J. Appl. Math. & Informatics Vol. 28(2010), No. 1 - 2, pp. 487 - 498
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ON THE FUNCTIONS OF BOUNDED $\kappa\phi$-VARIATIONS(I)

JAEKEUN PARK

ABSTRACT. For some ϕ-sequences ϕ_1, ϕ_2 and ϕ_3, and κ-function κ_1, κ_2 and κ_3 with $\kappa_1^{-1}(x)\kappa_2^{-1}(x) \geq \kappa_3^{-1}(x)$ for $x \geq 0$, the Luxemburg norm is lower semi-continuous on $\kappa\phi BV_0$, and some specialized equivalent conditions are considered.

AMS Mathematics Subject Classification: 26A45, 26A42; Secondary 42A20.
Keywords and phrases: Bounded variation, Bounded ϕ-variation, Bounded κ-variation.

1. Introductions and preliminaries

In defining a function of bounded variation on the closed interval $I_a^b = [a, b]$ we considered the supremum of $\sum \mid f(I_n) \mid$ for every collection $\{I_n\}$ of nonoverlapping subintervals of I_a^b such that $I_a^b = \bigcup I_n$, where $I_n = [x_n, y_n], f(I_n) = f(y_n) - f(x_n)$.

A function f is of bounded variation on I_a^b if $V_a^b(f) = \sup\Big(\sum \mid f(I_n) \mid\Big)$ is finite, that is, if there exists a positive constant c such that, for every collection $\{I_n\}$ of subintervals of I_a^b with $I_a^b = \bigcup I_n$, $\sum \mid f(I_n) \mid \leq c$.

The introduction of the *κ-function* κ can be viewed as a rescaling of length of subintervals in I_a^b such that the length of I_a^b is 1 if $\kappa(1) = 1$. From now on we are requiring that κ has the following properties on a closed interval I_0^1:

(a) κ is continuous with $\kappa(0) = 0$ and $\kappa(1) = 1$,
(b) κ is concave and strictly increasing, and
(c) $\lim_{x \to 0^+} (\kappa(x)/x) = \infty$.[1]

Note that, for κ-functions $\kappa_i (i = 1, 2, 3)$, the product $\kappa_1\kappa_2$ and the composite $\kappa_1 \circ \kappa_2$ are κ-functions and

$$\frac{1}{\kappa(x_1) + \kappa(x_2)} \leq \frac{1}{\kappa(x_1 + x_2)} \leq \frac{1}{\kappa(x_1)} + \frac{1}{\kappa(x_2)} \tag{1}$$

Received June 8, 2009. Accepted October 15, 2009.

for x_1, $x_2 \geq 0$.

We shall say that $\kappa_i (i = 1,2,3)$ satisfy the *Δ_κ-condition (briefly $\kappa_i \in \Delta_\kappa (i = 1,2,3)$)* if κ-functions κ_1, κ_2 and κ_3 satisfy $\kappa_1^{-1}(x)\kappa_2^{-1}(x) \geq \kappa_3^{-1}(x)$ for $x \geq 0$.[6]

Let $\phi = \{\phi_n\}$ be a sequences of convex functions such that $\phi_n : [0,\infty) \to [0,\infty), \phi_n(0) = 0, \phi_n(x) > 0$ for $x > 0$. We say that ϕ is a *ϕ-sequence* if $\phi_{n+1}(x) \leq \phi_n(x)$ for $x \geq 0$ and $\sum \phi_n(x) = \infty$ for all $x > 0$.[8] Note that, for $x_1, x_1 \in I_a^b$, $\phi(x_1) + \phi(x_2) \leq \phi(x_1 + x_2)$ and $\phi^{-1}(x_1 + x_2) \leq \phi^{-1}(x_1) + \phi^{-1}(x_2)$

For ϕ-sequences $\phi_1 = \{\phi_{1n}\}$ and $\phi_2 = \{\phi_{2n}\}$, if there exist two positive constants C and T such that

$$\phi_{1n}(t) \leq \phi_{2n}(Ct) \tag{2}$$

for $t \geq T$ and any n; we write $\phi_1 \prec \phi_2$, *i.e., ϕ_2 dominate ϕ_1 near infinity.* Here if $T = 0$ and $\phi_1 \prec \phi_2$, then ϕ_2 dominate ϕ_1 globally.

If $\phi_1 \prec \phi_2$ and $\phi_2 \prec \phi_1$ hold simultaneously we shall say that ϕ_1, ϕ_2 are equivalent. Note that, for a ϕ-sequence $\phi = \{\phi_n\}$, the function $\phi(t) = \{\phi_n(t)\}$ and $\phi_1(t) = \{\phi_{1n}(t)\} = \{\phi_n(kt)\}$ for $k > 0$ are equivalent.

A ϕ-sequence $\phi = \{\phi_n\}$ is said to satisfy *the Δ_2-condition(the Δ_2-condition globally) denoted $\phi \in \Delta_2$ ($\phi \in \Delta_2$ globally)* if, for $x \geq x_0 \geq 0 (x_0 = 0)$,

$$\phi_n(2x) \leq C\phi_n(x) \tag{3}$$

for any n and some absolute constant C, respectively.

For given I_n in I_a^b, for the simplicity of notations let $I_{n,a}^b = | I_n |/| I_a^b |$.

