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APPROXIMATION OF SOLUTIONS FOR GENERALIZED WIENER-HOPF EQUATIONS AND GENERALIZED VARIATIONAL INEQUALITIES

GUANGHUI GU AND YONGFU SU*

ABSTRACT. The purpose of this article is to introduce a new generalized class of the Wiener-Hopf equations and a new generalized class of the variational inequalities. Using the projection technique, we show that the generalized Wiener-Hopf equations are equivalent to the generalized variational inequalities. We use this alternative equivalence to suggest and analyze an iterative scheme for finding the solution of the generalized Wiener-Hopf equations and the solution of the generalized variational inequalities. The results presented in this paper may be viewed as significant and improvement of the previously known results. In special, our results improve and extend the resent results of M.A. Noor and Z.Y.Huang[M.A. Noor and Z.Y.Huang, Wiener-Hopf equation technique for variational inequalities and nonexpansive mappings, Appl. Math. Comput.(2007), doi:10.1016/j.amc.2007.02.117].

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1. Introduction and preliminaries

Variational inequalities introduced by Stampacchia [17] in the early sixties have enjoyed vigorous growth for the last thirty years. Variational inequality theory describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics and engineering sciences [1-20]. A general variational inequality is introduced by Noor in 1988 [1]. It turned out that the odd-order and nonsymmetric free, moving, unilateral obstacle and equilibrium can be studied via the general variational inequality approach. Projection methods and their variant forms including the Wiener-Hopf equations are being used to develop various numerical methods for solving variational inequalities. The purpose of this article is to introduce a new

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generalized class of the Wiener-Hopf equations and a new generalized class of the variational inequalities. Using the projection technique, we show that the generalized Wiener-Hopf equations are equivalent to the generalized variational inequalities. We use this alternative equivalence to suggest and analyze an iterative scheme for finding the solution of the generalized Wiener-Hopf equations and the solution of the generalized variational inequalities. The results presented in this paper may be viewed as significant and improvement of the previously known results. In special, our results improve and extend the resent results of M.A. Noor and Z.Y.Huang[20] in several respects:

(i) The results of this paper improve and extend the results in [20];

(ii) We have used more meticulous method of proofs;

(iii) We have modified some fuzzy contents in [20].

Let K be a nonempty closed and convex subset in a Hilbert space H, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let $T : H \to H$ be a nonlinear operator, $S : K \to K$ and $A : H \to H$ be two nonexpansive mappings. Let P_K be the metric projection of H onto the K.

In 1964. Stampacchia [17] considered the problem of finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall\, v \in K, \tag{1}$$

which is known as the classical variational inequality. It is obvious that, the (1) is equivalent to

$$\langle \rho Tu, v-u\rangle \geq 0, \quad \forall\, v \in K \tag{2}$$

where $\rho > 0$ is a any positive real constant.

Related to the variational inequality (1), the following original Wiener-Hopf equation introduced and studied by Shi [16]. Shi considered the problem of finding $z \in H$ such that

$$TP_K z + \rho^{-1} Q_K z = 0, \tag{3}$$

where $Q_K = I - P_K$. It is well known that, the classical variational inequality (1) is equivalent to the original Wiener-Hopf equation (3).

The Wiener-Hopf equation technique has been used to study the sensitivity analysis and asymptotical stability of the variational inequalities, see [2,5,6,15, 16, 20]. It has been shown that, the Wiener-Hopf equation technique is more flexible and general than the projection method and its variant form.

Very resent, in 2007 Noor and Huang [20] introduced and studied the following Wiener-Hopf equation which involving a nonexpansive mapping $S : K \to K$. They considered the problem of finding $z \in H$ such that

$$TSP_K z + \rho^{-1} Q_K z = 0. \tag{4}$$

They also considered the problem of finding the fixed points of S with together the Wiener-Hopf equation (4) and variational inequality (1).

