rangle, & \langle \bar{6}, 7, 0, 8\rangle, & \langle 6, \bar{0}, \bar{4}, \bar{1}\rangle, & \langle \bar{6}, 0, 4, 1\rangle, \\
\langle 7, \bar{8}, \bar{1}, \bar{0}\rangle, & \langle \bar{7}, 8, 1, 0\rangle, & \langle 7, \bar{1}, \bar{5}, \bar{2}\rangle, & \langle \bar{7}, 1, 5, 2\rangle, \\
\langle 8, \bar{0}, \bar{2}, \bar{1}\rangle. & \langle \bar{8}, 0, 2, 1\rangle. & \langle 8, \bar{2}, \bar{6}, \bar{3}\rangle. & \langle \bar{8}, 2, 6, 3\rangle.
\end{array}
$$

Case (2). Suppose $n = 4k+3$ with $k \geq 1$. Then we have $\frac{n-1}{2} = 2k+1$ and $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm(2k+1)\}$. With differences $\pm 1, \pm 2$ and ± 3, take f-sequences $\rho_1 = (1,1,1,-3)$, $\rho_2 = (1,3,-2,-2)$, $\rho_3 = (3,2,-3,-2)$, and we obtain the corresponding s-sequences $\sigma_{\rho_1} = \langle 0,1,2,3 \rangle$, $\sigma_{\rho_2} = \langle 0,1,4,2 \rangle$, $\sigma_{\rho_3} = \langle 0,3,5,2 \rangle$. Take flags $\phi_1^* = (\phi^-, \phi^+, \phi^+, \phi^-)$, $\phi_2^* = (\phi^-, \phi^-, \phi^+, \phi^-)$, and $\phi_3^* = (\phi^+, \phi^+, \phi^-, \phi^+)$. Then, the n-classes generated from the pairs $\phi_i^*(\sigma_{\rho_i})$, $i = 1,2,3$, are as below.

$$
\begin{array}{lll}
\langle\ \overline{0},\quad 1,\quad 2,\ \overline{3}\ \rangle, & \langle\ \overline{0},\quad \overline{1},\quad 4,\ \overline{2}\ \rangle, & \langle\ 0,\quad 3,\ \overline{5},\ 2\ \rangle, \\
\langle\ \overline{1},\quad 2,\quad 3,\ \overline{4}\ \rangle, & \langle\ \overline{1},\quad \overline{2},\quad 5,\ \overline{3}\ \rangle, & \langle\ 1,\quad 4,\ \overline{6},\ 3\ \rangle, \\
\langle\ \overline{2},\quad 3,\quad 4,\ \overline{5}\ \rangle, & \langle\ \overline{2},\quad \overline{3},\quad 6,\ \overline{4}\ \rangle, & \langle\ 2,\quad 5,\ \overline{7},\ 4\ \rangle, \\
\qquad\quad \vdots & \qquad\quad \vdots & \qquad\quad \vdots \\
\langle\overline{n-2},\ n-1,\ 0,\ \overline{1}\ \rangle, & \langle\overline{n-2},\ \overline{n-1},\ 2,\ \overline{0}\ \rangle, & \langle n-2,\ 1,\ \overline{3},\ 0\ \rangle, \\
\langle\ \overline{n-1},\quad 0,\quad 1,\ \overline{2}\ \rangle. & \langle\ \overline{n-1},\quad \overline{0},\quad 3,\ \overline{1}\ \rangle. & \langle n-1,\ 2,\ \overline{4},\ 1\ \rangle.
\end{array}
$$

Then, as explained earlier, it can be check that each edge of distance d appears exactly once in the γ_4-cycles above for $d = 1,2,3$.

If $k \geq 2$, $\mathcal{D}_n \setminus \{\pm 1, \pm 2, \pm 3\} = \{\pm 4, \pm 5, \ldots, \pm(2k+1)\}$ and this set is nonempty. Partition this set into $k-1$ subsets $\{\pm 4, \pm 5\}$, $\{\pm 6, \pm 7\}$, $\ldots$, $\{\pm 2k, \pm(2k+1)\}$. With each subset $\{2i, 2i+1\}$ for $i = 2,3,\ldots,k$, take f-sequence $\eta_i = (2i, 2i+1, -2i, -(2i+1))$. As in Case (1), if we generate the two standard n-classes from each of these η_i, then every edge of distance d appears exactly once in the 4-cycles of these $2(k-1)$ standard n-classes, for $d = 4, 5, \ldots, 2k+1$. If $k = 1$, we do not have these cycles.

Let $\mathcal{C}$ be the union of these $2(k-1)$ standard n-classes and the three classes above. Then, $\mathcal{C}$ is a circulant γ_4-decomposition of $K_{n(2)}$, invariant under τ_*.

Case (3). Suppose $n = 4k+2$ with $k \geq 1$. Then $\frac{n}{2} = 2k+1$ and $\mathcal{D}_k = \{\pm 1, \pm 2, \ldots, \pm 2k, 2k+1\}$. With differences $\pm 1, \pm(k+1)$ and $2k+1$, take the f-sequences $\rho_1 = (1, 2k+1, -1, 2k+1)$, $\rho_2 = (1, k+1, -1, -(k+1))$, $\rho_3 = (2k+1, k+1, -1, k+1)$, and obtain s-sequences $\sigma_{\rho_1} = \langle 0, 1, 2k+2, 2k+1 \rangle$, $\sigma_{\rho_2} = \langle 0, 1, k+2, k+1 \rangle$, $\sigma_{\rho_3} = \langle 0, 2k+1, 3k+2, 3k+1 \rangle$. Take flags $\phi_1^* = (\phi^+, \phi^-, \phi^-, \phi^+)$, $\phi_2^* = (\phi^-, \phi^-, \phi^+, \phi^-)$, and $\phi_3^* = (\phi^-, \phi^+, \phi^+, \phi^+)$. From $\phi_1^*(\sigma_{\rho_1})$, we generate the $\frac{n}{2}$-class $C_1 = \{\tau_*^j(\phi_1^*(\sigma_{\rho_1})) \mid j = 0, 1, \ldots, \frac{n}{2}-1\}$. From $\phi_2^*(\sigma_{\rho_2})$ and $\phi_2^*(\sigma_{\rho_2})$, we generate n-classes C_2 and C_3, respectively. We recall that $\frac{n}{2}-1 = 2k$ and $n-1 = 4k+1$. These classes are listed in three columns as below:

