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GEODESIC FORMULA OF A CERTAIN CLASS OF PSEUDORIEMANNIAN 2-STEP NILPOTENT GROUPS AND JACOBI OPERATORS ALONG GEODESICS IN PSEUDORIEMANNIAN 2-STEP NILPOTENT GROUPS

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Abstract. In this paper, we obtain geodesic formula of a certain class of Pseudoriemmanian 2-step nilpotent groups and show a constancy of represenation matrix of Jacobi oprerators along geodesics in Pseudoriemmanian 2-step nilpotent groups with one dimensional center.

1. Introduction

Nilpotent Lie groups play an important role in many areas of mathematics, and 2-step nilpotent groups have a special significance. These are nonabelian Lie groups that come as close as possible to being abelian, but they admit interesting phenomena that do not arise in abelian groups. During the 1990's many works were done in 2-step nilpotent Lie groups with a left invariant Riemmannian metric or Lorentzian metric. Eberlein [1] and Guederi [2] did general studies on 2-step nilpotent Lie groups with a left invariant Riemmannian metric and 2-step nilpotent Lie groups with a left invariant Lorentzian metric, respectively. But relatively little was known of the Pseudoriemmanian case. For example , Eberlein and Guederi succeeded in obtaining general geodesic formulae of 2-step nilpotent groups with a left invariant Riemmanian metric and with a left invariant Lorentzian metric respectively. But until now a geodesic formula of Pseudoriemmanian 2-step nilpotent groups has not been obtained. In this paper, we obtain a geodesic formula of a special famliy of Pseudoriemmanian 2-step nilpotent groups and show a constancy of representation matrix of Jacobi oprerators along geodesics in Pseudoriemmanian 2-step nilpotent groups with one dimensional center, which is a slight generalization of a result in [6].

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Throughout, N will denote a connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ having center Z. We shall use $\langle\, ,\rangle$ to denote either an inner product on $\mathfrak{n}$ or the induced left-invariant pseudoriemannian (indefinite) metric tensor on N.

In this paper we make the additional assumption that the center Z of $\mathfrak{n}$ is nondegenerate. Let $Z^{\perp}$ be the orthogonal complement of Z in $\mathfrak{n}$ relative $\langle\, ,\rangle$, and so write $\mathfrak{n}$ as an orthogonal direct sum $\mathfrak{n} = Z \oplus Z^{\perp}$. For each $z \in Z$, we define a skew-adjoint operator $J_z : Z^{\perp} \longrightarrow Z^{\perp}$ given by

$$\langle J_z(x), y\rangle = \langle z, [x, y]\rangle.$$

Notice that this map was first introduced by A.Kaplan[5] to study Riemmanian 2-step nilmanifolds of Heisenberg type. Next, let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$ (the identity in N) and $\gamma'(0) = x_0 + z_0$, where $x_0 \in Z^{\perp}$ and $z_0 \in Z$. Using exponential coordinates, one can write

$$\gamma(t) = exp(X(t) + Z(t)),$$

where $x(t) \in Z^{\perp}$ and $z(t) \in Z$ for all $t \in R$, and

$$X'(0) = x_0,\ Z'(0) = z_0.$$

The following lemma that is due to A.Kaplan[5] in the case of Riemmmanian metrics is intact in the case of a Pseudoriemmanian metric with nondegenerate center.

Lemma 1.1. *The curve $\gamma(t)$ is a geodesic in N if and only if the following system of equations is satisfied*

$$X''(t) = J_{z_0} X'(t), \tag{1.1}$$

$$Z'(t) + \frac{1}{2}[X'(t), X(t)] = z_0, \tag{1.2}$$

for all $t \in R$.