In this paper, we consider a new generalized variational inequality problem of finding $u \in K$ such that

$$\langle u - A[u - \rho Tu], v-u\rangle \geq 0, \quad \forall\, v \in K, \ \rho > 0, \tag{5}$$

which involving the nonexpansive mapping A. When $A = I$ is the identity operator, the new generalized variational inequality (5) reduce the special form (1). Related to the variational inequality (5), in this paper, we also consider a new generalized Wiener-Hopf equation problem of finding $z \in H$ such that

$$z = A[SP_K z - \rho TSP_K z], \ \rho > 0, \tag{6}$$

where $S : K \to K$ is a nonexpansive mapping. When $S = I$ is the identity operator, the above generalized Wiener-Hopf equation (6) reduce the following form

$$z = A[P_K z - \rho TP_K z]. \ \rho > 0. \tag{7}$$

When $A = I$ is the identity operator and let $Q_K = I - SP_K$, the Wiener-Hopf equation (7) reduce the original Wiener-Hopf equation (3) which was introduced and studied by Shi [16].

We shall show that, the generalized variational inequality (5) is equivalent to the generalized Wiener-Hopf equation (6). This equivalence has played a fundamental and basic role in developing some efficient and robust methods for solving variational inequalities and related optimization problems.

Definition 1.1. A mapping $T : H \to H$ is called $\mu-$Lipschitzian if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \le \mu\|x - y\|, \quad \forall \ x, y \in H.$$

Definition 1.2. A mapping $T : H \to H$ is called $\alpha-$inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha\|Tx - Ty\|^2, \quad \forall \ x, y \in H.$$

Definition 1.3. A mapping $T : H \to H$ is called $r-$strongly monotone if there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge r\|x - y\|^2, \quad \forall \ x, y \in H.$$

Definition 1.4. A mapping $T : K \to H$ is called relaxed $(\gamma, r)-$cocoercive if there exists a constants $\gamma > 0, r \ge 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge (-\gamma)\|Tx - Ty\|^2 + r\|x - y\|^2, \quad \forall \ x, y \in K.$$

Remark. Clearly a $r-$strongly monotone mapping must be relaxed (γ, r) - cocoercive mapping, or a $\gamma-$inverse strongly monotone must be a relaxed (γ, r) - cocoercive mapping whenever $r = 0$, but the converse is not true. Therefore the class of the relaxed $(\gamma, r)-$cocoercive mappings is most general class, and hence Definition 1.6 both Definition 1.4 and Definition 1.5 as special cases.

The following Lemma is well known.

Lemma 1.5. *Suppose $\{a_n\}$ is a nonnegative real sequence satisfying the following condition*

$$a_{n+1} \le (1 - \lambda_n)a_n,$$

with $\lambda_n \in [0, 1]$, $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then $a_n \to 0$ as $n \to \infty$.

2. Main results

Theorem 2.1.

(i) If element $u \in K$ is a solution of the generalized variational inequality (5), then $z = A[u - \rho Tu]$ satisfies the Wiener-Hopf equation (7) and $u = P_K z$;

(ii) If $z \in H$ satisfies the Wiener-Hopf equation (7), then $u = P_K z$ is a solution of the generalized variational inequality(5).

Proof. (i) If $u \in K$ is a solution of the generalized variational inequality (5), then

$$u = P_K A[u - \rho Tu],$$

which leads to $u = P_K z$. Hence

$$z = A[u - \rho Tu] = A[P_K z - \rho T P_K z].$$

Namely, $z = A[u - \rho Tu]$ satisfies the Wiener-Hopf equation (7). In addition, it is easy to see that, u is a solution of the generalized variational inequality (5) if and only if $u = P_K A[u - \rho Tu]$. This together with $z = A[u - \rho Tu]$ implies $u = P_K z$.

(ii) If $z \in H$ satisfies the Wiener-Hopf equation (7), then we have

$$P_K z = P_K A[P_K z - \rho T P_K z].$$

Let $u = P_K z$, hence

$$u = P_K A[u - \rho Tu]$$

It is obvious that, u is a solution of the generalized variational inequality (5). This completes the proof. □

Corollary 2.2.

(i) If element $u \in K$ is a solution of the variational inequality (1), then $z = [u - \rho Tu]$ satisfies the Wiener-Hopf equation (3);

(ii) If $z \in H$ satisfies the Wiener-Hopf equation (3), then $u = P_K z$ is a solution of the variational inequality(1).

Proof. Observe that, take A is equal the identity operator I, the generalized variational inequality (5) become the variational inequality (1) and the Wiener-Hopf equation (7) become the Wiener-Hopf equation (3). By using Theorem 2.1 we obtain the conclusion of Corollary 2.2.

