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(m, n)-FOLD p-IDEALS IN WEAK BCC-ALGEBRAS

Bushra Karamdin and Shaban Ali Bhatti

Abstract. Various characterizations of (m, n)-fold p-ideals of weak BCC-algebras are presented.

1. Introduction

BCC-algebras, introduced by Y. Komori (see [10] and [11]), are an algebraic model of BIK$^+$-logic, i.e., implicational logic whose axioms scheme are the principal type-scheme of the combinators B, I, and K, and whose inference rules are modus ponens and modus ponens 2, where $p \to q$ is inferred from $p \to (r \to q)$ and r. Several years later some authors introduced independently more extensive algebraic system using different names. This new algebraic systems have the same partial order as BCC-algebras and BCK-algebras but has no minimal element. Such obtained system is called a BZ-algebra [7, 15] or a weak BCC-algebra [2, 4, 13]. From the mathematical point of view the last name is more corrected but more popular is the first.

Many mathematicians studied such algebras as BCI-algebras, B-algebras, difference algebras, implication algebras, G-algebras, Hilbert algebras, d-algebras and many others. All these algebras have one distinguished element and satisfy some common identities playing a crucial role in these algebras and, in fact, are generalization or a special case of weak BCC-algebras. So, results obtained for weak BCC-algebras are in some sense fundamental for these algebras, especially for BCC/BCH/BCI/BCK-algebras.

A very important role in the theory of such algebras plays ideals. Many types of ideals in these algebras have been studied with various relations between them (see for example [5] and [16]). In [14] X.H.Zhang, J.Hao and S.A. Bhatti studied p-ideals of BCI-algebras. In [8] Y.Huang and Z.Chen introduced the foldness of some ideals in BCK-algebras. In [12] Kordi and Moussavi studied (m, n)-fold p-ideals and fuzzy (m, n)-fold p-ideals in BCI-algebras.

This paper is a continuation of our study of p-ideals initiated in [5].

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2. Preliminaries

Definition 2.1. A *weak BCC-algebra X* is an abstract algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms

(i) $((x * y) * (z * y)) * (x * z) = 0,$
(ii) $x * x = 0,$
(iii) $x * 0 = x,$
(iv) $x * y = y * x = 0 \longrightarrow x = y.$

A weak BCC-algebra satisfying the identity

(v) $0 * x = 0,$

is called a *BCC-algebra*. A BCC-algebra with the condition

(vi) $(x * (x * y)) * y = 0$

is called a *BCK-algebra*.

One can prove (see [2] or [3]) that a BCC-algebra is a BCK-algebra if and only if it satisfies the identity

(vii) $(x * y) * z = (x * z) * y.$

An algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms (i), (ii), (iii), (iv) and (vi) is called a *BCI-algebra*. A BCI-algebra satisfies also (vii) (cf. [9]). A weak BCC-algebra is a BCI-algebra if and only if it satisfies (vii).

A (weak) BCC-algebra which is not a BCK-algebra (respectively, BCI-algebra) is called *proper*. A proper BCC-algebra has at least four elements. Moreover, for every $n \geq 4$ there exists at least one proper BCC-algebra (cf. [2, 3]). Analogous results are valid for weak BCC-algebras (cf. [4]).

In all these algebras one can define a natural partial order $\leq$ putting

$$x \leq y \longleftrightarrow x * y = 0.$$

In all BCC/BCK-algebras we have $0 \leq x$ for every $x \in X$. Moreover, from (i) it follows that in any (weak) BCC-algebra

$$x \leq y \longrightarrow z * y \leq z * x, \tag{1}$$

$$x \leq y \longrightarrow x * z \leq y * z \tag{2}$$

for all $x, y, z \in X$.

In BCC-algebras we also have

$$x * y \leq x \tag{3}$$

for all $x, y \in X$ (cf. [3]). In weak BCC-algebras it is not true.

