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FINITENESS THEOREMS FOR 2-UNIVERSAL HERMITIAN LATTICES OVER SOME IMAGINARY QUADRATIC FIELDS

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Abstract. A positive definite Hermitian lattice is said to be 2-universal if it represents all positive definite binary Hermitian lattices. We find some finiteness theorems which ensure 2-universality of Hermitian lattices over several imaginary quadratic number fields.

1. Introduction

Since the celebrated Lagrange's Four Square Theorem, an important subject is positive definite quadratic forms which represent all positive integers. This subject was studied by many mathematicians, for instance Jacobi, Pepin, Liouville, etc. In particular, Ramanujan found all 54 positive definite integral quaternary diagonal forms represent all positive integers. Dickson called such quadratic forms *universal*.

In 1948, Willerding investigated the universal classically integral quaternary forms and found 178 such forms in her dissertation [11]. In her dissertation she had to check as many as 1046 quaratic forms and the methods are too complicated to be verified. Indeed, her list contains some mistakes, but the correction was performed in the different directions.

In 1993, Conway and Schneeberger proved the so-called *Fifteen Theorem* which ensures the universality of positive definite classically integral quadratic forms. It states that if a positive definite quadratic form represents 1, 2, 3, 5, 6, 7, 10, 14, and 15, then it represents all positive integers. This astounding theorem enables them to check Willerding's list. Consequently, they obtain 204 positive definite universal quaternary classically integral quadratic forms.

After several years, Bhargava proposed a simpler and more elegant proof than Conway and Schneeberger's original one. Besides he insisted that for every infinite subset S of $\mathbb{N}$, there is a finite subset S_0 of S such that if a positive definite quadratic form represents all elements of S_0, then it represents

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all elements of S. For instance, if a positive definite quadratic form represents odd integers from 1 through 33, then it represents all odd integers. Such criterions are called *Finiteness Theorems*.

This criterion was generalized and was proved in the view of a natural analogue of Bhargava's results for representation of forms by forms by M.-H. Kim, B. M. Kim and B.-K. Oh [4]. That is, for every infinite set S of positive definite quadratic forms of rank n, there exists a finite subset S_0 of S such that if a quadratic form represents all elements of S_0, then it represents all elements of S. Since a positive integer a can be considered as a quadratic form ax^2, this is a generalization of Bhargava's assertion.

In particular they found a criterion for quadratic forms which represent all binary quadratic forms [5]: if a quadratic form represents all of

$$\begin{array}{lll} x^2+y^2, & 2x^2+3y^2, & 3x^2+3y^2, \\ 2x^2+2xy+2y^2, & 2x^2+2xy+3y^2, & 2x^2+2xy+4y^2, \end{array}$$

then it represents all binary quadratic forms.

The idea of finiteness theorems has been applied to Hermitian lattices. Recently, the author and the collaborators succeeded in proving the *Fifteen Theorem for Universal Hermitian Lattices*. By this theorem it is enough to see whether a Hermitian lattice represents 1, 2, 3, 5, 6, 7, 10, 13, 14, and 15 for its universality. In this article we investigate the finiteness theorem for 2-universality which means a Hermitian lattice represents all positive definite binary Hermitian lattices.

M.-H. Kim and the author obtained all ternary and quaternary 2-universal Hermitian lattices over imaginary quadratic fields (notations will be introduced in the next section):

$$\begin{array}{ll} \mathbb{Q}(\sqrt{-1}): & \langle 1,1,1\rangle, \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ \mathbb{Q}(\sqrt{-2}): & \langle 1,1\rangle \perp \begin{pmatrix} 2 & -1+\sqrt{-2} \\ -1-\sqrt{-2} & 2 \end{pmatrix} \\ \mathbb{Q}(\sqrt{-3}): & \langle 1,1,1\rangle, \langle 1,1,2\rangle \\ \mathbb{Q}(\sqrt{-7}): & \langle 1,1,1\rangle \\ \mathbb{Q}(\sqrt{-11}): & \langle 1,1\rangle \perp \begin{pmatrix} 2 & \frac{1+\sqrt{-11}}{2} \\ \frac{1-\sqrt{-11}}{2} & 2 \end{pmatrix} \end{array}$$

We give some criteria for 2-universality of Hermitian lattices over several imaginary quadratic fields.