$$
\begin{array}{lll}
\langle\ 0,\quad \overline{1},\quad \overline{2k+2}, 2k+1 \rangle, & \langle\ \overline{0},\quad \overline{1},\quad k+2,\ \overline{k+1}\ \rangle, & \langle\ \overline{0},\quad 2k+1, 3k+2, 3k+1 \rangle, \\
\langle\ 1,\quad \overline{2},\quad \overline{2k+3}, 2k+2 \rangle, & \langle\ \overline{1},\quad \overline{2},\quad k+3,\ \overline{k+2}\ \rangle, & \langle\ \overline{1},\quad 2k+2, 3k+3, 3k+2 \rangle, \\
\qquad\quad \vdots & \qquad\quad \vdots & \qquad\quad \vdots \\
\langle 2k, \overline{2k+1},\ \overline{0},\quad 4k+1 \rangle. & \langle\ \overline{2k},\quad \overline{2k+1}, 3k+2, \overline{3k+1} \rangle, & \langle\ \overline{2k},\quad 4k+1,\quad k,\quad k-1\ \rangle, \\
 & \langle \overline{2k+1}, \overline{2k+2}, 3k+3, \overline{3k+2} \rangle, & \langle \overline{2k+1}, 4k+2,\ k+1,\quad k\ \rangle, \\
 & \qquad\quad \vdots & \qquad\quad \vdots \\
 & \langle\ \overline{4k+1},\quad \overline{0},\quad k+1,\quad \overline{k}\ \rangle. & \langle\ \overline{4k+1},\quad 2k,\ 3k+1,\quad 3k\ \rangle.
\end{array}
$$

We have the following observations for diagonal edges. Put $p-q = \frac{n}{2} = 2k+1$.

(i) The diagonal edge $p\,q$ appears as the fourth edge of a cycle in C_1, once for each $p = 0, 1, \ldots, 2k$.

(ii) The diagonal edge $\bar{p}\,\bar{q}$ appears as the second edge of a cycle in C_1, once for each $p = 0, 1, \ldots, 2k$.

(iii) The diagonal edge $\bar{p}\,q$ appears as the first edge of a cycle in C_3, once for each $p = 0, 1, \ldots, 4k, 4k+1$. Note that the edge $p\,\bar{q}$ can be written as $\overline{p'}\,q'$, where $p' = p+2k+1$ and $q' = q+2k+1$.

If $q-p = 1$, then the edge $p\,\bar{q}$ appears exactly once in a cycle in C_1, in the form $p\,\bar{q}$ as the first edge of a cycle for $p = 0, 1, \ldots, 2k$, and in the form $\bar{q}\,p$ as the third edge of a cycle for $p = 2k+1, \ldots, 4k+1$. In a similar way as before, it can be checked that each edge of distance 1 or distance $k+1$ appears exactly once in the cycles of C_2 or C_3. Furthermore, all these classes including the $\frac{n}{2}$-class are invariant under τ_*. In fact, we see that τ_* maps the last cycle of C_1 to the first cycle of C_1, except that the order of the vertices is reversed.

If $k \geq 2$ then $\mathcal{D}_n \setminus \{\pm 1, \pm(k+1), \pm(2k+1)\}$ and this set is nonempty. In this case, we partition this set into $k-1$ subsets $\{\pm\alpha_i, \pm\beta_i\}$ so that $\alpha_i + \beta_i \not\equiv 0$ (mod n) for $i = 1, 2, \ldots, k-1$. This is always possible since we can take α_i and β_i in such a way that $2 \leq \alpha_i \leq 2k$, and $2 \leq \beta_i \leq 2k$. Now, put $\eta_i = (\alpha_i, \beta_i, -\alpha_i, -\beta_i)$ for $i = 1, 2, \ldots, k-1$, and generate $2(k-1)$ standard n-classes from these η_i as in Case (1). Then each edge of distance d appears exactly once in these 4-cycles if d is in the set $\{\alpha_i, \beta_i \mid 1 \leq i \leq k-1\}$, that is, if $d \neq 1, k+1, 2k+1$. We already know that these classes are invariant under τ_*.

Thus, if we let $\mathcal{C}$ be the union of these $2(k-1)$ standard n-classes and the classes C_1, C_2 and C_3 above, then $\mathcal{C}$ is a circulant γ_4-decomposition of $K_{n(2)}$, invariant under τ_*.