For a κ-function κ and ϕ-sequence ϕ, a function f is said to be *of bounded variation, of bounded κ-variation, of bounded ϕ-variation* and of *bounded $\kappa\phi$-variation* on I_a^b if the *total variation*:

$$V(f) = V(f : a, b) = \sup\Big\{\sum(| f(I_n) |)\Big\},$$

$$\kappa V(f) = \kappa V(f : a, b) = sup\Big\{\sum(| f(I_n) |)/\sum\kappa(| I_{n,a}^b |)\Big\},$$

$$V_\phi(f) = V_\phi(f : a, b) = \sup\Big\{\sum\phi_n(| f(I_n) |)\Big\},$$

$$\kappa V_\phi(f) = \kappa V_\phi(f : a, b) = \sup\Big\{\sum\phi_n(| f(I_n) |)/\sum\kappa(| I_{n,a}^b |)\Big\}.$$

are respectively finite, where $I_n = [x_n, y_n], | I_n |= y_n - x_n$ and $I_a^b = \bigcup I_n$. The supremum may be taken either over all partitions of I_a^b or over all collections of nonoverlapping subintervals of I_a^b[7, 8, 9].

We denote by BV, κBV, ϕBV and $\kappa\phi BV$ the collection of all bounded, bounded κ-variation, bounded ϕ-variation and bounded $\kappa\phi$-variation on I_a^b, respectively. Also denote by κBV_0, ϕBV_0 and $\kappa\phi BV_0$ the collection of all bounded, bounded κ-variation, bounded ϕ-variation and bounded $\kappa\phi$-variation on I_a^b and $f(a) = 0$.(resp.).[3,4,5,6,8]

Let

$$\kappa BV^* = \Big\{ f \mid f \quad \text{is of} \quad \text{bounded} \quad \kappa\text{-variation} \Big\},$$
$$\phi BV^* = \Big\{ f \mid f \quad \text{is of} \quad \text{bounded} \quad \phi\text{-variation} \Big\},$$
$$\kappa\phi BV^* = \Big\{ f \mid f \quad \text{is of} \quad \text{bounded} \quad \kappa\phi\text{-variation} \Big\} [1,2,8].$$

The space $\kappa\phi BV$ is a Banach space with the norm $\mid f(a) \mid + \mid\mid\mid f - f(a) \mid\mid\mid$.

A function f is said to *$\kappa\phi$-decreasing, κ-decreasing* and *ϕ-decreasing* on I_a^b if there exists a positive constant C such that $\phi_n(\mid f(I) \mid) \leq C\kappa(\mid I_{n,a}^b \mid)$, $\mid f(I) \mid \leq C\kappa(\mid I_{n,a}^b \mid)$ and $\phi_n(\mid f(I) \mid) \leq C(\mid I_{n,a}^b \mid)$ for any interval $I \subset I_a^b$, respectively.

Just as every decreasing function is of bounded variation, we have every κ-decreasing function is of bounded κ-variation. If a function f is $\kappa\phi$-decreasing on I_a^b, then f is of bounded $\kappa\phi$-variation and, for any $a \leq x_0 < b$ and $a < y_0 \leq b$, $f(x_0^+)$ and $f(y_+)$ exist. If a function f is $\kappa\phi$-decreasing on I_a^b, f is of bounded $\kappa\phi$-variation.[2]

2. Some properties of the space $\kappa\phi BV$

For any κ-function κ and ϕ-sequence $\phi = \{\phi_n\}$, the spaces ϕBV_0 and $\kappa\phi BV_0$ are linear over I_a^b, but ϕBV_0^* and $\kappa\phi BV_0^*$ are not, in general.

Theorem 1. *For any κ-function κ and any ϕ-sequence $\phi = \{\phi_n\}$, ϕBV_0^* and $\kappa\phi BV_0^*$ are linear spaces over I_a^b iff $\phi = \{\phi_n\} \in \Delta_2$ globally.*

Proof. Firstly suppose that $\phi = \{\phi_n\} \in \Delta_2$ globally. There exists an $n \in N$ such that $r \leq 2^n$ for nonnegative r. The monotonicity of ϕ-sequences $\phi = \{\phi_n\}$ together with the repeated use of the Δ_2-condition globally imply

$$\phi_n\Big(\mid rf(I_n) \mid \Big) \leq \phi_n\Big(2^n \mid f(I_n) \mid \Big) \leq k^n \phi_n(\mid f(I_n) \mid),$$

which implies that if $f \in \phi BV^*$, then $cf \in \phi BV^*$ for any $c > 0$.

Note that $\phi_1 BV^* \subset \phi_2 BV^*$ if and only if there exists a constant C such that $\phi_{2n}(t) \leq C\phi_{1n}(t)$. Here let $\phi_{1n}(t) = \phi_n(t), \phi_{2n}(t) = \phi_n(rt)$ and $C = k^n$. If $f \in \phi BV^*$, then by $\phi = \{\phi_n\} \in \Delta_2$ globally, we have

$$\sum \phi_n(\mid 2f(I_n) \mid) \leq C \sum \phi_n(\mid f(I_n) \mid) < \infty,$$

which implies that $2f \in \phi BV^*$.

Since $\phi = \{\phi_n\}$ is a ϕ-sequences, for $f_1, f_2 \in \phi_1 BV^*$ we have:

$$\sum \phi_n\Big(\mid (f_1 + f_2)(I_n) \mid \Big) \leq \frac{1}{2} \sum \Big\{ \phi_n(\mid 2f_1(I_n) \mid) + \phi_n(\mid 2f_2(I_n) \mid) \Big\} \leq \infty,$$

which implies that $f_1 + f_2 \in \phi_1 BV^*$, hence ϕBV^* is a linear set.

Conversely suppose that ϕBV^* is a linear spaces. Then for $f \in \phi BV^*$ also $2f \in \phi BV^*$ which means that $\phi BV^* \subset \phi_2 BV^*$ with $\phi_2(t) = \phi(2t)$. Hence by the above mentioned note, it follows that there exists a positive constant C such that $\phi_2(t) = \phi(2t) \le C\phi(t)$, i.e., $\phi = \{\phi_n\} \in \Delta_2$ globally.