In the Riemannian case the solution to this equation was obtained by P. Eberlein and he obtained the following which also holds for Pseudoriemannian 2-step milpotent Lie groups with nondegenerate center. Let L_n denote left translation in N by $n \in N$.

$$\gamma'(t) = L^*_{\gamma(t)}(e^{tJ_{z_0}} x_0 + z_0).$$

As in [1], we shall identify all tangent spaces with $\mathfrak{n} = T_eN$. Thus we write as

$$\gamma'(t) = z_0 + e^{tJ} x_0.$$

Recall that the Jacobi operator along $\gamma(t)$ is defined by

$$R_{\gamma'(t)}(\cdot) = R(\cdot, \gamma'(t))\gamma'(t),$$

where R denotes Pseudoriemannian curvature tensor. The following formula of Jacobi operator of 2-step nilpotent Lie group can be found in [6].

$$\begin{aligned} R_{\gamma'(t)}(x+z) &= \frac{3}{4}J_{[x,x']}x' + \frac{1}{2}J_z Jx' - \frac{1}{4}JJ_z x' - \frac{1}{4}J^2 x \\ &\quad - \frac{1}{2}[x, Jx'] + \frac{1}{4}[x', Jx] + \frac{1}{4}[x', J_z x']. \end{aligned} \tag{1.3}$$

for all $z \in Z$ and $x \in Z^{\perp}$, where $x' = e^{tJ}x_0$ and $J = J_{z_0}$. In the Riemannian (positive definite) case, J^2 is always diagonalizable. This makes possible the completely explicit integration of the geodesic equations (compare [1]). But in the Pseudoriemannian case, J^2 is not always diagonalizable. So it seems difficult to obtain the integration of the geodesic equations, completely. First of all, we will integrate the geodesic equations of a 2-step nilpotent group with Pseudoriemannian metric when J^2 is diagonalizable.

2. Geodesics in 2-step nilpotent Lie Groups when J^2 is diagonalizable

N will denote a simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ having center Z. Suppose that $\mathfrak{n}$ is endowed with a nondegenerate inner product $\langle , \rangle$ that in turn induces a left-invariant pseudoriemmanian metric on the Lie group. Additionally assume that Z is nondegenerate with respect to the inner product $\mathfrak{n}$. Assume that $\gamma(t) = exp(X(t) + Z(t))$ is a geodesic in N emanating from the identity with $\gamma'(0) = x_0 + z_0 \in Z^{\perp} \oplus Z$ and that the map $J^2 = J^2_{z_0}$ is diagonalizable. Then by the classification of skewadjoint operators on Pseudoeuclidean space in [3], we may assume that $Z^{\perp}$ is decomposed into

$$\oplus_{k=1}^m \ker(J^2 + \theta_k^2 I) \oplus_{l=1}^n \ker(J - \mu_l I) \oplus_{l=1}^n \ker(J + \mu_l I) \oplus \ker J^2.$$

So we may write as

$$x_0 = x_1 + x_2$$

with $x_1 \in \ker J^2$ and $x_2 = \sum_{k=1}^m \xi_k + \sum_{l=1}^n \zeta_l + \sum_{l=1}^n \zeta'_l \in \oplus_{k=1}^m \ker(J^2 + \theta_k^2 I) \oplus_{l=1}^n \ker(J - \mu_l I) \oplus_{l=1}^n \ker(J + \mu_l I)$. Then from (1.1), we have

$$\begin{aligned} X(t) &= tx_1 + \frac{t^2}{2}Jx_1 + \sum_{k=1}^m (e^{tJ} - I)J^{-1}\xi_k + \sum_{l=1}^n (e^{tJ} - I)J^{-1}(\zeta_l + \zeta'_l) \\ &= tx_1 + \frac{t^2}{2}Jx_1 + (e^{tJ} - I)J^{-1}x_2. \end{aligned}$$

Sustituting this into (1.2) and integrating, we obtain

$$\begin{aligned} Z(t) &= \frac{1}{12}[x_1, Jx_1]t^3 + \frac{1}{4}[Jx_1, (e^{tJ} + I)J^{-1}x_2]t^2 \\ &\quad + \{\frac{1}{2}[x_1, (e^{tJ} + I)J^{-1}x_2] - [Jx_1, e^{tJ}J^{-2}x_2 + z_0]\}t \end{aligned}$$

$$+ [x_1, (I - e^{tJ}J^{-2}x_2] + [Jx_1, (e^{tJ} - I)J^{-3}x_2]$$
$$+ \frac{1}{2}[e^{tJ}J^{-1}x_2, x_2] - \frac{1}{2}\int_0^t [e^{uJ}x_2, e^{uJ}J^{-1}x_2]du.$$

