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ITERATIVE ALGORITHM FOR COMPLETELY GENERALIZED QUASI-VARIATIONAL INCLUSIONS WITH FUZZY MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce and study a class of completely generalized quasi-variational inclusions with fuzzy mappings. A new iterative algorithm for finding the approximate solutions and the convergence criteria of the iterative sequences generated by the algorithm are also given. These results of existence, algorithm and convergence generalize many known results.

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1. Introduction

In 1994, Haussouni and Moudafi [6] introduced and studied a class of variational inclusions and developed a perturbed algorithm for finding approximate solutions of the variational inclusions. Adly [1], Huang [7], Kazmi [9] and Ding [5] extended the results in [6] to generalized variational inclusions and generalized quasi-variational inclusions.

In 1989, Chang and Zhu [2] introduced and studied a class of variational inequalities for fuzzy mappings. Since then, several classes of variational inequalities (inclusions) with fuzzy mappings were considered by Chang and Huang [3], Noor [11], Huang [8], Park and Jeong [12-17].

In this paper, we study a class of completely generalized quasi-variational inclusions with fuzzy mappings. An innovative iterative algorithm for finding approximate solutions is suggested and analyzed. The convergence criteria of the algorithm is also given. These results of existence of generalized quasi-variational

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inclusions generalize many known results of generalized quasi-variational inequalities with fuzzy mappings in literature [4,6,13-16].

2. Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{F}(\mathcal{H})$ be a collection of all fuzzy sets over H. A mapping $F : H \to \mathcal{F}(\mathcal{H})$ is said to be a fuzzy mapping. For each $x \in H$, $F(x)$ (denote it by F_x in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x.

A fuzzy mapping $F : H \to \mathcal{F}(\mathcal{H})$ is said to be closed if for each $x \in H$, the function $y \mapsto F_x(y)$ is upper semicontinuous, i.e., for any given net $\{y_\alpha\} \subset H$ satisfying $y_\alpha \to y_0 \in H$, $\limsup_\alpha F_x(y_\alpha) \le F_x(y_0)$. For $A \in \mathcal{F}(\mathcal{H})$ and $\lambda \in [0,1]$, the set $(A)_\lambda = \Big\{x \in H : A(x) \ge \lambda\Big\}$ is called a λ-cut set of A.

A closed fuzzy mapping $A : H \to \mathcal{F}(\mathcal{H})$ is said to be satisfy the condition $(*)$ if there exists a function $a : H \to [0,1]$ such that for each $x \in H$, $(A_x)_{a(x)}$ is a nonempty bounded subset of H. It is clear that if A is a closed fuzzy mapping satisfying the condition $(*)$, then for each $x \in H$, the set $(A_x)_{a(x)} \in CB(H)$, where $CB(H)$ denotes the family of all nonempty bounded closed subsets of H. In fact, let $\{y_\alpha\}_{\alpha\in\Gamma} \subset (A_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in H$. Then $(A_x)(y_\alpha) \ge a(x)$ for each $\alpha \in \Gamma$. Since A is closed, we have

$$A_x(y_0) \ge \limsup_{\alpha\in\Gamma} A_x(y_\alpha) \ge a(x).$$

This implies $y_0 \in (A_x)_{a(x)}$ and so $(A_x)_{a(x)} \in CB(H)$.

Let $A, B, C, D, E : H \to \mathcal{F}(\mathcal{H})$ be five closed fuzzy mappings satisfying the condition $(*)$. Then there exist five functions $a, b, c, d, e : H \to [0,1]$ such that for each $x \in H$, we have

$$(A_x)_{a(x)}, (B_x)_{b(x)}, (C_x)_{c(x)}, (D_x)_{d(x)}, (E_x)_{e(x)} \in CB(H).$$

Therefore we can define five set-valued mappings $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E} : H \to CB(H)$ by

$$\begin{aligned} \tilde{A}(x) &= (A_x)_{a(x)}, \tilde{B}(x) = (B_x)_{b(x)}, \tilde{C}(x) = (C_x)_{c(x)}, \\ \tilde{D}(x) &= (D_x)_{d(x)}, \tilde{E}(x) = (E_x)_{e(x)} \end{aligned}$$

for each $x \in H$. In the sequel, $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ and $\tilde{E}$ are called the set-valued mappings induced by the fuzzy mappings A, B, C, D and E, respectively.