Example 3.2. Let $n = 4k+2 = 10$. Then $k = 2$ and $\mathcal{D}_{10} = \{\pm 1, \pm 2, \pm 3, \pm 4, 5\}$. According to Case (3), we take $\rho_1 = (1, 5, -1, 5)$, $\rho_2 = (1, 3, -1, -3)$, $\rho_3 = (5, 3, -1, 3)$ and $\eta_1 = (2, 4, -2, -4)$. Then $\sigma_{\rho_1} = (0, 1, 6, 5)$, $\sigma_{\rho_2} = (0, 1, 4, 3)$, $\sigma_{\rho_3} = (0, 5, 8, 7)$, and $\sigma_{\eta_1} = (0, 2, 6, 4)$. The classes generated by the method in Case (3) are as follows:

$\langle 0, \bar{1}, \bar{6}, 5 \rangle,$	$\langle \bar{0}, \bar{1}, 4, \bar{3} \rangle,$	$\langle \bar{0}, 5, 8, 7 \rangle,$	$\langle 0, \bar{2}, \bar{6}, \bar{4} \rangle,$	$\langle \bar{0}, 2, 6, 4 \rangle,$
$\langle 1, \bar{2}, \bar{7}, 6 \rangle,$	$\langle \bar{1}, \bar{2}, 5, \bar{4} \rangle,$	$\langle \bar{1}, 6, 9, 8 \rangle,$	$\langle 1, \bar{3}, \bar{7}, \bar{5} \rangle,$	$\langle \bar{1}, 3, 7, 5 \rangle,$
$\langle 2, \bar{3}, \bar{8}, 7 \rangle,$	$\langle \bar{2}, \bar{3}, 6, \bar{5} \rangle,$	$\langle \bar{2}, 7, 0, 9 \rangle,$	$\langle 2, \bar{4}, \bar{8}, \bar{6} \rangle,$	$\langle \bar{2}, 4, 8, 6 \rangle,$
$\langle 3, \bar{4}, \bar{9}, 8 \rangle,$	$\langle \bar{3}, \bar{4}, 7, \bar{6} \rangle,$	$\langle \bar{3}, 8, 1, 0 \rangle,$	$\langle 3, \bar{5}, \bar{9}, \bar{7} \rangle,$	$\langle \bar{3}, 5, 9, 7 \rangle,$
$\langle 4, \bar{5}, \bar{0}, 9 \rangle.$	$\langle \bar{4}, \bar{5}, 8, \bar{7} \rangle,$	$\langle \bar{4}, 9, 2, 1 \rangle,$	$\langle 4, \bar{6}, \bar{0}, \bar{8} \rangle,$	$\langle \bar{4}, 6, 0, 8 \rangle,$
	$\langle \bar{5}, \bar{6}, 9, \bar{8} \rangle,$	$\langle \bar{5}, 0, 3, 2 \rangle,$	$\langle 5, \bar{7}, \bar{1}, \bar{9} \rangle,$	$\langle \bar{5}, 7, 1, 9 \rangle,$
	$\langle \bar{6}, \bar{7}, 0, \bar{9} \rangle,$	$\langle \bar{6}, 1, 4, 3 \rangle,$	$\langle 6, \bar{8}, \bar{2}, \bar{0} \rangle,$	$\langle \bar{6}, 8, 2, 0 \rangle,$
	$\langle \bar{7}, \bar{8}, 1, \bar{0} \rangle,$	$\langle \bar{7}, 2, 5, 4 \rangle,$	$\langle 7, \bar{9}, \bar{3}, \bar{1} \rangle,$	$\langle \bar{7}, 9, 3, 1 \rangle,$
	$\langle \bar{8}, \bar{9}, 2, \bar{1} \rangle,$	$\langle \bar{8}, 3, 6, 5 \rangle,$	$\langle 8, \bar{0}, \bar{4}, \bar{2} \rangle,$	$\langle \bar{8}, 0, 4, 2 \rangle,$
	$\langle \bar{9}, \bar{0}, 3, \bar{2} \rangle.$	$\langle \bar{9}, 4, 7, 6 \rangle.$	$\langle 9, \bar{1}, \bar{5}, \bar{3} \rangle.$	$\langle \bar{9}, 1, 5, 3 \rangle.$

Case (4). Suppose $n = 4k$ with $k \geq 2$. Then $\frac{n}{2} = 2k$ and $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm(2k-1), 2k\}$. With differences ± 1, ± 3, $\pm(2k-2)$ and $\pm 2k$, take the f-sequences $\rho_1 = (2k-2, 2k, -(2k-2), 2k)$, $\rho_2 = (1, 3, -1, -3)$, $\rho_3 = (3, 2k-2, 2k-2, 1)$ and $\rho_4 = (2k, -3, -(2k-2), 1)$, and obtain s-sequences $\sigma_{\rho_1} = \langle 0, 2k-2, 4k-2, 2k \rangle$, $\sigma_{\rho_2} = \langle 0, 1, 4, 3 \rangle$, $\sigma_{\rho_3} = \langle 0, 3, 2k+1, 4k-1 \rangle$ and $\sigma_{\rho_4} = \langle 0, 2k, 2k-3, 4k-1 \rangle$. Then, we take flags $\phi_1^* = (\phi^+, \phi^-, \phi^-, \phi^+)$, $\phi_2^* = (\phi^-, \phi^-, \phi^+, \phi^-)$, $\phi_3^* = (\phi^+, \phi^-, \phi^-, \phi^+)$, and $\phi_4^* = (\phi^-, \phi^+, \phi^+, \phi^+)$. From $\phi_1^*(\sigma_{\rho_1})$, we generate the $\frac{n}{2}$-class $\{\tau_*^j(\phi_1^*(\sigma_{\rho_1})) \mid j = 0, 1, \ldots, \frac{n}{2}-1\}$, and generate an n-class from $\phi_i^*(\sigma_{\rho_i})$ for each $i = 2, 3, 4$. As in Case (3), it can be checked that, in the γ_4-cycles of these classes, every edge of distance d appears exactly once for $d = 1, 3, 2k-2, 2k$, and that these classes are invariant under τ_*.