Theorem 2.3.

(i) If element $u \in K$ is a solution of the generalized variational inequality (5) and $u \in F(S)$, then $z = A[u - \rho Tu]$ satisfies the Wiener-Hopf equation (6) and $P_K z \in F(S)$;

(ii) If $z \in H$ satisfies the Wiener-Hopf equation (6) and $P_K z \in F(S)$, then $u = P_K z$ is a solution of the generalized variational inequality (5) and $u \in F(S)$.

Proof. (i) If $u \in K$ is a solution of the generalized variational inequality (5), then

$$u = P_K A[u - \rho Tu],$$

which leads to $u = P_K z = SP_K z$. Hence

$$z = A[u - \rho Tu] = S_K[SP_K z - \rho TSP_K z].$$

Namely, $z = A[u - \rho Tu]$ satisfies the Wiener-Hopf equation (6) and $P_K z = SP_K z$.

(ii) If z satisfies the Wiener-Hopf equation (6), then we have

$$z = A[SP_K z - \rho TSP_K z],$$

which leads to

$$P_K z = P_K A[SP_K z - \rho TSP_K z].$$

This together with $P_K z \in F(S)$ implies that

$$SP_K z = P_K A[SP_K z - \rho TSP_K z].$$

Let $u = P_K z$. Hence $u = P_K A[u - \rho Tu]$. It is obvious that, $u = P_K z$ is a solution of the generalized variational inequality (5) and $u \in F(S)$. This completes the proof. □

Algorithm 2.4. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the following scheme

$$u_n = P_K z_n, \quad z_{n+1} = (1 - \alpha_n) z_n + \alpha_n A[u_n - \rho T u_n]. \tag{8}$$

Theorem 2.5. *Let T be a relaxed $(\gamma, r)-$cocoercive and $\mu-$Lipschitzian mapping. Let $\{z_n\}$ be a sequence defined by Algorithm 2.4, for any initial point $z_0 \in H$, with conditions*

$$0 < \rho < \frac{2(r - \gamma\mu^2)}{\mu^2}, \quad \gamma\mu^2 < r,$$

$\{\alpha_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{z_n\}$ converges strongly to the unique solution of Wiener-Hopf equation (7), and $\{u_n\}$ converges strongly to the unique solution of variational inequality (5). Where $\{u_n\}$ and $\{z_n\}$ are defined by the Algorithm 2.4.

Proof. Let u^* be a any solution of generalized variational inequality (5), from Theorem 2.1, we know that,

$$z^* = A[u^* - \rho T u^*], \tag{9}$$

is a solution of Wiener-Hopf equation (7) and $u^* = P_K z^*$. From (8) and (9), we have

$$\begin{aligned} \|z_{n+1} - z^*\| &= \|(1 - \alpha_n) z_n + \alpha_n A[u_n - \rho T u_n] - (1 - \alpha_n) z^* + \alpha_n A[u^* - \rho T u^*]\| \\ &\le (1 - \alpha_n)\|z_n - z^*\| + \alpha_n \|A[u_n - \rho T u_n] - A[u^* - \rho T u^*]\| \\ &\le (1 - \alpha_n)\|z_n - z^*\| + \alpha_n \|u_n - u^* - (\rho T u_n - \rho T u^*)\|. \end{aligned} \tag{10}$$

From the relaxed $(\gamma, r)-$cocoercive and $\mu-$Lipschitzian on T, we have

$$\begin{aligned} \|u_n - u^* - (\rho T u_n - \rho T u^*)\|^2 &= \|u_n - u^*\|^2 - 2\rho\langle T u_n - T u^*, u_n - u^*\rangle + \rho^2 \|T u_n - T u^*\|^2 \\ &\le \|u_n - u^*\|^2 - 2\rho[-\gamma\|T u_n - T u^*\|^2 + r\|u_n - u^*\|^2]\rangle + \rho^2 \|T u_n - T u^*\|^2 \end{aligned}$$

$$\le \|u_n - u^*\|^2 + 2\rho\gamma\mu^2\|u_n - u^*\|^2 - 2\rho r\|u_n - u^*\|^2 + \rho^2\mu^2\|u_n - u^*\|^2$$
$$= (1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2)\|u_n - u^*\|^2 = \theta^2\|u_n - u^*\|^2, \tag{11}$$

where

$$\theta = \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}. \tag{12}$$

It follows from the condition of Theorem 2.5 that $\theta < 1$. Combining (10) and (11), we have

$$\|z_{n+1} - z^*\| \le (1 - \alpha_n)\|z_n - z^*\| + \alpha_n\theta\|u_n - u^*\|. \tag{13}$$