We say that two elements $x, y \in X$ are *comparable* if $x \leq y$ or $y \leq x$. An algebra X is *linearly ordered* if each its two elements are comparable. An element a of a weak BCC-algebra X is called an *atom* if $x \leq a$ implies $x = 0$ or $x = a$. A linearly ordered weak BCC-algebra (BCI-algebra) is a BCC-algebra (BCK-algebra, respectively).

The set

$$B(a) = \{x \in X | a \leq x\}$$

where a is an atom of X, is called a *branch* of X. An element a is called *initial* for $B(a)$. In the case when an initial element $a \neq 0$ is comparable with 0 we say that a branch $B(a)$ is *improper*. The set of all initial elements of proper branches of X is denoted by $I(X)$. The set of all elements comparable with 0, i.e., the set

$$B(0) = \{x \in X \mid 0 \leq x\}$$

is called a *BCK-part* of X. The following proposition is proved in [5].

Proposition 2.2. *Two elements x and y are in the same branch if and only if $x * y \in B(0)$.*

Corollary 2.3. *Comparable elements are in the same branch.*

Corollary 2.4. *$x * y \in B(0)$ if and only if $y * x \in B(0)$.*

3. Ideals

Definition 3.1. A nonempty subset A of a weak BCC-algebra X is called a *BCK-ideal*, if

(ix) $0 \in A$,

(x) $x * y \in A$ and $y \in A$ imply $x \in A$,

and a *BCC-ideal* if it satisfies (ix) and

(xi) $(x * y) * z \in A$ and $y \in A$ imply $x * z \in A$.

Putting $z = 0$ we can see that a BCC-ideal is a BCK-ideal. The converse is not true [6]. This means that a BCC-ideal is a BCK-ideal with some additional property.

Definition 3.2. A nonempty subset A of a weak BCC-algebra X is called a *p-ideal* of X if it contains 0 and

$$(x * z) * (y * z) \in A \quad \text{and} \quad y \in A \quad \text{imply} \quad x \in A. \tag{4}$$

Putting $z = 0$ in (4) we can see that every *p-ideal* is a BCK-ideal.

We use the following abbreviated notation: the expression $(...((x * y) * y) * ...) * y$, where y occurs n times is written as $x * y^n$. Similarly, $x^n * y$ denotes the expression $(x * (... * (x * (x * y))...)$, where x occurs n times.

Definition 3.3. A nonempty subset A of a weak BCC-algebra X is called an *(m,n)-fold p-ideal* of X if it contains 0 and

$$(x * z^m) * (y * z^n) \in A \quad \text{and} \quad y \in A \quad \text{imply} \quad x \in A. \tag{5}$$

An (n,n)-fold p-ideal is called an *n-fold p-ideal.* Since $(0,0)$-fold p-ideals coincide with BCK-ideals we will consider only (m,n)-fold p-ideals with $m \geq 1$ and $n \geq 1$. Note that for $m = n = 1$ the concept of $(1,1)$-fold p-ideals coincides with the concept of p-ideals studied in BCI-algebras (see for example [14]).

Example 3.4. Consider a weak BCC-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	b	b
a	a	0	b	b
b	b	b	0	0
c	c	c	a	0

Table 1

It is easy to show that $A = \{0, a\}$ is a 1-fold p-ideal (and n-fold p-ideal) of X, but it is not an (m, n)-fold p-ideal, where m is odd and n is even. This is because $(b * b^m) * (0 * b^n) \in A$ and $0 \in A$, but $b \notin A$.

Proposition 3.5. *Every (m, n)-fold p-ideal is a BCK-ideal.*

Proof. Putting $z = 0$ in (5), the result follows. □

The converse is not true as the following example shows.