By the simillar way as the above, we may be able to prove that $\kappa\phi BV_0^*$ is a linear space over I_a^b if and only if $\phi = \{\phi_n\} \in \Delta_2$ globally.

Let us consider $\kappa V_\phi(cf)$ as a function of variable c. If $\phi = \{\phi_n\}$ is a sequence of increasing convex function, $\phi_n(0) = 0, x \ge 0$, we have $\phi_n(cx) \le c\phi_n(x)$ for $0 \le c \le 1$, and $\phi_n(cx) \ge c\phi_n(x)$ for $c > 1$. Let $\kappa V_\phi(f) < \infty$ and let $0 < c \le 1$. Then $\kappa V_\phi(cf) \le c\kappa V_\phi(f) \to 0$ as $c \to 0$. With this in mind, we define norms as follow [8]:

$$\begin{aligned} &\text{(a)} \quad \text{For} \quad f \in \kappa BV_0, \quad \text{let} \quad |||\, f\, |||_\kappa = \kappa V_a^b(f), \\ &\text{(b)} \quad \text{For} \quad f \in \phi BV_0, \quad \text{let} \quad |||\, f\, |||_\phi = \inf\{c > 0 \mid V_\phi(\frac{f}{c}) \le 1\}, \\ &\text{(c)} \quad \text{For} \quad f \in \kappa\phi BV_0, \quad \text{let} \quad |||\, f\, |||_{\kappa\phi} = \inf\{c > 0 \mid \kappa V_\phi(\frac{f}{c}) \le 1\}. \end{aligned} \tag{4}$$

Theorem 2. *For a κ-function κ and ϕ-sequences $\phi_i = \{\phi_{in}\} (i = 1, 2)$, then the followings are equivalent;*

(i) $\kappa\phi_1 BV_0 \subset \kappa\phi_2 BV_0$;

(ii) $\phi_{2n} \prec \phi_{1n}$ *globally;* (5)

(iii) *there exists a constant C such that*

$$|||\, f\, |||_{\kappa\phi_2} \le C\, |||\, f\, |||_{\kappa\phi_1}.$$

Proof. Clearly that (i) is equivalent to (iii) is trivial. If $\phi_{2n} \prec \phi_{1n}$ globally, then there exists a constant $c > 0$ such that $\phi_{2n}(t) \le \phi_{1n}(ct)$ for any $t \ge 0$. Let $f \in \kappa\phi_1 BV_0$. Then there exists a $\alpha > 0$ such that $\alpha f \in \kappa\phi_1 BV_0$. By (4), we have $\phi_{2n}\Big(\frac{\alpha}{c} \mid f(I_n) \mid\Big) = \phi_{1n}(\alpha \mid f(I_n) \mid)$ and, consequently

$$\frac{\sum \phi_{2n}\Big(\frac{\alpha}{c} \mid f(I_n) \mid\Big)}{\sum \kappa(\mid I_{n,a}^b \mid)} = \frac{\sum \phi_{1n}(\alpha \mid f(I_n) \mid)}{\sum \kappa(\mid I_{n,a}^b \mid)} < \infty,$$

that is, $\frac{\alpha}{c} f \in \kappa\phi_2 BV_0$, which means that $f \in \kappa\phi_2 BV_0$, which implies that $\kappa\phi_1 BV_0 \subset \kappa\phi_2 BV_0$ if $\phi_{2n} \prec \phi_{1n}$ globally.

Let us suppose that condition (4) is not satisfied. Then there exists a strictly increasing sequence $\{t_n\}_{n=1}^\infty$ such that $0 < t_1 < t_2 < t_3 < \cdots$, $\lim_{n\to\infty} t_n = \infty$ and $\phi_{2n}(t_n) > \phi_{1n}(2^n n t_n)$, for any $n \in N$. By the definition of ϕ-sequences, we have:

$$\phi_{1n}(nt_n) = \phi_{1n}\left(\frac{1}{2^n} 2^n n t_n\right) \le \frac{1}{2^n}\phi_{1n}(2^n n t_n),$$

which implies that

$$\phi_{2n}(t_n) > \phi_{1n}(2^n n t_n) > 2^n \phi_{1n}(n t_n).$$

Let us choose a sequence of disjoint nonoverlapping subintervals I_n in I_a^b, $n \in N$ such that

$$\mid I_n \mid = \frac{b-a}{2^n}\frac{\phi_{1n}(t_1)}{\phi_{1n}(nt_n)};$$

this is possible since $\sum \mid I_n \mid < \sum \dfrac{b-a}{2^n} = b-a$. The function f defined by

$$f(x) = \begin{cases} nt_n, & \text{if } x \in I_n, n \in N \\ 0, & \text{otherwise in } I_a^b \end{cases}$$

is an element of the $\kappa\phi_1 BV_0$;

$$\frac{\sum \phi_{1n}(\mid f(I_n) \mid)}{\sum \kappa(\mid I_{n,a}^b \mid)} = \frac{\sum \phi_{1n}(nt_n) \mid I_n \mid}{\sum \kappa(\mid I_{n,a}^b \mid)} = \sum \frac{1}{2^n}\phi_{1n}(t_1) \mid I_a^b \mid < \infty.$$