From (8) and $u^* = P_K z^*$ we have

$$\|u_n - u^*\| \le \|P_K z_n - P_K z^*\| \le \|z_n - z^*\|. \tag{14}$$

Substituting the (14) into (13), we obtain

$$\|z_{n+1} - z^*\| \le (1 - \alpha_n)\|z_n - z^*\| + \alpha_n\theta\|z_n - z^*\|$$
$$\le [1 - \alpha_n(1 - \theta)]\|z_n - z^*\|,$$

and hence by Lemma 1.7, $\{z_n\}$ converges strongly to z^*. Therefore, observe (14) we also claim that, $\{u_n\}$ converges strongly to u^*. Because the choice of solution $u^* \in K$ is arbitrary, then we claim that, the generalized variational inequality (5) has unique solution $u^* \in K$. We also claim that, the Wiener-Hopf equation (7) has unique solution $z^* = A[u^* - \rho Tu^*]$. If not, then the Wiener-Hopf equation (7) has another solution $z^{**} \neq z^*$, by using the conclusion (ii) of Theorem 2.1, we know $u^* = P_K z^{**}$, so that $P_K z^* = P_K z^{**}$ and

$$z^* = A[P_K z^* - \rho T P_K z^*], \quad z^{**} = A[P_K z^{**} - \rho T P_K z^{**}],$$

which implies $z^* = z^{**}$. This is a contradiction. This completes the proof. □

From Theorem 2.1 and Theorem 2.5, we have the following result.

Theorem 2.6. *Under the conditions of Theorem 2.5, the following conclusions hold:*

(i) The generalized variational inequality (5) has a unique solution:

(ii) The Wiener-Hopf equation (7) has a unique solution;

(iii) The element $u \in K$ is a solution of the generalized variational inequality (5) if and only if $z \in H$ satisfies the Wiener-Hopf equation (7), where

$$z = A[u - \rho Tu], \quad u = P_K z.$$

Observe that $S : K \to K$ is a nonexpansive maping with fixed point set $F(S)$. By using the similar ways as in the Theorem 2.1 and Theorem 2.5, the following Theorem 2.7 and Theorem 2.8 are not hard to prove.

Theorem 2.7.

(i) If element $u \in F(S) \subset K$ is a solution of the generalized variational inequality (5), then $z = A[u - \rho Tu] = A[Su - \rho TSu]$ satisfies the Wiener-Hopf equation (6) and $u = P_K z = SP_K z$;

(ii) If $z \in H$ satisfies the Wiener-Hopf equation (6) and $P_K z \in F(S)$, then $u = P_K z = SP_K z$ is a solution of the generalized variational inequality(5).

Theorem 2.8. *Let T be a relaxed $(\gamma, r)-$cocoercive and $\mu-$Lipschitzian mapping. Let $\{z_n\}$ be a sequence defined by Algorithm 2.4, for any initial point $z_0 \in H$, with conditions*

$$0 < \rho < \frac{2(r - \gamma\mu^2)}{\mu^2}, \quad \gamma\mu^2 < r,$$

$\{\alpha_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(S) \cap GVI(5) \neq \emptyset$. Then $\{z_n\}$ converges strongly to the unique solution of Wiener-Hopf equation (7), and $\{u_n\}$ converges strongly to the unique common element of $F(S) \cap GVI(5)$. Where $GVI(5)$ denote the set of solutions of the generalized variational inequality (5) and $\{u_n\}, \{z_n\}$ are defined by the Algorithm 2.4.

Remark. If choice the nonexpansive mapping A in generalized variational inequalities and generalized Wiener-Hopf equations is equal the identity operator, the results of this paper reduce the results of M.A.Noor and Z.Y.Huang [20].

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Guanghui Gu,
Department of Mathematics, cangzhou Normal University, Cangzhou 061001, Hebei, China
e-mail: guguanghui@yahoo.com.cn

Yongfu Su ,
Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
e-mail: suyongfu@gmail.com