Example 3.6. Consider on the set $X = \{0, a, b, c\}$ the binary operation defined by the following table:

$*$	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

Table 2

The algebra $(X, *, 0)$ defined by this table is a proper weak BCC-algebra (cf. [4]). The set $A = \{0, a\}$ is a BCK-ideal. It is an (m, n)-fold p-ideal only in the case $m = n = 1$. Indeed, for $m \geq 2$, $n \geq 1$ we have $(b * b^m) * (0 * b^n) \in A$, $0 \in A$ and $b \notin A$, which means that A is not an (m, n)-fold p-ideal for $m \geq 2$, $n \geq 1$. It is not difficult to see that for $m = n = 1$ the condition (4) is satisfied. Hence A is a p-ideal. On the other hand, it is easy to show that the set $I = \{0\}$ is a BCK-ideal of X, but it is not an (m, n)-fold p-ideal. This is because $(a * a^m) * (0 * a^n) \in I$, $0 \in I$ and $a \notin I$.

As a simple consequence of the definition of a BCK-ideal we obtain:

Lemma 3.7. *If A is a BCC-ideal of a weak BCC-algebra X then for every $x \in X$ and $y \in A$ from $x \leq y$ it follows $x \in A$.*

Theorem 3.8. *An n-fold ideal is a k-fold ideal for any $k \leq n$.*

Proof. Indeed, by (i), for every $1 \leq k \leq n$ and $x, y, z \in X$ we have

$$(x * z^n) * (y * z^n) \leq (x * z^{n-1}) * (y * z^{n-1}) \leq \ldots \leq (x * z^k) * (y * z^k).$$

Thus, if A is an n-fold p-ideal of X and $(x * z^k) * (y * z^k) \in A$, then, by Lemma, also $(x * z^n) * (y * z^n) \in A$. This, for $y \in A$ implies $x \in A$. Hence A is a k-fold ideal. □

Proposition 3.9. *$B(0)$ is an n-fold p-ideal for any $n \geq 1$.*

Proof. Obviously $0 \in B(0)$. If $y \in B(0)$ and $(x * z^n) * (y * z^n) \in B(0)$, then $0 \leq y$ and $0 * z^n \leq y * z^n$ by (2). Thus, by (1) and (i), we have

$$(x * z^n) * (y * z^n) \leq (x * z^n) * (0 * z^n) \leq (x * z^{n-1}) * (0 * z^{n-1}) \leq \ldots \leq x * 0 = x,$$

i.e., $(x * z^n) * (y * z^n) \leq x$. Since, by the assumption, $0 \leq (x * z^n) * (y * z^n)$, the last means that $0 \leq x$. So, $x \in B(0)$. Hence $B(0)$ is an n-fold p-ideal. □

Theorem 3.10. *A BCC/BCK-ideal A is an n-fold p-ideal if and only if $B(0) \subset A$.*

Proof. If A is an n-fold p-ideal of a weak BCC-algebra X, then for every $x \in B(0)$ from $0 \leq x$ it follows

$$(x * x^n) * (0 * x^n) = 0 * 0 = 0 \in A,$$

which, according to (5), gives $x \in A$. Thus $B(0) \subseteq A$.

Conversely, if $B(0) \subseteq A$ and A is an ideal of X, then from $y \in A$ and $(x * z^n) * (y * z^n) \in A$, by (i), it follows $(x * z^n) * (y * z^n) \leq x * y$, which means that $(x * z^n) * (y * z^n)$ and $x * y$ are in the same branch (Corollary 2.3). Hence, $(x * y) * ((x * z^n) * (y * z^n)) \in B(0) \subseteq A$, by Proposition 2.2. Since $(x * z^n) * (y * z^n) \in A$ and A is a BCC-ideal (or a BCK-ideal), by Lemma 3.7 we have $x * y \in A$. Consequently, $x \in A$. So, A is an n-fold p-ideal. □

Corollary 3.11. *$B(0)$ is the least n-fold p-ideal for every $n \geq 1$.*

Corollary 3.12. *Any BCK/BCC-ideal containing an n-fold p-ideal also is an n-fold p-ideal.*

Proof. Let an n-fold p-ideal A be contained in an ideal B. Then $B(0) \subset A \subset B$, which completes the proof. □

Proposition 3.13. *A BCK/BCC-ideal A of a weak BCC-algebra is an n-fold p-ideal if and only if $x * (0 * (0 * x)) \in A$ for every $x \in B(0)$.*

Proof. Let A be an n-fold p-ideal on X. Since $0 * (0 * x) \leq x$ for every $x \in X$ (Lemma 3.6 in [5]), elements $0 * (0 * x)$ and x are in the same branch. Thus, $x * (0 * (0 * x)) \in B(0)$, by Proposition 2.2. This, by Theorem 3.10, gives $x * (0 * (0 * x)) \in A$.