If $k \geq 3$, the set $\mathcal{D}_n \setminus \{\pm 1, \pm 3, \pm(2k-2), 2k\}$ is nonempty. We partition this set into $k-2$ subsets $\{\pm\alpha_i, \pm\beta_i\}$ such that $\alpha_i + \beta_i \not\equiv 0 \pmod{n}$ for $i = 1, 2, \ldots, k-2$. As in Case (1), generate $2(k-2)$ standard n-classes from $\rho_i = (\alpha_i, \beta_i, -\alpha_i, -\beta_i)$ for $i = 1, 2, \ldots, k-2$. Then, each edge of distance d appears exactly once in the 4-cycles of these classes if $d \neq 1, 3, 2k-2, 2k$. These classes are already invariant under τ_*.

If we let $\mathcal{C}$ be the union of these $2(k-2)$ standard n-classes and the 4 classes above, $\mathcal{C}$ is a circulant γ_4-decomposition of $K_{n(2)}$, invariant under τ_*.

Example 3.3. Let $n = 4k = 8$. Then $k = 2$ and $\mathcal{D}_8 = \{\pm 1, \pm 2, \pm 3, 4\}$. By Case (4), we have $\rho_1 = (2, 4, -2, 4)$, $\rho_2 = (1, 3, -1, -3)$, $\rho_3 = (3, 2, 2, 1)$ and $\rho_4 = (4, -3, -2, 1)$, and $\sigma_{\rho_1} = (0, 2, 6, 4)$, $\sigma_{\rho_2} = (0, 1, 4, 3)$, $\sigma_{\rho_3} = (0, 3, 5, 7)$, and $\sigma_{\rho_4} = (0, 4, 1, 7)$. The classes generated by the method in Case (4) are as follows:

$\langle 0, \bar{2}, \bar{6}, 4 \rangle$,	$\langle \bar{0}, \bar{1}, 4, \bar{3} \rangle$,	$\langle 0, \bar{3}, \bar{5}, 7 \rangle$,	$\langle \bar{0}, 4, 1, 7 \rangle$,
$\langle 1, \bar{3}, \bar{7}, 5 \rangle$,	$\langle \bar{1}, \bar{2}, 5, \bar{4} \rangle$,	$\langle 1, \bar{4}, \bar{6}, 0 \rangle$,	$\langle \bar{1}, 5, 2, 0 \rangle$,
$\langle 2, \bar{4}, \bar{0}, 6 \rangle$,	$\langle \bar{2}, \bar{3}, 6, \bar{5} \rangle$,	$\langle 2, \bar{5}, \bar{7}, 1 \rangle$,	$\langle \bar{2}, 6, 3, 1 \rangle$,
$\langle 3, \bar{5}, \bar{1}, 7 \rangle$.	$\langle \bar{3}, \bar{4}, 7, \bar{6} \rangle$,	$\langle 3, \bar{6}, \bar{0}, 2 \rangle$,	$\langle \bar{3}, 7, 4, 2 \rangle$,
	$\langle \bar{4}, \bar{5}, 0, \bar{7} \rangle$,	$\langle 4, \bar{7}, \bar{1}, 3 \rangle$,	$\langle \bar{4}, 0, 5, 3 \rangle$,
	$\langle \bar{5}, \bar{6}, 1, \bar{0} \rangle$,	$\langle 5, \bar{0}, \bar{2}, 4 \rangle$,	$\langle \bar{5}, 1, 6, 4 \rangle$,
	$\langle \bar{6}, \bar{7}, 2, \bar{1} \rangle$,	$\langle 6, \bar{1}, \bar{3}, 5 \rangle$,	$\langle \bar{6}, 2, 7, 5 \rangle$,
	$\langle \bar{7}, \bar{0}, 3, \bar{2} \rangle$.	$\langle 7, \bar{2}, \bar{4}, 6 \rangle$.	$\langle \bar{7}, 3, 0, 6 \rangle$.

Summarizing the discussion in this section and combining with Lemma 3.1, we have the following lemma.

Lemma 3.2. *For every integer $n \geq 4$, $K_{n(2)}$ has a γ_4-decomposition, and the decomposition can be circulant if $n \geq 5$, in the sense that it is invariant under τ_*.*

Let $n \geq 5$ and t be any positive integer. For each $i = 1, 2, \ldots, t$, we blow up the partite set $A_i = \{i, \bar{i}\}$ of $K_{n(2)}$ to $A_i' = \{i_1, i_2, \ldots, i_t, \bar{i}_1, \bar{i}_2, \ldots, \bar{i}_t\}$, and use these sets as the partite sets of $K_{n(2t)}$. A gregarious 4-cycle of $K_{n(2t)}$ is then of the form $\langle \phi_0(a)_j, \phi_1(b)_k, \phi_2(c)_j, \phi_3(d)_k \rangle$ for some gregarious 4-cycle $\langle \phi_0(a), \phi_1(b), \phi_2(c), \phi_3(d) \rangle$ of $K_{n(2)}$ and j, k in $\{1, 2, \ldots, t\}$. We extend the

cyclic permutation $\tau = (0, 1, 2, \ldots, n-1)$ on $\mathbb{Z}_n$ to a mapping τ_{**} of 4-cycles of $K_{n(2t)}$ by defining

$$\tau_{**}\langle \phi_0(a)_j, \phi_1(b)_k, \phi_2(c)_j, \phi_3(d)_k \rangle = \langle \phi_0(\tau(a))_j, \phi_1(\tau(b))_k, \phi_2(\tau(c))_j, \phi_3(\tau(d))_k \rangle$$
$$= \langle \phi_0(a{+}1)_j, \phi_1(b{+}1)_k, \phi_2(c{+}1)_j, \phi_3(d{+}1)_k \rangle.$$

Let $\mathcal{C}$ be the circulant γ_4-decomposition of $K_{n(2)}$ constructed earlier in this section. We will apply the the standard "expanding points method" in [5] to $\mathcal{C}$ to produce a γ_4-decomposition $\mathcal{C}^*$ of $K_{n(2t)}$.