2. Notations and symbols

Let m be a positive square-free integer and $E = \mathbb{Q}(\sqrt{-m})$ with the ring $\mathcal{O} = \mathcal{O}_E = \mathbb{Z}[\omega]$ of integers, where $\omega = \omega_m = \sqrt{-m}$ or $\frac{1+\sqrt{-m}}{2}$ if $m \equiv 1, 2$ or $3 \pmod 4$, respectively.

For a prime p, we define $E_p := E \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Then the ring $\mathcal{O}_p$ of integers of E_p is defined $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. If p is inert or ramifies in E, then $E_p = \mathbb{Q}_p(\sqrt{-m})$ and $\alpha \otimes \beta = \alpha\beta$ with $\alpha \in E$ and $\beta \in \mathbb{Q}_p$. If p splits in E, then $E_p = \mathbb{Q}_p \times \mathbb{Q}_p$ and $\alpha \otimes \beta = (\alpha\beta, \overline{\alpha}\beta)$ where $\bar{\cdot}$ is the canonical involution in E. Thus E_p allows the unique involution $\overline{\alpha \otimes \beta} = \overline{\alpha} \otimes \beta$ [10], [3].

Definition 1. *A* Hermitian space *is a finite-dimensional vector space V over E equipped with a sesqui-linear map $H : V \times V \to E$ satisfying the following conditions:*

(1) $H(\mathbf{x}, \mathbf{y}) = \overline{H(\mathbf{y}, \mathbf{x})}$,
(2) $H(a\mathbf{x}, \mathbf{y}) = aH(\mathbf{x}, \mathbf{y})$ *for* $a \in E$,
(3) $H(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}) = H(\mathbf{x}_1, \mathbf{y}) + H(\mathbf{x}_2, \mathbf{y})$,

We simply denote $H(\mathbf{v}, \mathbf{v})$ by $H(\mathbf{v})$.

From (1)-(3) follow

$(2')$ $H(\mathbf{x}, b\mathbf{y}) = \bar{b}H(\mathbf{x}, \mathbf{y})$ and $(3')$ $H(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) = H(\mathbf{x}, \mathbf{y}_1) + H(\mathbf{x}, \mathbf{y}_2)$.

Definition 2. *A* Hermitian lattice *or briefly a lattice L is an $\mathcal{O}$-module with a sesqui-linear map H such that $H(L, L) \subseteq \mathcal{O}$.*

Through localization at prime p, we can define a Hermitian space over E_p and a Hermitian $\mathcal{O}_p$-lattice.

If L is free with a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then we define

$$M_L := (H(\mathbf{v}_i, \mathbf{v}_j))_{n \times n}$$

and call it the *Gram matrix* of L. We often identify M_L with the lattice L. If M_L is diagonal, we simply write $L = \langle a_1, \dots, a_n \rangle$, where $a_i = H(\mathbf{v}_i)$ for $i = 1, 2, \dots, n$. The determinant of M_L is called the *discriminant* of L, denoted by dL. By EL, we mean the Hermitian space $V = E \otimes_{\mathcal{O}} L$ where L is nested. We define the *rank* of L by $\operatorname{rank} L := \dim_E EL$.