Let $M : H \to H$, $N : H \times H \to H$ and $f, g : H \to H$ be single-valued mappings. Let $A, B, C, D, E : E \to \mathcal{F}(\mathcal{H})$ be fuzzy mappings and let $a, b, c, d, e : H \to [0,1]$ be given functions. Let $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a functional such that for fixed $z \in H$, $x \to \varphi(x,z)$ is a proper convex lower semicontinuous functional with $g(H) \cap \text{dom}\partial\varphi(\cdot, z) \neq \phi$. Let

(1) $b(x,y)$ is linear in first argument,

(2) $b(x, y)$ is bounded, i.e., there exists a constant $\nu > 0$ such that

$$b(x, y) \leq \nu \|x\| \|y\|,$$

(3) for all $x, y, z \in H$

$$b(x, y) - b(x, z) \leq b(x, y - z).$$

In this paper, we consider the following completely generalized quasi-variational inclusion problem with fuzzy mappings (CGQVIP):

Find $x, u, v, w, y, z \in H$ such that $A_x(u) \geq a(x)$, $B_x(v) \geq b(x)$, $C_x(w) \geq c(x)$, $D_x(y) \geq d(x)$, $E_x(z) \geq e(x)$ and

$$\begin{aligned} &\Big\langle M(f(u)) - N(v, w), h - g(x) \Big\rangle + b(y, h) - b(y, g(x)) \\ &\quad \geq \varphi(g(x), z) - \varphi(h, z), \quad \forall h \in H. \end{aligned} \tag{2.1}$$

(I) If $M = I$ is the identity mapping, then (CGQVIP) reduces to the following generalized quasi-variational inclusion problem with fuzzy mapping:

Find $x, u, v, w, y, z \in H$ such that $A_x(u) \geq a(x)$, $B_x(v) \geq b(x)$, $C_x(w) \geq c(x)$, $D_x(y) \geq d(x)$, $E_x(z) \geq e(x)$ and

$$\begin{aligned} &\Big\langle f(u) - N(v, w), h - g(x) \Big\rangle + b(y, h) - b(y, g(x)) \\ &\quad \geq \varphi(g(x), z) - \varphi(h, z), \quad \forall h \in H. \end{aligned} \tag{2.2}$$

(II) If $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E} : H \to CB(H)$ are set-valued mappings, we can define the fuzzy mapping $A, B, C, D, D, E : H \to \mathcal{F}(\mathcal{H})$ by

$$x \to \chi_{\tilde{A}(x)}, \quad x \to \chi_{\tilde{B}(x)}, \quad x \to \chi_{\tilde{C}(x)}, \quad x \to \chi_{\tilde{D}(x)}, \quad x \to \chi_{\tilde{E}(x)},$$

where $\chi_{\tilde{A}(x)}, \chi_{\tilde{B}(x)}, \chi_{\tilde{C}(x)}, \chi_{\tilde{D}(x)}, \chi_{\tilde{E}(x)}$ are the characteristic functions of $\tilde{A}(x)$, $\tilde{B}(x)$, $\tilde{C}(x)$, $\tilde{D}(x)$ and $\tilde{E}(x)$, respectively. Taking $a(x) = b(x) = c(x) = d(x) = e(x) = 1$ for $x \in H$, the problem (2.1) is equivalent to the following problem:

Find $x \in H$, $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$, $w \in \tilde{C}(x)$, $y \in \tilde{D}(x)$, and $z \in \tilde{E}(x)$ such that

$$\begin{aligned} &\Big\langle M(f(u)) - N(v, w), h - g(x) \Big\rangle + b(y, h) - b(y, g(x)) \\ &\quad \geq \varphi(g(x), z) - \varphi(h, z), \quad \forall h \in H. \end{aligned}$$