On the other hand, f is not an element of the space $\kappa\phi_2 BV_0$; for an arbitrary α with $0 < \alpha \leq 1$ we have that $\alpha f \in \kappa\phi_2 BV_0$ because it suffices to show an $m \in N$ such that $\alpha > \frac{1}{m}$ and we have

$$\begin{aligned}\frac{\sum_{n=1} \phi_{2n}(\alpha \mid f(I_n) \mid)}{\sum \kappa(\mid I_{n,a}^b \mid))} &= \frac{\sum_{n=1} \phi_{2n}(\alpha n t_n) \mid I_n \mid}{\sum \kappa(\mid I_{n,a}^b \mid)} \\ &\geq \frac{\sum_{n=m}^{\infty} \phi_{2n}(t_n) \mid I_n \mid}{\sum \kappa(\mid I_{n,a}^b \mid)} \\ &> \frac{\sum_{n=m}^{\infty} 2^n \phi_{1n}(nt_n) \mid I_n \mid}{\sum \kappa(\mid I_{n,a}^b \mid)} \\ &= \frac{\sum_{n=m}^{\infty} 2^n \phi_{1n}(t_1) \mid I_a^b \mid}{\sum \kappa(\mid I_{n,a}^b \mid)} = \infty\end{aligned}$$

Consequently, condition (4) is necessary for the inclusion $\kappa\phi_1 BV_0 \subset \kappa\phi_2 BV_0$ and the theorem is proved.

Lemma 1. [4] *For any κ-function κ and any ϕ-sequence $\phi = \{\phi_n\}$, we have;*

$$\begin{aligned} &(a) \quad \kappa V_\phi(f/\| f \|_{\kappa\phi}) \leq 1, \\ &(b) \quad \text{if} \quad \| f \|_{\kappa\phi} \leq 1, \quad \text{then} \quad \kappa V_\phi(f) \leq \| f \|_{\kappa\phi}, \\ &(c) \quad \text{if} \quad \| f \|_{\kappa\phi} \geq 1, \quad \text{then} \quad \kappa V_\phi(f) \geq \| f \|_{\kappa\phi}. \end{aligned} \tag{6}$$

Proof. (a) Take $k > |||\, f \,|||_{\kappa\phi}$. Then for any finite collection $I = \{I_n\}$, we have;

$$\frac{\sum \phi_n(|\, f(I_n) \,|/k)}{\sum \kappa(|\, I^b_{n,a} \,|)} \le \kappa V_\phi\Big(\frac{f}{k}\Big) \le 1.$$

Hence, by Fatou's lemma we have:

$$\kappa V_\phi\left(\frac{f}{|||\, f \,|||_{\kappa\phi}}\right) = \sup\left\{\frac{\sum \phi_n(\lim_{k\to|||f|||_{\kappa\phi}} |\, f(I_n)/k \,|)}{\sum \kappa(|\, I^b_{n,a} \,|)}\right\}$$
$$= \sup \lim_{k\to|||f|||_{\kappa\phi}} \left\{\frac{\sum \phi_n(|\, f(I_n)/k \,|)}{\sum \kappa(|\, I^b_{n,a} \,|)}\right\} \le 1$$

(b) For any finite collection $I = \{I_n\}$, since $|||\, f \,|||_{\kappa\phi} \le 1$ and $\phi_n(\alpha t) \le \alpha\phi_n(t)$ for $\alpha \in I^1_0$, letting $\alpha = |||\, f \,|||_{\kappa\phi} \le 1$ and $t = \dfrac{|\, f(I_n) \,|}{|||\, f \,|||_{\kappa\phi}}$, then we have:

$$\frac{\sum \phi_n(|\, f(I_n) \,|)}{\sum \kappa(|\, I^b_{n,a} \,|)} \le |||\, f \,|||_{\kappa\phi} \frac{\sum \phi_n(|\, f(I_n) \,| /|||\, f \,|||_{\kappa\phi})}{\sum \kappa(|\, I^b_{n,a} \,|)}$$
$$\le |||\, f \,|||_{\kappa\phi} \kappa V_\phi\left(\frac{f}{|||\, f \,|||_{\kappa\phi}}\right) \le |||\, f \,|||_{\kappa\phi}.$$

(c) Since $|||\, f \,|||_{\kappa\phi} \ge 1$ and $\phi_n(\beta t) \ge \beta\phi_n(t)$ for $\beta > 1$, letting $\beta = |||\, f \,|||_{\kappa\phi} - \varepsilon$ and $t = \dfrac{|\, f(I_n) \,|}{|||\, f \,|||_{\kappa\phi} - \varepsilon}$ for sufficiently small ε, then

$$\frac{\sum \phi_n(|\, f(I_n) \,|)}{\sum \kappa(|\, I^b_{n,a} \,|)} \ge (|||\, f \,|||_{\kappa\phi} - \varepsilon) \frac{\sum \phi_n(|\, f(I_n) \,| /(|||\, f \,|||_{\kappa\phi} - \varepsilon))}{\sum \kappa(|\, I^b_{n,a} \,|)}$$
$$\ge (|||\, f \,|||_{\kappa\phi} - \varepsilon)\kappa V_\phi\left(\frac{|\, f(I_n) \,|}{|||\, f \,|||_{\kappa\phi} - \varepsilon}\right) \ge |||\, f \,|||_{\kappa\phi} - \varepsilon,$$

where the latter inequality follows from the definition of the Luxemburg norm. Since $\varepsilon > 0$ is arbitrary the assertion (c) is proved, which completes the proof. □

Lemma 2. [3] *If a function f is $\kappa\phi$-decreasing on I^b_a, then for any $a \le x_0 < b$ and $a < y_0 \le b$, $f(x_0^+)$ and $f(y_{0-})$ exist.*

Corollary 1. *If a function f is κ-decreasing, ϕ-decreasing or $\kappa\phi$-decreasing on I^b_a, then f is of bounded κ-variation, bounded ϕ-variation or bounded $\kappa\phi$-variation on I^b_a, respectively.*

Corollary 2. [3] *If a function f is $\kappa\phi$-decreasing on I^b_a, then*

(*i*) *f is a continuous function on I^b_a;*

(*ii*) *f is of bounded $\kappa\phi$-variation on I^b_a.*

Proof. Let $a \le x \le x_0 < y \le b$. Then $\phi_1(| f(I_{x_0}^x) |) \le \kappa(| I_{x_0}^x |/| I_a^b |)$ and $\phi_1(| f(I_{x_0}^y) |) \le \kappa(| I_{x_0}^y |/| I_a^b |)$. Letting $x \to x_0$ and $y \to x_0$, show $\phi_1(| f(x^+) - f(x_0) |) \le 0$ and $\phi_1(| f(x_-) - f(x_0) |) \le 0$. By the above lemma, $f(x_0^+) = f(x_0) = f(x_{0-})$, and by the definition of $\kappa\phi$-decreasing function, f is of bounded $\kappa\phi$-variation on I_a^b.