It is well known that an $\mathcal{O}$-lattice L can be written as

$$L = \mathfrak{a}_1\mathbf{v}_1 + \cdots + \mathfrak{a}_n\mathbf{v}_n \tag{1}$$

for vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in L$ and ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n \subseteq \mathcal{O}$. The expression (1) of an $\mathcal{O}$-lattice L, which is not necessarily free, can be transformed to the form

$$L = \mathcal{O}\mathbf{w}_1 + \cdots + \mathcal{O}\mathbf{w}_{n-1} + \mathfrak{a}\mathbf{w}_n \tag{2}$$

for some vectors $\mathbf{w}_1, \dots, \mathbf{w}_n \in L$ and an ideal $\mathfrak{a} \subseteq \mathcal{O}$ [9, 81:5]. If L is not free, or equivalently, if $\mathfrak{a}$ is not principal, then $\mathfrak{a}$ is generated by two elements, say $\alpha, \beta \in \mathcal{O}$. Therefore (2) may be rewritten as

$$L = \mathcal{O}\mathbf{w}_1 + \mathcal{O}\mathbf{w}_2 + \cdots + \mathcal{O}\mathbf{w}_{n-1} + \mathcal{O}\alpha\mathbf{w}_n + \mathcal{O}\beta\mathbf{w}_n.$$

We may treat L as if it were a free lattice with basis $\{\mathbf{w}_1, \dots, \mathbf{w}_{n-1}, \alpha\mathbf{w}_n, \beta\mathbf{w}_n\}$. The rank of L, however, is still n not $n+1$. In this case the formal Gram matrix of L is defined as

$$M_L = \begin{pmatrix} H(\mathbf{w}_1, \mathbf{w}_1) & \dots & H(\mathbf{w}_1, \alpha\mathbf{w}_n) & H(\mathbf{w}_1, \beta\mathbf{w}_n) \\ \vdots & \ddots & \vdots & \vdots \\ H(\alpha\mathbf{w}_n, \mathbf{w}_1) & \dots & H(\alpha\mathbf{w}_n, \alpha\mathbf{w}_n) & H(\alpha\mathbf{w}_n, \beta\mathbf{w}_n) \\ H(\beta\mathbf{w}_n, \mathbf{w}_1) & \dots & H(\beta\mathbf{w}_n, \alpha\mathbf{w}_n) & H(\beta\mathbf{w}_n, \beta\mathbf{w}_n) \end{pmatrix}.$$

Let ℓ and L be two (free or nonfree) Hermitian $\mathcal{O}$-lattices whose (formal) Gram matrices are $M_\ell \in M_{m\times m}(\mathcal{O})$ and $M_L \in M_{n\times n}(\mathcal{O})$ respectively. We say that L *represents* ℓ, denoted by $\ell \to L$, if there exists a suitable $X \in M_{m\times n}(\mathcal{O})$ such that $M_\ell = XM_LX^*$, where X^* is the conjugate transpose of X.

3. Finiteness theorems for 2-universality

Conway, Schneeberger, and Bhargava's Fifteen Theorem states: *A positive definite quadratic $\mathbb{Z}$-lattice is universal if it represents every element in the set*

$$A = \{\,1, 2, 3, 5, 6, 7, 10, 14, 15\,\}.$$

Shortly after, an analogous criterion for 2-universality was proved [4], which states: *A positive definite quadratic $\mathbb{Z}$-lattice is 2-universal if it represents every element in the set*

$$B = \left\{ \langle 1,1\rangle_{\mathbb{Z}}\,, \langle 2,3\rangle_{\mathbb{Z}}\,, \langle 3,3\rangle_{\mathbb{Z}}\,, \begin{pmatrix}2&1\\1&2\end{pmatrix}_{\mathbb{Z}}, \begin{pmatrix}2&1\\1&3\end{pmatrix}_{\mathbb{Z}}, \begin{pmatrix}2&1\\1&4\end{pmatrix}_{\mathbb{Z}} \right\},$$

where $\langle a,b\rangle_{\mathbb{Z}} = \begin{pmatrix}a&0\\0&b\end{pmatrix}_{\mathbb{Z}}$ and $\begin{pmatrix}a&b\\b&c\end{pmatrix}_{\mathbb{Z}} = ax^2+2bxy+cy^2$. We refer the readers to [7] and [5] for recent developments in this direction. The sets A and B are called *minimal* in the sense that no proper subset ensures (2-)universality.