(III) If $b(x, y) = 0$ for all $(x, y) \in H \times H$, then problem (2.2) is reduces to the following problem:

Find $x, u, v, w, y, z \in H$ such that $A_x(u) \geq a(x)$, $B_x(v) \geq b(x)$, $C_x(w) \geq c(x)$, $D_x(y) \geq d(x)$, $E_x(z) \geq e(x)$ and

$$\Big\langle f(u) - N(v, w), h - g(x) \Big\rangle \geq \varphi(g(x), z) - \varphi(h, z), \quad \forall h \in H. \tag{2.3}$$

The problem (2.3) includes many generalized quasivariational inclusion problems considered in [4,6].

(vi) If $K : H \to 2^H$ is a set-valued mapping such that each $K(x)$ is a closed convex subset of H and for each fixed $z \in H$, $\varphi(\cdot, z) = I_{K(z)}(\cdot)$ is the indicate function of $K(z)$, then the problem (2.2) reduces to the following problem:

Find $x, u, v, w, y, z \in H$ such that $A_x(u) \geq a(x)$, $B_x(v) \geq b(x)$, $C_x(w) \geq c(x)$, $D_x(y) \geq d(x)$, $E_x(z) \geq e(x)$ and $g(x) \in K(z)$,

$$\Big\langle f(u) - N(u, v), h - g(x) \Big\rangle + b(y, h) - b(y, g(x)) \geq 0, \quad \forall h \in K(z).$$

Definition 2.1. A mapping $f : H \to H$ is called

(i) Lipschitz continuous if there exists a constant $l > 0$ such that

$$\|f(x) - f(y)\| \leq l\|x - y\|, \quad \forall x, y \in H,$$

(ii) strongly monotone if there exists a constant $r > 0$ such that

$$\langle f(x) - f(y), x - y \rangle \geq r\|x - y\|^2, \quad \forall x, y \in H.$$

Definition 2.2. A set-valued mapping $A : H \to CB(H)$ is said to be Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\tilde{H}(A(x), A(y)) \leq \delta\|x - y\|, \quad \forall x, y \in H,$$

where $\tilde{H}(\cdot, \cdot)$ denotes the Hausdorff metric on $CB(H)$.

Definition 2.3. Let $E : H \to 2^H$ and $N : H \times H \to H$ be mappings.

(i) E is said to be α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle u_1 - u_2, x_1 - x_2 \rangle \geq \alpha\|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in H, u_1 \in E(x_1), u_2 \in E(x_2),$$

(ii) $N(\cdot, \cdot)$ is said to be α-relaxed Lipschitz with respect to E in the first argument if there exists a constant $\beta > 0$ such that

$$\Big\langle N(u_1, \cdot) - N(u_2, \cdot), x_1 - x_2 \Big\rangle \leq -\beta\|x_1 - x_2\|^2,$$
$$\forall x_1, x_2 \in H, u_1 \in E(x_1), u_2 \in E(x_2).$$

(iii) $N(\cdot, \cdot)$ is said to be γ-Lipschitz continuous in the first argument if there exists a constant $\gamma > 0$ such that

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \gamma\|u_1 - u_2\|, \quad \forall u_1, u_2 \in H.$$

In a similar way, we can define the ξ-Lipschitz continuous of $N(\cdot, \cdot)$ in the second argument.

3. Existence and algorithms of solutions

In this section, by using the resolvent operator technique, we first transfer the problem (2.1) to a fixed point problem. Next, an existence theorem of solutions for the problem (2.1) is proved and a new iterative algorithm to compute approximate solutions of the problem (2.1) is suggested and analyzed. The convergence of the iterative sequence generated by the new algorithm is also proved.

In order to prove our main theorems we need the following concepts and results (see [18]).