Theorem 3. *If a function f is κ-decreasing or ϕ-decreasing on I_a^b, then*
(*i*) *f is bounded on I_a^b,*
(*ii*) *f is of bounded $\kappa\phi$-variation on I_a^b,*
(*iii*) *${}_\kappa V_\phi(f) \le V_\phi(f)$ on I_a^b.*

Proof. If f is of bounded κ-variation on I_a^b, then there is a positive constant C such that, for any $I = \{I_n\}$, $\sum | f(I_n) | \le C \sum \kappa(| I_{n,a}^b |)$.

If $\phi = \{\phi_n\}$ is a ϕ-sequence, then for any M, there is a positive k such that $\phi_1(k) = M$ and $\phi_n(x) \le Mx$ for all $x \le k$.

Now if f is of bounded κ-variation on I_a^b,, then f is bounded, i.e., $| f(x) | \le M/2$. So,

$$\sum \phi_n(f(I_n)) \le \sum M \mid f(x) \mid \le CM \sum \kappa(| I_{n,a}^b |),$$

Hence f is of bounded $\kappa\phi$-variation.

For any x, $a \le x \le b$, consider $\{I_n\} = \{I_1, I_2\} = \{I_a^x, I_x^b\}$, then

$$\frac{\phi_1(| f(I_a^x) |)}{\kappa\Big(| I_a^x |/| I_a^b |\Big) + \kappa\Big(| I_x^b |/| I_a^b |\Big)} \le \frac{\phi_1(| f(I_a^x) |) + \phi_2(| f(I_x^b) |)}{\kappa\Big(| I_a^x |/| I_a^b |\Big) + \kappa\Big(| I_x^b |/| I_a^b |\Big)} \le \kappa V_\phi(f),$$

thus $\phi_1(| f(I_a^x) |) \le 2{}_\kappa V_\phi(f)$. Since $\phi_1(x) \to \infty$ as $x \to \infty$, f is bounded.

If f is of bounded ϕ-variation on I_a^b, then there exists a positive constant M such that, for any $I = \{I_n\}$,

$$\sum \phi_n(| f(I_n) |) \le M = M\kappa(1) \le M \sum \kappa(| I_{n,a}^b |),$$

which implies that f is of bounded $\kappa\phi$-variation and ${}_\kappa V_\phi(f) \le V_\phi(f)$.

Lemma 3. *For $f \in \phi BV_0$, letting*

$$\| f \|_\phi = \inf_{k>0} \frac{1}{k} \left\{ 1 + \sum_n \phi_n(k \mid f(I_n) \mid) \right\}, \tag{7}$$

then

$$||| f |||_\phi \le \| f \|_\phi \le 2||| f |||_\phi,$$

that is, the norms $||| \cdot |||_\phi$ and $\| \cdot \|_\phi$ are equivalent.

Proof. If we let k tend to $\| f \|_\phi$ in $\sum_n \phi_n\Big(\frac{1}{k} \mid f(I_n) \mid\Big) \le 1$, then we obtain, by Fatou's lemma and (6) lemma 1, $\sum_n \phi_n\Big(\frac{1}{\| f \|_\phi} \mid f(I_n) \mid\Big) \le 1$, which implies that $||| f |||_\phi \le \| f \|_\phi$.

Let $w = \dfrac{f}{|||\, f\, |||_\phi}$. Then since $\sum_n \phi_n\Big(\dfrac{|\, f(I_n)\,|}{|||\, f\, |||_{\;\phi}}\Big) \leq 1$, we have $\|\, w\,\|_\phi \leq \sum_n \phi_n(|\, w(I_n)\,|) + 1 \leq 2$, which implies that $\|\, f\,\|_\phi \leq 2|||\, f\,|||_\phi$. Hence the assertion holds.

Theorem 4. $\Big(\kappa\phi BV_0, |||\cdot|||_{\kappa\phi}\Big)$ *is a normed linear spaces when equivalent functions are identified. Moreover,*

$$|||\, f\,|||_{\kappa\phi} \leq 1 \quad \textit{if and only if} \quad \kappa V_\phi(f) \leq |||\, f\,|||_{\kappa\phi}$$

Proof. Since ϕ_n is increasing for each n, $\kappa V_\phi(f/t_2) \leq \kappa V_\phi(f/t_1)$ if $t_1 \leq t_2$.

Thus $\kappa V_\phi\Big(f/|||\, f\,|||_{\kappa\phi}\Big) \leq 1$, we have; $\Big\{t \mid \kappa V_\phi(f/t) \leq 1\Big\} = \Big[|||\, f\,|||_{\kappa\phi}, \infty\Big)$. Clearly $|||\, 0\,|||_{\kappa\phi} = 0$.