Conversely, if $x * (0 * (0 * x)) \in A$ for any $x \in B(0)$, then $0 \leq x$ implies $0 * x = 0$, and consequently, $0 * (0 * x) = 0$. Hence $x = x * (0 * (0 * x)) \in A$, so $B(0) \subset A$. Theorem 3.10 completes the proof. □

Corollary 3.14. *If A is an n-fold p-ideal of a weak BCC-algebra X, then*

$$B(a) \cap A \neq \emptyset \Longrightarrow B(a) \subset A$$

for every $a \in I(X)$.

Proof. Let $x \in B(a) \cap A$ for some $a \in I(X)$ and an n-fold p-ideal A. If $y \in B(a)$, then $a \leq y$, whence, by (2), we obtain $0 = a * x \leq y * x$. Thus $y * x \in B(0) \subset A$. Since A is a BCK-ideal and $x \in A$, we have $y \in A$. This proves $B(a) \subset A$. □

Corollary 3.15. *An n-fold ideal A together with an element $x \in A$ contains whole branch containing this element.*

Corollary 3.16. *For any n-fold p-ideal A from $x \leq y$ and $x \in A$ it follows $y \in A$.*

Theorem 3.17. *A BCK/BCC-ideal A of a weak BCC-algebra X is its n-fold p-ideal if and only if the following implication*

$$(x * z^n) * (y * z^n) \in A \Longrightarrow x * y \in A$$

is valid for all $x, y, z \in X$.

Proof. Since the first condition of Definition 2.1 can be written in the form

$$(x * y) * (z * y) \leq x * z,$$

we have

$$(x * z^n) * (y * z^n) \leq (x * z^{n-1}) * (y * z^{n-1}) \leq \ldots \leq (x * z) * (y * z) \leq x * y.$$

So, if $(x * z^n) * (y * z^n) \in A$ and A is an n-fold p-ideal, then $x * y \in A$ by Corollary 3.16.

The converse statement is obvious. □

A special class of weak BCC-algebras form *group-like weak BCC-algebras* (called also *anti-grouped*), i.e., weak BCC-algebras X with the property $X = I(X)$. Such algebras are uniquely characterized by some groups (see [1] and [15]). Below we present a simple characterization of such weak BCC-algebras.

Theorem 3.18. *A weak BCC-algebra X is group-like if and only if for some $n \geq 1$ and all $x, z \in X$ the following implication*

$$(x * z^n) * (0 * z^n) = 0 \Longrightarrow x = 0$$

is valid.

Proof. Assume that X is a weak group-like BCC-algebra. Then $X = I(X)$ which means that $x \leq y$ implies $x = y$. So, for all $x, y, z \in X$ we have $(x * z^n) * (y * z^n) = x * y$ (see the proof of Theorem 3.17). In particular $0 = (x * z^n) * (0 * z^n) = x * 0 = x$. So, the above implication is valid.

Conversely, if the above implication is valid for all $x, z \in X$, then

$$0 = (x * z^n) * (0 * z^n) \leq x * 0 = x,$$

means that $0 \leq x$ implies $x = 0$. Thus $B(0) = \{0\}$. Hence for all $x \leq y$ we have $x * y = 0$ and $y * x = 0$ (Corollary 2.4). Therefore $x = y$. Consequently $X = I(X)$. □

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Bushra Karamdin
Department of Mathematics
University of the Punjab
Quaid-e-Azam Campus, Lahore-54590, Pakistan
E-mail address: ayeshafatima5@hotmail.com

Shaban Ali Bhatti
Department of Mathematics
University of Education, Township Campus
Lahore-54590, Pakistan