Definition 3.1. Let X be a Banach space with the dual space X^* and let $\varphi : X \to \mathbb{R}\cup\{+\infty\}$ be a proper functional. Then φ is said to be subdifferentiable at a point $x \in X$ if there exists an $f^* \in X^*$ such that

$$\varphi(y) - \varphi(x) \geq \langle f^*, y - x\rangle, \quad \forall y \in X,$$

where f^* is called a subgradient of φ at x. The set of all subgradient of φ at x is denoted by $\partial\varphi(x)$. The mapping $\partial\varphi : X \to 2^{X^*}$ defined by

$$\partial\varphi(x) = \Big\{f^* \in X^* : \varphi(y) - \varphi(x) \geq \langle f^*, y - x\rangle, \quad \forall y \in X\Big\}$$

is said to be the subdifferential of φ.

Definition 3.2. Let H be a Hilbert space and let $G : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^G : H \to H$ defined by

$$J_\rho^G(x) = (I + \rho G)^{-1}(x), \quad \forall x \in H$$

is said to be the resolvent operator of G, where I is the identity mapping on H.

Lemma 3.1([18]). *Let X be a reflexive Banach space endowed with a strictly convex norm and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Then $\partial\varphi : X \to 2^{X^*}$ is a maximal monotone mapping.*

Lemma 3.2([18]). *Let $G : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J_\rho^G : H \to H$ of G is nonexpansive, i.e.,*

$$\|J_\rho^G(x) - J_\rho^G(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Lemma 3.3([5]). *Let $b : H \times H \to \mathbb{R}$ be a real function satisfying the condition (1)-(3) mentioned before. Then for each $y \in H$ there exists a unique point $k(y) \in H$ such that*

$$b(x, y) = \langle k(y), x\rangle, \quad \forall x \in H$$

and

$$\|k(y) - k(z)\| \leq \nu\|y - z\|, \quad \forall y, z \in H,$$

i.e., the mapping $k : H \to H$ is Lipschitz continuous.

Theorem 3.1. *(x, u, v, w, y, z) is a solution of the problem (2.1) if and only if (x, u, v, w, y, z) satisfies the following relation:*

$$g(x) = J_\rho^{\partial\varphi(\cdot,z)}\Big[g(x) - \rho(M(f(u)) - N(v, w) + k(y))\Big], \tag{3.1}$$

where

$$u \in \tilde{A}(x), v \in \tilde{B}(x), w \in \tilde{C}(x), y \in \tilde{D}(x), z \in \tilde{E}(x), \langle k(y), x\rangle = b(x,y)$$

for all $x \in H$ and $\rho > 0$ is a constant.

Proof. Suppose that (x,u,v,w,y,z) is a solution of the problem (2.1). Then $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$, $w \in \tilde{C}(x)$, $y \in \tilde{D}(x)$ and $z \in \tilde{E}(x)$ satisfy

$$\begin{aligned}&\Big\langle M(f(u)) - N(v,w), h - g(x)\Big\rangle + b(y,h) - b(y, g(x))\\ &\quad \geq \varphi(g(x), z) - \varphi(h,z), \quad \forall h \in H. \qquad (3.2)\end{aligned}$$

By Lemma 3.3, we see that

$$\begin{aligned}b(h,y) - b(g(x),y) &= b(h - g(x), y)\\ &= \langle k(y), h - g(x)\rangle\end{aligned}$$

for all $h \in H$. Hence the relation (3.2) holds if and only if

$$\varphi(h,y) - \varphi(g(x),y) \geq \langle N(v,w) - M(f(u)) - k(y), h - g(x)\rangle, \quad \forall h \in H. \qquad (3.3)$$

The relation (3.3) holds if and only if

$$N(v,w) - M(f(u)) - k(y) \in \partial\varphi(\cdot, z)(g(x)). \qquad (3.4)$$

From the definition of $J_\rho^{\varphi(\cdot,z)}$ the relation (3.4) holds if and only if

$$g(x) = J_\rho^{\varphi(\cdot,z)}\Big[g(x) - \rho(M(f(u)) - N(v,w) + k(y))\Big],$$

where $\langle k(y), x\rangle = b(x,y)$ for all $x \in H$ and $\rho > 0$ is a constant. Hence we get that (x,u,v,w,y,z) is a solution of the problem (2.1) if and only if $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$, $w \in \tilde{C}(x)$, $y \in \tilde{D}(x)$ and $z \in \tilde{E}(x)$ satisfy (3.1). This completes the proof of Theorem 3.1. □

Remark 3.1. We observe that (3.1) can be rewritten as following:

$$x = x - g(x) + J_\rho^{\partial\varphi(\cdot,z)}\Big[g(x) - \rho(M(f(u)) - N(v,w) + k(y))\Big].$$

This fixed point formulation enables us to suggest the following iterative algorithm for solving the problem (2.1).