First, suppose n is odd. Then, from Case (1) and Case (2), $\mathcal{C}$ has $\frac{n-1}{2}$ n-classes $C^{(1)}, C^{(2)}, \ldots, C^{(\frac{n-1}{2})}$ of 4-cycles such that $\mathcal{C} = \bigcup_{i=1}^{\frac{n-1}{2}} C^{(i)}$. If $C^{(i)}$ is the n-class generated by the 4-cycle $\langle \phi_0(s_0), \phi_1(s_1), \phi_2(s_2), \phi_3(s_3) \rangle$ of $K_{n(2)}$, then, for $j = 1, 2, \ldots, t$ and $k = 1, 2, \ldots, t$,

$$C_{jk}^{(i)} = \{ \tau_{**}^r \langle \phi_0(s_0)_j, \phi_1(s_1)_k, \phi_2(s_2)_j, \phi_3(s_3)_k \rangle \mid 0 \le r \le n{-}1 \}$$
$$= \{ \langle \phi_0(s_0{+}r)_j, \phi_1(s_1{+}r)_k, \phi_2(s_2{+}r)_j, \phi_3(s_3{+}r)_k \rangle \mid 0 \le r \le n{-}1 \}$$

is the n-class of $K_{n(2t)}$ generated by the 4-cycle $\langle \phi_0(s_0)_j, \phi_1(s_1)_k, \phi_2(s_2)_j, \phi_3(s_3)_k \rangle$. Note that this class is invariant under τ_{**} as $C^{(i)}$ is invariant under τ_*. Thus, if we let

$$\mathcal{C}^* = \bigcup \{ C_{jk}^{(i)} \mid 0 \le r \le n{-}1,\ 1 \le j \le t,\ 1 \le k \le t \},$$

then $\mathcal{C}^*$ is a γ_4-decomposition of $K_{n(2t)}$ and invariant under τ_{**}.

Now, suppose n is even. Then, from Case (3) and Case (4), $\mathcal{C}$ has one $\frac{n}{2}$-class $C^{(1)}$ and $\frac{n-2}{2}$ n-classes $C^{(2)}, \ldots, C^{(\frac{n}{2})}$ of 4-cycles such that $\mathcal{C} = \bigcup_{i=1}^{\frac{n}{2}} C^{(i)}$. We proceed similarly to the case when n is odd, and we can show that $K_{n(2t)}$ has a γ_4-decomposition invariant under τ_{**}.

Combining this with Lemmas 1.2 and 3.2, we have the following theorem.

Theorem 3.3. *For every integer $n \ge 4$ and $t \ge 1$, $K_{n(2t)}$ has a γ_4-decomposition, and the decomposition can be circulant if $n \ge 5$, in the sense that it is invariant under τ_{**}.*

The notations preceding Theorem 3.1 will be needed in the next section.

4. Multipartite graphs with one partite set of different size

Let $K_{h,n(m)}$ denote $K(h, m, m, \ldots, m)$ with m repeated n times. We assume h is even, and then m is also even by Lemma 1.1. So let $m = 2t$. In the light of Theorem 3.1, we may assume that $h \ne m$, and then we may assume $n \ge 4$ due to Lemma 1.3.

Let the partite sets of $K_{h,n(2t)}$ be $A_i = \{ i_1, i_2, \ldots, i_t, \bar{i}_1, \bar{i}_2, \ldots, \bar{i}_t \}$ for $i = 0, 1, \ldots, n-1$ and $B = \{ a_1, b_1, a_2, b_2, \ldots, a_{h/2}, b_{h/2} \}$. An edge of $K_{h,n(2t)}$ is called an α-*edge* if it joins two vertices in $\bigcup_{i=0}^{n-1} A_i$, and a β-*edge* if it involves a vertex in B. A gregarious 4-cycle of $K_{h,n(2t)}$ is said to be of type I if it involves α-edges only, and of type II if it involves β-edges.

The graph $K_{h,n(2t)}$ has $\frac{n(n-1)(2t)^2}{2} = 2n(n-1)t^2$ α-edges and $2hnt$ β-edges. Note that a 4-cycle of type II consists of two α-edges and two β-edges. Thus, for $K_{h,n(2t)}$ to have a γ_4-decomposition, we need to have $2hnt \leq 2n(n-1)t^2$, and so $h \leq (n-1)t$. Regard $K_{n(2t)}$ as a subgraph of $K_{h,n(2t)}$. Thus, a 4-cycle in $K_{n(2t)}$ is a 4-cycle of type II in $K_{h,n(2t)}$.

Suppose $n \geq 5$ and let h is odd such that $h \leq (n-1)t$. Assume n is odd. Let $\mathcal{C}^*$ be the γ_4-decomposition of $K_{n(2t)}$ obtained from $\mathcal{C}$ as in the preceding section. Using notations there, we have

$$\mathcal{C}^* = \bigcup \{C_{jk}^{(i)} \mid 1 \leq i \leq \tfrac{n-1}{2}, 1 \leq j \leq t, 1 \leq k \leq t\}.$$