Also we have

$$\begin{aligned}
&\int_0^t [e^{uJ}x_2, e^{uJ}J^{-1}x_2]du \\
&= \int_0^t [e^{uJ}(\sum_{k=1}^m \xi_k + \sum_{l=1}^n \zeta_l + \sum_{l=1}^n \zeta_l'), e^{uJ}J^{-1}(\sum_{k=1}^m \xi_k + \sum_{l=1}^n \zeta_l + \sum_{l=1}^n \zeta_l')]du \\
&= \sum_{k=1}^m \sum_{k'=1}^m \int_0^t [e^{uJ}\xi_k, e^{uJ}J^{-1}\xi_{k'}]du \\
&\quad + \sum_{k=1}^m \sum_{l=1}^n \int_0^t [e^{uJ}\xi_k, e^{uJ}J^{-1}(\zeta_l + \zeta_l')]du \\
&\quad + \sum_{k=1}^m \sum_{l=1}^n \int_0^t [e^{uJ}(\zeta_l + \zeta_l'), e^{uJ}J^{-1}\xi_k]du \\
&\quad + \sum_{l=1}^n \sum_{l'=1}^n \int_0^t [e^{uJ}(\zeta_l + \zeta_l'), e^{uJ}J^{-1}(\zeta_{l'} + \zeta_{l'}')]du \\
&= \sum_{k\neq k'=1}^m \int_0^t [e^{uJ}\xi_k, e^{uJ}J^{-1}\xi_{k'}]du + \sum_{k=1}^m [\xi_k, J^{-1}\xi_k]t \\
&\quad + \sum_{k=1}^m \sum_{l=1}^n \int_0^t [e^{uJ}\xi_k, e^{uJ}J^{-1}(\zeta_l + \zeta_l')]du \\
&\quad + \sum_{k=1}^m \sum_{l=1}^n \int_0^t [e^{uJ}(\zeta_l + \zeta_l'), e^{uJ}J^{-1}\xi_k]du \\
&\quad + \sum_{l\neq l'=1}^n \int_0^t [e^{uJ}(\zeta_l + \zeta_l'), e^{uJ}J^{-1}(\zeta_{l'} + \zeta_{l'}')]du - \sum_{l=1}^n \frac{2}{\mu_l}[\zeta_l, \zeta_l']t.
\end{aligned}$$

Thus

$$Z(t) = \sum_{i=0}^3 z_i(t)t^i,$$

where

$$z_3(t) = \frac{1}{12}[x_1, Jx_1],$$

$$z_2(t) = \frac{1}{4}[Jx_1, (e^{tJ} + I)J^{-1}x_2],$$

$$z_1(t) = \frac{1}{2}[x_1, (e^{tJ}+I)J^{-1}x_2] - \frac{1}{2}[Jx_1, e^{tJ}J^{-2}x_2] + z_0 + \sum_{l=1}^{n}\frac{1}{\mu_l}[\zeta_l, \zeta_l'] + \frac{1}{2}\sum_{k=1}^{m}[J^{-1}\xi_k, \xi_k],$$

$$\begin{aligned} z_0(t) &= [x_1, (I-e^{tJ})J^{-2}x_2] - [Jx_1, (I-e^{tJ})J^{-3}x_2] + \frac{1}{2}[e^{tJ}J^{-1}x_2, J^{-1}x_2] \\ &- \frac{1}{2}\sum_{k\neq k'=1}^{m}\frac{1}{(\theta_{k'}^2-\theta_k^2)}([e^{tJ}J\xi_k, e^{tJ}J^{-1}\xi_{k'}] - [e^{tJ}\xi_k, e^{tJ}\xi_{k'}]) \\ &+ \frac{1}{2}\sum_{k\neq k'=1}^{m}\frac{1}{(\theta_{k'}^2-\theta_k^2)}([J\xi_k, J^{-1}\xi_{k'}] - [\xi_k, \xi_{k'}]) \\ &- \sum_{k=1}^{m}\sum_{l=1}^{n}\frac{1}{(\theta_k^2+\mu_l^2)}([e^{tJ}(\zeta_l+\zeta_l'), e^{tJ}\xi_k] - \frac{\mu_l^2-\theta_k^2}{2}[e^{tJ}J^{-1}(\zeta_l+\zeta_l'), e^{tJ}J^{-1}\xi_k]) \\ &+ \sum_{k=1}^{m}\sum_{l=1}^{n}\frac{1}{(\theta_k^2+\mu_l^2)}([\zeta_l+\zeta_l'), \xi_k] - \frac{\mu_l^2-\theta_k^2}{2}[J^{-1}(\zeta_l+\zeta_l'), J^{-1}\xi_k]) \\ &+ \sum_{l\neq l'=1}^{n}\frac{\mu_l}{\mu_{l'}(\mu_l^2-\mu_{l'}^2)}([e^{tJ}\zeta_l, e^{tJ}\zeta_{l'}] + [e^{tJ}\zeta_l', e^{tJ}\zeta_{l'}'] - [\zeta_l, \zeta_{l'}] - [\zeta_l', \zeta_{l'}']). \end{aligned}$$