Algorithm 3.1. Let $A, B, C, D, E : H \to \mathcal{F}(\mathcal{H})$ be closed fuzzy mappings satisfying the condition $(*)$ and $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E} : H \to CB(H)$ be the set-valued mappings induced by the fuzzy mappings A, B, C, D, E, respectively.

Let $N : H \times H \to H$ and $M, f, g : H \to H$ be single-valued mappings. Let $b : H \times H \to \mathbb{R}$ satisfy the conditions (1)-(3) and $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be proper convex lower semicontinuous on H in the first argument with $g(H) \cap$

$\mathrm{dom}(\partial\varphi(\cdot,z)) \neq \phi$ for all $z \in H$. For given $x_0 \in H$, $u_0 \in \tilde{A}(x_0)$, $v_0 \in \tilde{B}(x_0)$, $w_0 \in \tilde{C}(x_0)$, $y_0 \in \tilde{D}(x_0)$, $z_0 \in \tilde{E}(x_0)$, let

$$x_1 = x_0 - g(x_0) + J_\rho^{\partial\varphi(\cdot,z_0)}\Big[g(x_0) - \rho(M(f(u_0)) - N(v_0,w_0) + k(y_0))\Big].$$

By Nadler's theorem ([10]), there exist $u_1 \in \tilde{A}(x_1)$, $v_1 \in \tilde{B}(x_1)$, $w_1 \in \tilde{C}(x_1)$, $y_1 \in \tilde{D}(x_1)$ and $z_1 \in \tilde{E}(x_1)$ such that

$$\begin{aligned}
\|u_0 - u_1\| &\le (1+1)\tilde{H}(\tilde{A}(x_0),\tilde{A}(x_1)),\\
\|v_0 - v_1\| &\le (1+1)\tilde{H}(\tilde{B}(x_0),\tilde{B}(x_1)),\\
\|w_0 - w_1\| &\le (1+1)\tilde{H}(\tilde{C}(x_0),\tilde{C}(x_1)),\\
\|y_0 - y_1\| &\le (1+1)\tilde{H}(\tilde{D}(x_0),\tilde{D}(x_1)),\\
\|z_0 - z_1\| &\le (1+1)\tilde{H}(\tilde{E}(x_0),\tilde{E}(x_1)).
\end{aligned}$$

Let

$$x_2 = x_1 - g(x_1) + J_\rho^{\partial\varphi(\cdot,z_1)}\Big[g(x_1) - \rho(M(f(u_1)) - N(v_1,w_1) + k(y_1))\Big].$$

Continuing this way, we can define the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ which satisfy the following conditions:

$$x_{n+1} = x_n - g(x_n) + J_\rho^{\partial\varphi(\cdot,z_n)}\Big[g(x_n) - \rho(M(f(u_n)) - N(v_n,w_n) + k(y_n))\Big], \tag{3.6}$$

$$\begin{aligned}
u_n \in \tilde{A}(x_n), \quad \|u_n - u_{n+1}\| &\le \left(1+\frac{1}{n+1}\right)\tilde{H}(\tilde{A}(x_n),\tilde{A}(x_{n+1})),\\
v_n \in \tilde{B}(x_n), \quad \|v_n - v_{n+1}\| &\le \left(1+\frac{1}{n+1}\right)\tilde{H}(\tilde{B}(x_n),\tilde{B}(x_{n+1})),\\
w_n \in \tilde{C}(x_n), \quad \|w_n - w_{n+1}\| &\le \left(1+\frac{1}{n+1}\right)\tilde{H}(\tilde{C}(x_n),\tilde{C}(x_{n+1})),\\
y_n \in \tilde{D}(x_n), \quad \|y_n - y_{n+1}\| &\le \left(1+\frac{1}{n+1}\right)\tilde{H}(\tilde{D}(x_n),\tilde{D}(x_{n+1})),\\
z_n \in \tilde{E}(x_n), \quad \|z_n - z_{n+1}\| &\le \left(1+\frac{1}{n+1}\right)\tilde{H}(\tilde{E}(x_n),\tilde{E}(x_{n+1}))
\end{aligned} \tag{3.7}$$

for $n = 0, 1, 2, \cdots$, where $\rho > 0$ is a constant.