Let $E = \{(i,k) \mid 1 \leq i \leq \tfrac{n-1}{2}, 1 \leq k \leq t\}$, and let F be a subset of E with $\frac{h}{2}$ elements. This is possible since $\frac{h}{2} \leq \frac{(n-1)t}{2} = |E|$. Since $|B| = h$, we can partition B into $\frac{h}{2}$ subsets $\{a_{(i,k)}, b_{(i,k)}\}$ for $(i,k) \in F$, using elements of F as indices for the vertices in B. For each $(i,k) \in F$ and $j = 1, 2, \ldots, t$, assume

$$C_{jk}^{(i)} = \{\langle \phi_0(s_0+r)_j, \phi_1(s_1+r)_k, \phi_2(s_2+r)_j, \phi_3(s_3+r)_k \rangle \mid 0 \leq r \leq n-1\},$$

and split each 4-cycle in $C_{jk}^{(i)}$ into two paths of length 2 and join the end vertices of the paths to $a_{(i,k)}$ and $b_{(i,k)}$, to produce two 4-cycles

$$\langle a_{(i,k)}, \phi_0(s_0+r)_j, \phi_1(s_1+r)_k, \phi_2(s_2+r)_j \rangle, \quad \langle b_{(i,k)}, \phi_2(s_2+r)_j, \phi_3(s_3+r)_k, \phi_0(s_0+r)_j \rangle$$

of type II. The following figure depicts this process.
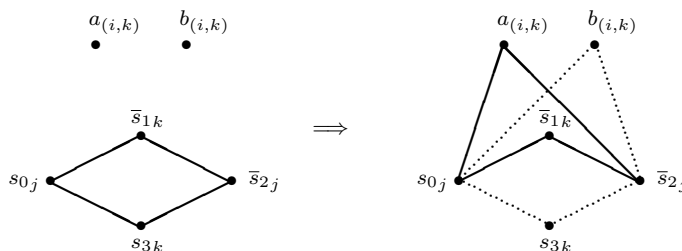
Let $C_{jk}^{(i)*}$ be the set of these 4-cycles of type II obtained from 4-cycles in $C_{jk}^{(i)}$ and put $\mathcal{F} = \bigcup \{C_{jk}^{(i)*} \mid (i,k) \in F, 1 \leq j \leq t\}$. By the edges of cycles in $\mathcal{F}$, the vertex $a_{(i,k)}$ is joined to each vertex of the set

$$N_{a_{(i,k)}} = \{\phi_0(r)_j, \phi_2(r)_j \mid 0 \leq r \leq n-1, 1 \leq j \leq t\}.$$

Note that, at the constructions of γ_4-cycles in Section 3, we always have $(\phi_0, \phi_2) = (\phi^+, \phi^-)$ or $(\phi_0, \phi_2) = (\phi^-, \phi^+)$. Thus, $N_{a_{(i,k)}} = \{r_j, \bar{r}_j \mid r = 0, 1, \ldots, n-1, j = 1, 2, \ldots, t\} = \bigcup_{r=0}^{n-1} A_r$. That is, every α-edge involving $a_{(i,k)}$ is used by 4-cycles in $\mathcal{F}$. The same is true for the α-edges involving $b_{(i,k)}$. Note that, we keep the 4-cycles in $C_{jk}^{(i)}$ untouched if $(i,k) \notin F$. Thus, the class

$$\mathcal{C}^{**} = \left(\bigcup \{C_{jk}^{(i)} \mid (i,k) \in E \setminus F, 1 \leq j \leq t\}\right) \bigcup \mathcal{F}$$

is a γ_4-decomposition of $K_{h,n(2t)}$.

If n is even, it gets more complicated. In this case, we have

$$\mathcal{C}^* = \bigcup \{C_{jk}^{(i)} \mid 1 \leq i \leq \tfrac{n}{2}, \ 1 \leq j \leq t, \ 1 \leq k \leq t\},$$

where $C_{jk}^{(1)}$ are n-classes and $C_{jk}^{(i)}$ are $\tfrac{n}{2}$-classes for $i = 2, 3, \ldots, \tfrac{n}{2}$. Put $E = \{(i,k) \mid 2 \leq i \leq \tfrac{n}{2}, \ 1 \leq k \leq t\}$. Recall that $\tfrac{h}{2} \leq \tfrac{(n-1)t}{2}$. Assume $\tfrac{h}{2} \leq \tfrac{(n-2)t}{2}$. Let F be a subset of E with $\tfrac{h}{2}$ elements, and repeat the above procedure except that we put

$$\mathcal{C}^{**} = \left(\bigcup \{C_{jk}^{(i)} \mid (i,k) \in E \backslash F, \ 1 \leq j \leq t\}\right) \bigcup \left(\bigcup \{C_{jk}^{(1)} \mid 1 \leq j \leq t, \ 1 \leq k \leq t\}\right) \bigcup \mathcal{F}.$$

Then, $\mathcal{C}^{**}$ is a γ_4-decomposition of $K_{h,n(2t)}$.

Summarizing the discussion above, we have the following theorem.

Theorem 4.1. *Suppose $n \geq 5$ and h is even with $h \leq (n-1)t$. If n is odd or n is even with $h \leq (n-2)t$, then $K_{h,n(2t)}$ is γ_4-decomposable.*

Corollary 4.2. *Suppose $n \geq 5$ and h is even with $h \leq n-1$. Then, $K_{ht,n(2t)}$ is γ_4-decomposable for every $t \geq 1$.*

Proof. Since h is even, $h \leq n-1$ implies $h \leq n-2$ if n is even. By the above theorem, $K_{h,n(2)}$ is γ_4-decomposable, and then, by Lemma 1.2, $K_{ht,n(2t)}$ is also γ_4-decomposable for every integer $t \geq 1$. $\qquad\square$

Example 4.1. Consider $K_{4,8(2)}$ with partite sets $B = \{a_1, b_1, a_2, b_2\}$ and $A_i = \{i, \bar{i}\}$ for $i = 0, 1, \ldots, 7$. We have a γ_4-decomposition of $K_{8(2)}$ in Example 3.3. Take the last two n-classes of them and proceed as explained above. Then, we obtain four n-classes of 4-cycles of type II in $K_{4,8(2)}$, listed at the last four columns in the following table. These four n-classes of $K_{4,8(2)}$ together with the $\tfrac{n}{2}$-class and the unused n-class above yields a γ_4-decomposition of $K_{4,8(2)}$ as below.