Corollary 2.1. ([1]) *Let N be a simply connected, 2-step nilpotent Riemannian group with Lie algebra $\mathfrak{n}$ having center Z. Let*

$$\gamma(t) = exp(X(t)+Z(t)),$$

where $X(t)\in Z^\perp$ and $Z(t)\in Z$ for all $t\in R$ be a geodesic in N such that $\gamma(0)=e$ (the identity in N) and $\gamma'(0)=x_0+z_0$, where $x_0\in Z^\perp$ and $z_0\in Z$. Then $X(t)$ and $Z(t)$ are given by

$$X(t) = tx_1 + (e^{tJ}-I)J^{-1}x_2$$

and

$$\begin{aligned} Z(t) &= t(z_0 + \frac{1}{2}[x_1, (e^{tJ}+I)J^{-1}x_2] + \frac{1}{2}\sum_{k=1}^{m}[J^{-1}\xi_k, \xi_k]) \\ &+ [x_1, (I-e^{tJ})J^{-2}x_2] + \frac{1}{2}[e^{tJ}J^{-1}x_2, J^{-1}x_2] \\ &- \frac{1}{2}\sum_{k\neq k'=1}^{m}\frac{1}{(\theta_{k'}^2-\theta_k^2)}([e^{tJ}J\xi_k, e^{tJ}J^{-1}\xi_{k'}] - [e^{tJ}\xi_k, e^{tJ}\xi_{k'}]) \\ &+ \frac{1}{2}\sum_{k\neq k'=1}^{m}\frac{1}{(\theta_{k'}^2-\theta_k^2)}([J\xi_k, J^{-1}\xi_{k'}] - [\xi_k, \xi_{k'}]), \end{aligned}$$

where $x_0 = x_1 + x_2$, $x_1 \in \ker J$ *and* $x_2 = \sum_{k=1}^{m} \xi_k \in \oplus_{k=1}^{m} \ker(J^2 + \theta_k I)$.

Corollary 2.2. *Let* N *be a simply connected, pseudo-Heisenberg group* ($J_z^2 = -\langle z, z\rangle I$ *for every* $z \in Z$) *with Lie algebra* $\mathfrak{n}$ *having center* Z. *Let*

$$\gamma(t) = exp(X(t) + Z(t)),$$

where $x(t) \in Z^{\perp}$ *and* $z(t) \in Z$ *for all* $t \in R$ *be a geodesic in* N *such that* $\gamma(0) = e$ *(the identity in* N *) and* $\gamma'(0) = x_0 + z_0$, *where* $x_0 \in Z^{\perp}$ *and* $z_0 \in Z$. *Then* $X(t)$ *and* $Z(t)$ *are given by*

(1) $\langle z_0, z_0\rangle > 0$

$$X(t) = (e^{tJ} - I)J^{-1}x_0$$

and

$$Z(t) = t(z_0 + \frac{1}{2}[J^{-1}x_0, x_0]) + \frac{1}{2}[e^{tJ}J^{-1}x_0, J^{-1}x_0],$$

(2) $\langle z_0, z_0\rangle < 0$

$$X(t) = (e^{tJ} - I)J^{-1}x_0$$

and

$$Z(t) = t(z_0 + \frac{1}{\sqrt{-\langle z_0, z_0\rangle}}[\zeta, \zeta']) + \frac{1}{2}[e^{tJ}J^{-1}x_0, J^{-1}x_0],$$

where $\zeta \in \ker(J - \sqrt{-\langle z_0, z_0\rangle}I)$ *and* $\zeta' \in \ker(J + \sqrt{-\langle z_0, z_0\rangle}I)$.