Now we establish the existence of solutions for the problem (2.1) and convergence of the iterative sequence generated by the Algorithms 3.1.

Theorem 3.2. *Let $A, B, C, D, E : H \to \mathcal{F}(\mathcal{H})$ be closed fuzzy mappings satisfying the condition $(*)$ and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E} : H \to CB(H)$ be set-valued mappings induced by the fuzzy mappings A, B, C, D and E, respectively.*

Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$ be λ_A-Lipschitz continuous, λ_B-Lipschitz continuous, λ_C-Lipschitz continuous, λ_D-Lipschitz continuous and λ_E-Lipschitz continuous, respectively. Let $M : H \to H$ be ε-Lipschitz continuous, $f : H \to H$

be γ-Lipschitz continuous, $g : H \to H$ be δ-strongly monotone and σ-Lipschitz continuous.

Let $N : H \times H \to H$ be α-relaxed Lipschitz continuous with respect to $\tilde{B}$ and β-Lipschitz continuous in the first argument and ξ-Lipschitz continuous in the second argument.

Let $b : H \times H \to \mathbb{R}$ be a function satisfying the conditions (1)-(3) mentioned before and let $k : H \to H$ be η-Lipschitz continuous. Let $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be proper convex lower semicontinuous on H in the first argument with $g(H) \cap dom(\partial\varphi(\cdot, z)) \neq \phi$ for all $z \in H$ such that

$$\left\| J_\rho^{\partial\varphi(\cdot,y)}(x) - J_\rho^{\partial\varphi(\cdot,z)}(x) \right\| \leq \mu \|y - z\|, \quad \forall x, y, z \in H. \tag{3.8}$$

Suppose that there exists a constant $\rho > 0$ satisfying

$$\left| \rho - \frac{\alpha - (1-k)q}{p^2 - q^2} \right| < \frac{\sqrt{\left[\alpha - (1-k)q\right]^2 - (p^2 - q^2)k(2-k)}}{p^2 - q^2}, \tag{3.9}$$

$$\begin{aligned} k &= 2\sqrt{1 - 2\sigma + \delta^2} + \mu\lambda_E < 1, \\ p &= \beta\lambda_B > \varepsilon\gamma\lambda_A + \xi\lambda_C + \eta\lambda_D = q, \\ \alpha &> (1-k)q + \sqrt{(p^2 - q^2)k(2-k)}. \end{aligned}$$

Then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 3.1 converge strongly to x, u, v, w, y and z, respectively and (x, u, v, w, y, z) is a solution of the completely generalized quasi-variational inclusion problem with fuzzy mappings (2.1).

Proof. By the Algorithm 3.1, Lemma 3.2 and the condition (3.8), we have

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \Big\| x_n - g(x_n) + J_\rho^{\partial\varphi(\cdot,z_n)}[g(x_n) - \rho(M(f(u_n)) - N(v_n, w_n) + k(y_n)) \\ &\quad - (x_{n-1} - g(x_{n-1}) + J_\rho^{\partial\varphi(\cdot,z_{n-1})}[g(x_{n-1}) - \rho(M(f(u_{n-1})) \\ &\quad - N(v_{n-1}, w_{n-1}) + k(y_{n-1}))]) \Big\| \\ &\leq \Big\| x_n - x_{n-1} - (g(x_n) - g(x_{n-1})) \Big\| \\ &\quad + \Big\| J_\rho^{\partial\varphi(\cdot,z_n)}[g(x_n) - \rho(M(f(u_n)) - N(v_n, w_n) + k(y_n))] \\ &\qquad - J_\rho^{\partial\varphi(\cdot,z_n)}[g(x_{n-1}) - \rho(M(f(u_{n-1})) - N(v_{n-1}, w_{n-1}) + k(y_{n-1}))] \Big\| \end{aligned}$$