If $f \neq 0$, let $x \in I_a^b$ be such that $f(x) \neq 0$, then, by $V_\phi(f) = \phi_1(|\, f(I_a^b)\,|)$, for suitable $I = \{I_n\}$ we have;

$$\kappa V_\phi\left(\frac{f}{t}\right) = \sup \frac{\phi_1(|\, f(I_a^b)/t\,|)}{\sum \kappa(|\, I_{n,a}^b\,|)} = \frac{\phi_1(|\, f(I_n)/t\,|)}{\sum \kappa(|\, I_{n,a}^b\,|)} \longrightarrow \infty$$

as $t \longrightarrow 0$. Thus there is $t > 0$ such that $\kappa V_\phi(f/t) > 1$, which implies that $|||\, f\,|||_{\kappa\phi} \neq 0$.

For $c \neq 0$,

$$\begin{aligned} |||\, cf\,|||_{\kappa\phi} &= \inf\Big\{t > 0 \mid \kappa V_\phi(cf/t) \leq 1\Big\} \\ &= |\, c\,| \inf\Big\{s > 0 \mid \kappa V_\phi(f/s) \leq 1\Big\} = |\, c\,|\; |||\, f\,|||_{\kappa\phi}. \end{aligned}$$

Now for $f_1, f_2 \in \kappa\phi BV_0$, let $a_i > |||\, f_i\,|||_{\kappa\phi}, i = 1, 2$. Then $0 < a_i < \infty$, and let $b = a_1 + a_2 > 0$. Since $f_1 + f_2 \in \kappa\phi BV_0$, $|||\, f_1 + f_2\,|||_{\kappa\phi} < \infty$. Consider

$$\begin{aligned} &\sum \phi_n\left(\frac{|\,(f_1+f_2)(I_n)\,|}{b}\right) \\ &\leq \sum \phi_n\left(\frac{a_1}{b}\frac{|\, f_1(I_n)\,|}{a_1} + \frac{a_2}{b}\frac{|\, f_2(I_n)\,|}{a_2}\right) \\ &\leq \frac{a_1}{b}\sum \phi_n\left(\frac{|\, f_1(I_n)\,|}{a_1}\right) + \frac{a_2}{b}\sum \phi_n\left(\frac{|\, f_2(I_n)\,|}{a_2}\right) \\ &\leq \sum \kappa(|\, I_{n,a}^b\,|). \end{aligned}$$

Letting $a_i \to |||\, f_i\,|||_{\kappa\phi}, i = 1, 2$. Then $|||\, f_1 + f_2\,|||_{\kappa\phi} \leq |||\, f_1\,|||_{\kappa\phi} + |||\, f_2\,|||_{\kappa\phi}$.

For the last part, let $a = |||\, f\,|||_{\kappa\phi}, f \in \kappa\phi BV_0$ and $a > 0$, since$a = 0$ is the trivial case. Then by definition, $f/a \in \kappa\phi BV_0$. If $a \leq 1$, then

$$\kappa V_\phi(f) \leq \kappa V_\phi(f/a) \leq 1 \tag{8}$$

so that $|||\, cf\,|||_{\kappa\phi} \leq 1$ implies the left side of (8) is bounded by 1. On the other hand, if $f \in \kappa\phi BV_0$ then $|||\, f\,|||_{\kappa\phi} \leq 1$ hold. Note that if $a > 1$, then

$\kappa V_\phi(f/a) \leq 1$ but $\kappa V_\phi(f) = \infty$ is possible. Thus only $0 \leq a \leq 1$ is relevant here, which completes the proof.

Corollary 3. *$\left(\kappa\phi BV_0, ||| f |||_{\kappa\phi}\right)$, $\left(\kappa BV_0, ||| f |||_{\kappa}\right)$, and $\left(\phi BV_0, ||| f |||_{\phi}\right)$ are respectively Banach space.*

Proof. Let f and g be functions in $\kappa\phi BV_0$ such that $||| f - g |||_{\kappa\phi} \leq \varepsilon$. Then $||| f - g |||_{\kappa\phi} \leq 1$, so, by part (b) of the Lemma 1,

$$\kappa V_\phi\Big((f-g)/\varepsilon\Big) \leq ||| f - g |||_{\kappa\phi} \leq 1.$$

Now for $x \in I_a^b$ and $I = \{I_a^x, I_x^b\}$, we have;

$$\phi_1\left(\Big| \frac{f(x)-g(x)}{\varepsilon} \Big|\right) \leq \kappa V_\phi\left(\frac{f-g}{\varepsilon}\right)\left(\kappa(\frac{| I_a^x |}{| I_a^b |}) + \kappa(\frac{| I_x^b |}{| I_a^b |})\right) \leq 2,$$

that is, $| f(x) - g(x) | \leq \varepsilon\phi_1^{-1}(2)$.

Using this, we see that if $\{f_n\}$ is a Cauchy sequence in this Luxemburg norm $||| \cdot |||_{\kappa\phi}$, it is also a Cauchy sequence in the supremum norm, so that there exists a function f such that f_n converges to f uniformly. Hence there exists an $n_0 = n_0(\varepsilon) \in N$ such that, for every $n > n_0$ and $m > n_0$, $||| f_n - f_m |||_{\kappa\phi} \leq \varepsilon$.

Since $f_n(a) = 0$, we have $f(a) = 0$.

Let $\varepsilon > 0$ be given and let$I = \{I_n\}$ be a finite collection of nonoverlapping subintervals of I_a^b. Then, by Fatou's lemma, we have:

$$\begin{aligned}\sum_k \phi_k\Big(| (f_n - f)(I_k) | /\varepsilon\Big) &= \sum_k \lim_{m\to\infty} \phi_k(| (f_n - f)(I_k) | /\varepsilon)\\ &= \lim_{m\to\infty} \sum_k \phi_k\Big(| (f_n - f)(I_k) | /\varepsilon\Big) \leq \sum \kappa(| I_{k,a}^b |),\end{aligned}$$

which means that $\kappa V_\phi((f_n - f)/\varepsilon) \leq 1$. Hence $||| f_n - f ||| < \varepsilon$ for sufficiently large n and $f \in \kappa\phi BV_0$, i.e., f_n and $f_n - f \in \kappa\phi BV_0$ and, consequently, also $f = f_n - (f_n - f) \in \kappa\phi BV_0$, which implies that

$$\lim_n ||| f_n - f |||_{\kappa\phi} = 0,$$

which completes the proof.