(3) $\langle z_0, z_0\rangle = 0$

$$X(t) = tx_0 + \frac{t^2}{2}Jx_0$$

and

$$Z(t) = \frac{t^3}{12}[x_0, Jx_0].$$

Corollary 2.3. *Let* N *be a simply connected, Heisenberg group (Riemannian and* $J_z^2 = -\langle z, z\rangle I$ *for every* $z \in Z$) *with Lie algebra* $\mathfrak{n}$ *having center* Z. *Let*

$$\gamma(t) = exp(X(t) + Z(t)),$$

where $X(t) \in Z^{\perp}$ *and* $Z(t) \in Z$ *for all* $t \in R$ *be a geodesic in* N *such that* $\gamma(0) = e$ *(the identity in* N *) and* $\gamma'(0) = x_0 + z_0$, *where* $x_0 \in Z^{\perp}$ *and* $z_0 \in Z$. *Then* $X(t)$ *and* $Z(t)$ *are given by*

$$X(t) = (e^{tJ} - I)J^{-1}x_0,$$

and

$$Z(t) = t(z_0 + \frac{1}{2}[J^{-1}x_0, x_0]) + \frac{1}{2}[e^{tJ}J^{-1}x_0, J^{-1}x_0].$$

The following proposition gives information for a rank of map J in some class of Pseudoriemannian 2-step nilpotent groups, which is a genralization of proposition 2.2 in [4].

Proposition 2.4. *Let N be a Pseudoriemannian 2-step nilpotent Lie group with nondegenerate center and let $\mathfrak{n} = Z \oplus Z^{\perp}$ be its Lie algebra. Suppose that $J_z^2 = -\langle z, z\rangle A$, where A is a nondegenerate symmetric operator on $Z^{\perp}$ for every $z \in Z$. Then* $\dim Z$ *is even and rank*$J_z = \frac{1}{2}\dim Z$ *for nonzero null z.*

Proof. Let $0 \neq z \in Z$ be null and assume that $J_z x = 0$. Write $z = z_t - z_s$ where z_t is timelike, z_s is spacelike, and $z_t \perp z_s$. By assumption, $J_{z_t}x = J_{z_s}x$. By a standard calculation $\langle J_{z_t}x, J_{z_s}x\rangle = \langle z_t, z_s\rangle\langle Ax, x\rangle = 0$ on the one hand, and $\langle J_{z_t}x, J_{z_s}x\rangle = \langle J_{z_t}x, J_{z_t}x\rangle = \langle z_t, z_t\rangle\langle Ax, x\rangle$ on the other. Hence x is null with respect to a new nondegerate scalar product on $Z^{\perp}$ defined by $<< x, y >>= \langle Ax, y\rangle$ for $x, y \in Z^{\perp}$ and $\ker J_z$ is a null subspace of $Z^{\perp}$ with respect to this product. Thus there exists a complementary null subspace Z'. Consider the subspace $\ker J_z \oplus Z'$. Note that $J_z Z \subseteq \ker J_z$ since $J_z^2 = 0$. Now, $J_z(\ker J_z \oplus Z') = J_z Z'$ and $\dim J_z Z' = \dim Z' = \dim\ker J_z$ so $J_z Z' = J_z Z = \ker J_z$. Therefore $Z = \ker J_z \oplus Z'$ and rank$J_z = \dim Z' = \frac{1}{2}\dim Z$. □

This proposition implies that rank$J_z = \frac{1}{2}\dim Z^{\perp}$ for a nonzero null vector $z \in Z$ for a pseudo-Heigenberg group N.

3. Jacobi operators along geodesics

In this section we will show that the representation matrix of a Jacobi operator is constant along a geodesic in a Pseudoriemannian 2-step nilpotent Lie group with a nondegenerate one dimensional center. Suppose that N is a Pseudoriemannian 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ having one dimensional center Z. We need the following lemma.