$\langle 0, \bar{2}, \bar{6}, 4 \rangle,$	$\langle \bar{0}, \bar{1}, 4, \bar{3} \rangle,$	$\langle a_1, 0, \bar{3}, \bar{5} \rangle,$	$\langle b_1, \bar{5}, 7, 0 \rangle,$	$\langle a_2, \bar{0}, 4, 1 \rangle,$	$\langle b_2, 1, 7, \bar{0} \rangle,$
$\langle 1, \bar{3}, \bar{7}, 5 \rangle,$	$\langle \bar{1}, \bar{2}, 5, \bar{4} \rangle,$	$\langle a_1, 1, \bar{4}, \bar{6} \rangle,$	$\langle b_1, \bar{6}, 0, 1 \rangle,$	$\langle a_2, \bar{1}, 5, 2 \rangle,$	$\langle b_2, 2, 0, \bar{1} \rangle,$
$\langle 2, \bar{4}, \bar{0}, 6 \rangle,$	$\langle \bar{2}, \bar{3}, 6, \bar{5} \rangle,$	$\langle a_1, 2, \bar{5}, \bar{7} \rangle,$	$\langle b_1, \bar{7}, 1, 2 \rangle,$	$\langle a_2, \bar{2}, 6, 3 \rangle,$	$\langle b_2, 3, 1, \bar{2} \rangle,$
$\langle 3, \bar{5}, \bar{1}, 7 \rangle.$	$\langle \bar{3}, \bar{4}, 7, \bar{6} \rangle,$	$\langle a_1, 3, \bar{6}, \bar{0} \rangle,$	$\langle b_1, \bar{0}, 2, 3 \rangle,$	$\langle a_2, \bar{3}, 7, 4 \rangle,$	$\langle b_2, 4, 2, \bar{3} \rangle,$
	$\langle \bar{4}, \bar{5}, 0, \bar{7} \rangle,$	$\langle a_1, 4, \bar{7}, \bar{1} \rangle,$	$\langle b_1, \bar{1}, 3, 4 \rangle,$	$\langle a_2, \bar{4}, 0, 5 \rangle,$	$\langle b_2, 5, 3, \bar{4} \rangle,$
	$\langle \bar{5}, \bar{6}, 1, \bar{0} \rangle,$	$\langle a_1, 5, \bar{0}, \bar{2} \rangle,$	$\langle b_1, \bar{2}, 4, 5 \rangle,$	$\langle a_2, \bar{5}, 1, 6 \rangle,$	$\langle b_2, 6, 4, \bar{5} \rangle,$
	$\langle \bar{6}, \bar{7}, 2, \bar{1} \rangle,$	$\langle a_1, 6, \bar{1}, \bar{3} \rangle,$	$\langle b_1, \bar{3}, 5, 6 \rangle,$	$\langle a_2, \bar{6}, 2, 7 \rangle,$	$\langle b_2, 7, 5, \bar{6} \rangle,$
	$\langle \bar{7}, \bar{0}, 3, \bar{2} \rangle.$	$\langle a_1, 7, \bar{2}, \bar{4} \rangle.$	$\langle b_1, \bar{4}, 6, 7 \rangle.$	$\langle a_2, \bar{7}, 3, 0 \rangle.$	$\langle b_2, 0, 6, \bar{7} \rangle.$

We remark that no γ_4-decompositions of graphs in this section are circulant since the graphs are not regular, that is, the graphs have vertices of distinct degrees.

5. Join of γ_4-decomposable graphs

Throughout this section, let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ be pairwise disjoint sets, and put $h_i = |A_i|$ for $i = 1, 2, \ldots, m$ and $k_j = |B_j|$ for $j = 1, 2, \ldots, n$.

Lemma 5.1. *If* $K(A_1, A_2, \ldots, A_n)$ *is* γ_4-*decomposable, then* $h_p \leq \sum_{i \neq p} h_i$ *for each* $p = 1, 2, \ldots, n$.

Proof. There are $h_p \cdot (\sum_{i \neq p} h_i)$ edges which join a vertex in A_p to a vertex in $\bigcup_{i \neq p} A_i$. We say that these edges are of type I$_p$. There are $\sum_{1 \leq i < j \leq m, \, i, j \neq p} h_i h_j$ edges which join vertices in $\bigcup_{i \neq p} A_i$ among themselves. We say that these edges are of type II$_p$. Any gregarious 4-cycle having a vertex in A_p involves two edges of type I$_p$ and two edges of type II$_p$. Thus, to have a γ_4-decomposition, we need more edges of type II$_p$ than edges of type I$_p$. Therefore, we have

$$h_p(\textstyle\sum_{i \neq p} h_i) \leq \sum_{1 \leq i < j \leq m, \, i, j \neq p} h_i h_j \leq (\textstyle\sum_{i \neq p} h_i)^2,$$

which yields the desired inequality. $\qquad\square$

The conclusion of the above lemma is not true if we allow arbitrary 4-cycles in the decomposition. For example, $K(6, 2, 2)$ and $K(10, 2, 2, 2, 2)$ are decomposable into (not necessarily gregarious) 4-cycles, but they do not satisfy the conclusion in the above lemma.

The following lemma is a simple restatement of a well-known result on perfect matchings in multipartite graphs.

Lemma 5.2. *Let* $m \geq 2$ *and* $\sum_{i=1}^{m} h_i$ *be even. Then,* $\bigcup_{i=1}^{m} A_i$ *can be partitioned into subsets consisting of of two vertices from distinct* A_i *and* A_j *if and only if* $h_p \leq \sum_{i \neq p} h_i$ *for each* $p = 1, 2, \ldots, m$.