We can find the criteria for 2-universality over several imaginary quadratic fields.

Theorem 1. *A Hermitian lattice over $\mathbb{Q}(\sqrt{-1})$ is 2-universal if it represents $\langle 1,1\rangle$ and $\begin{pmatrix}2&1\\1&2\end{pmatrix}$.*

Proof. Let L be a 2-universal lattice over $\mathbb{Q}(\sqrt{-1})$. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_3, \mathbf{v}_4\}$ be bases of $\langle 1,1\rangle$ and $\begin{pmatrix}2&1\\1&2\end{pmatrix}$, respectively. Then L contains a lattice generated by all $\mathbf{v}_i$'s. This lattice can be obtained by using the following positive semi-definite 4×4-matrix

$$\begin{pmatrix}1&0&*&*\\0&1&*&*\\ *&*&2&1\\ *&*&1&2\end{pmatrix}.$$

It is isometric to $\langle 1,1,1,0\rangle$, $\langle 1,1,1,1\rangle$, or $\langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ according to suitable entries. Thus L contains $\langle 1,1,1\rangle$ or $\langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Both lattices are 2-universal [8]. $\square$

Theorem 2. *A Hermitian lattice over* $\mathbb{Q}(\sqrt{-2})$ *is* 2*-universal if it represents* $\langle 1,1\rangle$ *and* $\begin{pmatrix} 2 & -1+\omega_2 \\ -1+\overline{\omega}_2 & 2 \end{pmatrix}$.

Proof. The positive semi-definite matrix

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ * & * & 2 & -1+\omega_2 \\ * & * & -1+\overline{\omega}_2 & 2 \end{pmatrix}$$

gives only one lattice $\langle 1,1\rangle \perp \begin{pmatrix} 2 & -1+\omega_2 \\ -1+\overline{\omega}_2 & 2 \end{pmatrix}$. This lattice is 2-universal [8]. $\square$

Theorem 3. *A Hermitian lattice over* $\mathbb{Q}(\sqrt{-3})$ *is* 2*-universal if it represents* $\langle 1,1\rangle$ *and* $\langle 1,2\rangle$.

Proof. Let L be a Hermitian lattice and assume that $\langle 1,1\rangle \to L$ and $\langle 1,2\rangle \to L$. Since $\langle 1,1\rangle$ is unimodular, it splits L, that is, $L = \langle 1,1\rangle \perp L_0$ for some sublattice L_0 of L. In order for L to represent $\langle 1,2\rangle$, L_0 should represent 1, 2, or $\langle 1,2\rangle$. In any case, L contains either $\langle 1,1,1\rangle$ or $\langle 1,1,2\rangle$. Both are 2-universal [8]. $\square$

Remark 1. The sets A for universal $\mathbb{Z}$-lattices and B for 2-universal $\mathbb{Z}$-lattices are unique minimal sets in the respective criteria. In Theorem 3, however, $\langle 1,2\rangle$ can be replaced by $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. So, the set $\{\langle 1,1\rangle, \langle 1,2\rangle\}$ is a minimal but not a unique set ensuring the 2-universality.

Theorem 4. *A Hermitian lattice over* $\mathbb{Q}(\sqrt{-11})$ *is* 2*-universal if it represents* $\langle 1,1\rangle$ *and* $\begin{pmatrix} 2 & \omega_{11} \\ \overline{\omega}_{11} & 2 \end{pmatrix}$.

Proof. The positive semi-definite matrix

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ * & * & 2 & \omega_{11} \\ * & * & \overline{\omega}_{11} & 2 \end{pmatrix}$$

gives only one lattice $\langle 1,1\rangle \perp \begin{pmatrix} 2 & \omega_{11} \\ \overline{\omega}_{11} & 2 \end{pmatrix}$. This lattice is 2-universal [8]. $\square$

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