Lemma 3.1. *For any x_1 and x_2 in $Z^{\perp}$, we have $[e^{tJ}x_1, e^{tJ}x_2] = [x_1, x_2]$ for every $t \in R$, where $J = J_{z_0}$ for some $z_0 \in Z$.*

Proof. Suppose that z is an orthonormal basis for Z, then we have

$$\begin{aligned}\left[e^{tJ}x_1, e^{tJ}x_2\right] &= \pm\langle [e^{tJ}x_1, e^{tJ}x_2], z\rangle z \\ &= \pm\langle J_z e^{tJ}x_1, e^{tJ}x_2\rangle z = \pm\langle e^{-tJ}J_z e^{tJ}x_1, x_2\rangle z \\ &= \pm\langle J_z x_1, x_2\rangle z = \pm\langle [x_1, x_2], z\rangle z = [x_1, x_2].\end{aligned}$$

□

From (1.3) and this lemma it follows that

$$\begin{aligned}R_{\gamma'(t)}(x_1' + z_1) &= \frac{3}{4}J_{[x_1', x']}x' + \frac{1}{2}J_z J x' - \frac{1}{4}J J_z x' - \frac{1}{4}J^2 x_1' \\ &\quad - \frac{1}{2}[x_1', Jx'] + \frac{1}{4}[x', Jx_1'] + \frac{1}{4}[x', J_z x'] \\ &= \frac{3}{4}J_{[x_1, x_0]}x' + \frac{1}{2}J_z J x' - \frac{1}{4}J J_z x' - \frac{1}{4}J^2 x_1' \\ &\quad - \frac{1}{2}[x_1, Jx_0] + \frac{1}{4}[x_0, Jx_1] + \frac{1}{4}[x_0, J_z x_0]\end{aligned}$$

for a vector $z \in Z$ and $x_1 \in Z^\perp$, where $x' = e^{tJ}x_0$, $x_1' = e^{tJ}x_1$ and $J = J_{z_0}$.

Thus we have

$$\begin{aligned}\langle R_{\gamma'(t)}(x_1' + z_1), x_2' + z_2\rangle = \langle &\frac{3}{4}J_{[x_1,x_0]}x_0 + \frac{1}{2}J_zJx_0 - \frac{1}{4}JJ_zx_0 - \frac{1}{4}J^2x_1, x_2\rangle \\ &+ \langle -\frac{1}{2}[x_1, Jx_0] + \frac{1}{4}[x_0, Jx_1] + \frac{1}{4}[x_0, J_zx_0], z_2\rangle.\end{aligned}$$

for a vectior $z_2 \in Z$ and $x_2 \in Z^\perp$, where $x_2' = e^{tJ}x_2$. We can see that this inner product is independent from the parameter t, which means that this inner product is constant along the geodesic $\gamma(t)$. So we have the following proposition.

Proposition 3.2. *Let N be a Pseudoriemannian 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ having an one dimensional center Z and $x_1 + z_1, x_2 + z_2, \cdots, x_{m+1} + z_{m+1}$ be a basis for T_eN, where $z_i \in Z$ and $x_i \in Z^\perp$. Then the Jacobi operator $R_{\gamma'(t)}$ along a geodesic $\gamma(t)$ in N has a constant representation matrix with respect to a basis $x_1' + z_1, x_2' + z_2, \cdots, x_{m+1}' + z_{m+1}$ for $T_{\gamma(t)}N$, where $x_i' = e^{tJ}x_i$.*

References

[1] P. Eberlein, *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sci. École Norm. Sup. **27** (1994), 611–660.

[2] M. Guederi, *Lorentz Geometry of 2-Step Nilpotent Lie Groups*, Geom. Dedicata **100** (2003), 11–51.

[3] C. Jang and P. E. Parker, *Skewadjoint Operators on Pseudoeuclidean spaces*, preprint.

[4] C. Jang and P. E. Parker, *Pseudo H-type 2-step Nilpotent Lie Groups*, Houston J. Math. **31** (2005), 765–786.

[5] A. Kaplan, *Riemannian nilmanifolds attached to Clifford modules*, Geom. Dedicata **11** (1981), 127–136.

[6] K. Park, *Jacobi operators along geodesics in 2-step nilmanifolds*, Pusan Kyoungnam Math. Jour. **12** (1996), 175–182.

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