Theorem 5. *If the functions $f\Big(| f | \geq a > 0\Big)$ and g are in $\kappa\phi BV_0$ on I_a^b, then the function $\frac{1}{f}$ and the product $h = fg$ are in $\kappa\phi BV_0$.*

Proof. For some positive constant c the function cf is of bounded $\kappa\phi$-variation for any closed interval $I = \{I_n\} = \{I_{x_{n-1}}^{x_n}\}$. Hence we have;

$$\sum \phi_n\left(\Big| \frac{ca^2}{f}(I_n) \Big|\right) = \sum \phi_n\left(ca^2 \Big| \frac{f(I_n)}{f(x_n)f(x_{n-1})} \Big|\right) \leq \sum \phi_n(| cf(I_n) |),$$

that is, the function ca^2/f is in $\kappa\phi BV_0^*$, which implies that the function $1/f$ is in $\kappa\phi BV_0$.

To prove the second part, let the functions $f(\mid f \mid \geq a > 0)$ and g be in $\kappa\phi BV_0$. Then there are some positive constants c_1 and c_2 such that functions $c_1 f$ and $c_2 g$ are of bounded $\kappa\phi$-variation and $\kappa V_\phi(f) < \infty, \kappa V_\phi(g) < \infty$.

So, since $\phi_n(x) \to \infty$ as $x \to \infty$, we have $\phi_1(\mid f(I_x^y) \mid) \leq \kappa V_\phi(f)$ and $\phi_1(\mid g(I_x^y) \mid) \leq \kappa V_\phi(g)$, that is, f and g are bounded.

Let $a > \mid\mid\mid f \mid\mid\mid_{\kappa\phi}$ and $b > \mid\mid\mid g \mid\mid\mid_{\kappa\phi}$. Then $0 < a < \infty$ and $0 < b < \infty$, and let $k = a + b > 0$ and $c = \min\{a, b\}$. For any closed interval $I = \{I_n\} = \{I_{x_{n-1}}^{x_n}\}$,

$$\begin{aligned}
\sum_n \phi_n\left(\frac{c}{k} \mid (fg)(I_n) \mid\right) &= \sum_n \phi_n\left(\frac{c}{k} \mid f(x_n)g(x_n) - f(x_{n-1})g(x_{n-1}) \mid\right) \\
&\leq \sum_n \phi_n\left(\frac{c}{k} \mid f(I_n)g(x_n) \mid + \frac{c}{k} \mid f(x_{n-1})g(I_n) \mid\right) \\
&\leq \sum_n \phi_n\left(\frac{a}{k} \mid bf(I_n) \mid + \frac{b}{k} \mid ag(I_n) \mid\right) \\
&\leq \sum_n \phi_n\left(\frac{b}{k} \mid af(I_n) \mid + \frac{a}{k} \mid bg(I_n) \mid\right) \\
&\leq \frac{b}{k} \sum_n \phi_n(\mid af(I_n) \mid) + \frac{a}{k} \sum_n \phi_n(\mid bg(I_n) \mid) < \infty,
\end{aligned}$$

which implies that $\frac{c}{k} fg$ is in $\kappa\phi BV_0^*$. Thus the product fg is in $\kappa\phi BV_0$.

Note that if $\varepsilon < 1$ and $\mid\mid\mid f_n - f \mid\mid\mid_{\kappa\phi} < \varepsilon$, then $\kappa V_\phi(f_n - f) < \varepsilon$, that is, if $\varepsilon < 1$ and $\mid\mid\mid (f_n - f)/\varepsilon \mid\mid\mid_{\kappa\phi} < 1$, then $\kappa V_\phi((f_n - f)/\varepsilon) \leq 1$.

Since convergence in norm implies uniform convergence, the continuous functions form a closed subset of $\kappa\phi BV_0$ and so $\kappa\phi BV_0 \bigcap C$ is itself a Banach space. Also note that $\kappa\phi BV$ is a Banach space with norm $\mid\mid\mid f \mid\mid\mid_{\kappa\phi} + \mid f(a) \mid$.

Theorem 6. *For a ϕ-sequences $\phi = \{\phi_n\}$ and a κ-function κ, if $\phi = \{\phi_n\} \in \Delta_2$(globally), then $\kappa\phi BV_0{}^* = \kappa\phi BV_0$ and $\phi BV_0{}^* = \phi BV_0$(resp.).*

Proof. For a ϕ-sequences $\phi = \{\phi_n\}$ and a κ-function κ, $\phi BV_0{}^* \subset \phi BV_0$ and $\kappa\phi BV_0{}^* \subset \kappa\phi BV_0$, in general.

To show that $\phi BV_0{}^* \supset \phi BV_0$, let $f \in \phi BV_0$ and $\mid\mid\mid f \mid\mid\mid_\phi \neq 0$. Then $\sum \phi_n(\mid f(I_n) \mid / \mid\mid\mid f \mid\mid\mid_\phi) \leq 1$ implies $g = f / \mid\mid\mid f \mid\mid\mid_\phi \in \phi BV_0{}^*$. If $\phi = \{\phi_n\} \in \Delta_2$(globally), then $\phi BV_0{}^*$ is a linear spaces, which implies that $\mid\mid\mid f \mid\mid\mid_\phi g = f \in \phi BV_0{}^*$ and hence $\phi BV_0 \subset \phi BV_0{}^*$.