For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *join* $G_1 + G_2$ of the two graphs is the graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{v_1 v_2 \mid v_1 \in V_1, \ v_2 \in V_2\}$. With this notation, we have that $K(h_1, h_2, \ldots, h_m) = K(h_{i_1}, \ldots, h_{i_j}) + K(h_{i_{j+1}}, \ldots, h_{i_m})$ for any partition $\{\{i_1, \ldots, i_j\}, \{i_{j+1}, \ldots, i_m\}\}$ of $\{1, 2, \ldots, m\}$. The *union* $G_1 \cup G_2$ of graphs G_1 and G_2 is the graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Thus, we have that $G_1 + G_2 = G_1 \cup G_2 \cup K(V_1, V_2)$.

Lemma 5.3. *Let* $\sum_{i=1}^{m} h_i$ *and* $\sum_{j=1}^{n} k_j$ *be even with* $m \geq 2$ *and* $n \geq 2$. *If*

$$h_p \leq \textstyle\sum_{i \neq p} h_i \quad (p = 1, 2, \ldots, m) \quad and \quad k_q \leq \textstyle\sum_{j \neq q} k_j \quad (q = 1, 2, \ldots, n),$$

then the graph $K(\bigcup_{i=1}^{m} A_i, \bigcup_{i=1}^{n} B_i)$ *has a decomposition into 4-cycles such that no two vertices of a 4-cycle belong to the same subset* A_i *or* B_j.

Proof. By the above lemma, $\bigcup_{i=1}^{m} A_i$ has a partition Φ into subsets consisting of two elements from distinct A_i and A_j, and $\bigcup_{i=1}^{n} B_i$ has a partition Ψ into subsets consisting of two elements from distinct B_i and B_j. Note that

$$K(\textstyle\bigcup_{i=1}^{m} A_i, \bigcup_{i=1}^{n} B_i) = \bigcup \left\{ K(\{a_1, a_2\}, \{b_1, b_2\}) \mid \{a_1, a_2\} \in \Phi, \ \{b_1, b_2\} \in \Psi \right\}.$$

However, since each $K(\{a_1, a_2\}, \{b_1, b_2\})$ is simply the 4-cycle $\langle a_1, b_1, a_2, b_2 \rangle$, these 4-cycles constitute a desired decomposition. $\qquad\square$

Theorem 5.4. *Let $\sum_{i=1}^{m} h_i$ and $\sum_{j=1}^{n} k_j$ be even. If $K(A_1, \ldots, A_m)$ and $K(B_1, \ldots, B_n)$ are γ_4-decomposable, then so is $K(A_1, \ldots, A_m, B_1, \ldots, B_n)$.*

Proof. Because

$$K(A_1, \ldots, A_m, B_1, \ldots, B_n)$$
$$= K(A_1, \ldots, A_m) \bigcup K(B_1, \ldots, B_n) \bigcup K(\textstyle\bigcup_{i=1}^{m} A_i, \bigcup_{i=1}^{l} B_n),$$

we only need to show that $K(\bigcup_{i=1}^{m} A_i, \bigcup_{i=1}^{n} B_i)$ is γ_4-decomposable. This follows immediately from Lemmas 5.1 and 5.3. Thus, the theorem is proved. $\quad\square$

With the above theorem, we can build up various γ_4-decomposable graphs from known γ_4-decomposable graphs.

Note. Only after producing the main results of this paper, the authors became aware of the fact that Billington and Hoffman [3] have also considered gregarious 4-cycle decompositions (see [3]) and much of our results overlaps theirs. However, the approach and the decompositions are different.

References

[1] B. Alspach and H. Gavlas, *Cycle decompositions of K_n and $K_n - I$*, J. Combin. Theory Ser. B **81** (2001), 77–99.

[2] E. Billington and D. G. Hoffman, *Decomposition of complete tripartite graphs into gregarious 4-cycles*, Discrete Math. **261**(2003), 87–111.

[3] E. Billington and D. G. Hoffman, *Equipartite and almost-equipartite gregarious 4-cycle decompositions*, Discrete Math. **308** (2008), no.5–6, 696–714.

[4] E. Billington, D. G. Hoffman and C. A. Rodger, *Resolvable gregarious cycle decompositions of complete equipartite graphs*, Discrete Math. **308** (2008), no.13, 2844–2853.

[5] N. J. Cavenagh and E. J. Billington, *Decompositions of complete multipartite graphs into cycles of even length*, Graphs and Combinatorics **16** (2000), 49–65.

[6] G. Chartrand and L. Lesniak, *Graphs and digraphs*, 4th Ed., Chapman & Hall/CRC, Boca Raton, 2005.

[7] J. Liu, *The equipartite Oberwolfach problem and C_t-factorizations of complete equipartite graphs*, J. Combin. Designs **9** (2000), 42–49.

[8] J. Liu, *A generalization of the Oberwolfach problem with uniform tables*, J. Combin. Theory Ser. A **101** (2003), 20–34.

[9] M. Šajna, *On decomposing $K_n - I$ into cycles of a fixed odd length*, Descrete Math. **244** (2002), 435–444.

[10] M. Šajna, *Cycle decompositions III: complete graphs and fixed length cycles*, J. Combin. Designs **10** (2002), 27–78.

[11] D. Sotteau, *Decomposition of $K_{m,n}$ ($K_{m,n}^*$) into cycles (circuits) of length $2k$,* J. Combin. Theory Ser. B **30** (1981), 75–81.

Eun Kyung Kim
Department of Mathematics
Pusan National University
Busan, 609-735, Korea
E-mail address: ddbrsy9608@naver.com

Young Min Kim
Department of Mathematics
Pusan National University
Busan, 609-735, Korea
E-mail address: amusein@naver.com

Jung Rae Cho
Department of Mathematics
Pusan National University
Busan, 609-735, Korea
E-mail address: jungcho@pusan.ac.kr