To show that $\kappa\phi BV_0{}^* \supset \kappa\phi BV_0$. let $f \in \kappa\phi BV_0$ and $\mid\mid\mid f \mid\mid\mid_{\kappa\phi} \neq 0$. Then

$$\sum \phi_n(\mid f(I_n) \mid / \mid\mid\mid f \mid\mid\mid_{\kappa\phi}) / \sum \kappa(\mid I_n \mid / <\mid I_a^b \mid) \leq 1$$

implies $g = f / \mid\mid\mid f \mid\mid\mid_{\kappa\phi} \in \kappa\phi BV_0{}^*$. If $\phi = \{\phi_n\} \in \Delta_2$(globally), then $\kappa\phi BV_0{}^*$ is a linear spaces, which implies that $\mid\mid\mid f \mid\mid\mid_{\kappa\phi} g = f \in \kappa\phi BV_0{}^*$ and hence $\kappa\phi BV_0 \subset \kappa\phi BV_0{}^*$.

By the simillar way as the above, also we have $\phi BV_0{}^* = \phi BV_0$, which completes the proof.

Theorem 7. *Let $\{f_n\} \in \kappa\phi BV_0$ be a sequence such that f_n converges to f almost everywhere. with $f \in \kappa\phi BV_0$, and $\phi_n(x) = 0$ iff $x = 0$. Then*

$$||| f |||_{\kappa\phi} \leq \lim_{n\to\infty} \inf ||| f_n |||_{\kappa\phi},$$

that is, the Luxemburg norm is lower semi-continuous on $\kappa\phi BV_0$.

Proof. If $f = 0$ a.e. on I_a^b, the result is trivial. So we may assume that $f \neq 0$ a.e. on I_a^b.

Thus $||| f |||_{\kappa\phi} > 0$ so that $||| f_n |||_{\kappa\phi} > 0$ for all sufficiently large n.

If $k_0 = \lim_{n\to\infty} \inf ||| f_n |||_{\kappa\phi} = \infty$, the result is again true. We may suppose that $0 \leq k_0 < \infty$. But k_0 implies that there exists a subsequence $\{f_{n_i}\}$ such that $\lim_{i\to\infty} ||| f_{n_i} |||_{\kappa\phi} = 0$. Consequently from some i_0, using the convexity of $\phi = \{\phi_n\}$, we have; $||| f_{n_i} |||_{\kappa\phi} \leq 1$, $i \geq i_0$ and

$$\frac{1}{||| f_{n_i} |||_{\kappa\phi}} \sum \phi_n(| f_{n_i}(I_n) |) \leq \sum \phi_n \left(\frac{| f_{n_i}(I_n) |}{||| f_{n_i} |||_{\kappa\phi}} \right) \leq \sum \kappa(| I_{n,a}^b |).$$

Hence

$$\sum \phi_n(| f_{n_i}(I_n) |) / \sum \kappa(| I_{n,a}^b |) \leq ||| f_{n_i} |||_{\kappa\phi} \to 0 \quad \text{as} \quad i \to \infty.$$

Since $| f_{n_i} | \to | f |$ a.e., by Fatou's lemma and the fact that $\phi_n(x) > 0$ for $x > 0$ we have:

$$\begin{aligned} 0 < \frac{\sum \phi_n(| f(I_n) |)}{\sum \kappa(| I_{n,a}^b |)} &= \frac{\sum \lim_{i\to\infty} \phi_n(| f(I_{n_i}) |)}{\sum \kappa(| I_{n,a}^b |)} \\ &\leq \lim_{i\to\infty} \inf \frac{\sum \phi_n(| f_{n_i}(I_n) |)}{\sum \kappa(| I_{n,a}^b |)} = 0. \end{aligned}$$

This is a contradiction, which implies that $k_0 \neq 0$, that is, $0 < k_0 < \infty$. Finally let $0 < k_0 < t$ be arbitrary. Then $k_0 < k_i < t$ for some i so that

$$\begin{aligned} \frac{\sum \phi_n(| f(I_n) | /t)}{\sum \kappa(| I_{n,a}^b |)} &= \frac{\sum \lim_{i\to\infty} \phi_n(| f(I_{n_i}) | /t)}{\sum \kappa(| I_{n,a}^b |)} \\ &\leq \frac{\lim_{i\to\infty} \inf \sum \phi_n(| f(I_{n_i}) | /t)}{\sum \kappa(| I_{n,a}^b |)} \\ &\leq \frac{\lim_{i\to\infty} \inf \sum \phi_n(| f(I_{n_i}) | /k_i)}{\sum \kappa(| I_{n,a}^b |)} \leq 1. \end{aligned} \tag{9}$$

But then the definition of $||| \cdot |||_{\kappa\phi}$ and (9) imply that $||| f |||_{\kappa\phi} \leq t$. Since $t > k_0$ is arbitrary, we get

$$||| f |||_{\kappa\phi} \leq k_0 = \lim_{n\to\infty} \inf ||| f_n |||_{\kappa\phi},$$

so that the Luxemburg norm $|||\cdot|||_{\kappa\phi}$ is lower semi-continuous, which completes the proof.

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Jaekeun Park received his BS and MS degrees from Seoul National University and his Ph. D degree in Choongang University. Since 1977 he is a professor in the Faculty of Department of Mathematics in Korea Air Force Academy and Hanseo University, Korea. His research interests focus on the function spaces of ultradistributions and Fuzzy sets.

Department of Mathematics, Hanseo University, Seosan 356-706, Korea
e-mail: jkpark@anseo.ac.kr