$$
\begin{aligned}
&+ \Big\| J_\rho^{\partial\varphi(\cdot,z_n)}[g(x_{n-1}) - \rho(M(f(u_{n-1})) - N(v_{n-1},w_{n-1}) + k(y_{n-1}))] \\
&\quad - J_\rho^{\partial\varphi(\cdot,z_{n-1})}[g(x_{n-1}) - \rho(M(f(u_{n-1})) - N(v_{n-1},w_{n-1}) + k(y_{n-1}))]\Big\| \\
&\leq \Big\| x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\Big\| \\
&\quad + \Big\| g(x_n) - \rho(M(f(u_n)) - N(v_n,w_n) + k(y_n)) \\
&\quad - [g(x_{n-1}) - \rho(M(f(u_{n-1})) - N(v_{n-1},w_{n-1}) + k(y_{n-1}))]\Big\| + \mu\Big\| z_n - z_{n-1}\Big\| \\
&\leq 2\Big\| x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\Big\| + \rho\|M(f(u_n)) - M(f(u_{n-1}))\| \\
&\quad + \Big\| x_n - x_{n-1} + \rho(N(v_n,w_n) - N(v_{n-1},w_n))\Big\| \\
&\quad + \rho\Big\| N(v_{n-1},w_n) - N(v_{n-1},w_{n-1})\Big\| + \rho\|k(y_n) - k(y_{n-1})\| + \mu\|z_n - z_{n-1}\|.
\end{aligned} \tag{3.10}
$$

Since g is δ-strongly monotone and σ-Lipschitz continuous, we see that

$$
\Big\| x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\Big\| \leq \sqrt{1 - 2\delta + \delta^2}\|x_n - x_{n-1}\|. \tag{3.11}
$$

Since $M : H \to H$ is ε-Lipschitz continuous, $f : H \to H$ is γ-Lipschitz continuous and $\tilde{A} : H \to CB(H)$ is λ_A-Lipschitz continuous, we obtain

$$
\begin{aligned}
\Big\| M(f(u_n)) - M(f(u_{n-1}))\Big\| &\leq \varepsilon\gamma\left(1 + \frac{1}{n}\right)\tilde{H}(\tilde{A}(x_n), \tilde{A}(x_{n-1})) \\
&\leq \varepsilon\gamma\lambda_A\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\|.
\end{aligned} \tag{3.12}
$$

Since $N(\cdot,\cdot)$ is α-relaxed Lipschitz continuous with respect to $\tilde{B}$, β-Lipschitz continuous in the first argument and $\tilde{B}$ is λ_B-Lipschitz continuous, we obtain

$$
\begin{aligned}
&\Big\| x_n - x_{n-1} + \rho(N(v_n,w_n) - N(v_{n-1},w_n))\Big\|^2 \\
&= \|x_n - x_{n-1}\|^2 + 2\rho\Big\langle N(v_n,w_n) - N(v_{n-1},w_n), x_n - x_{n-1}\Big\rangle \\
&\quad + \rho^2\Big\| N(v_n,w_n) - N(v_{n-1},w_n)\Big\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 - 2\rho\alpha\|x_n - x_{n-1}\|^2 + \rho^2\beta^2\left[\left(1 + \frac{1}{n}\right)\tilde{H}(\tilde{B}(x_n), \tilde{B}(x_{n-1}))\right]^2 \\
&\leq \left(1 - 2\rho\alpha + \rho^2\beta^2\lambda_B^2\left(1 + \frac{1}{n}\right)^2\right)\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.13}
$$

Using ξ-Lipschitz continuous of $N(\cdot,\cdot)$ in the second argument and λ_C-Lipschitz continuous of $\tilde{C}$, we obtain

$$\left\|N(v_{n-1},w_n)-N(v_{n-1},w_{n-1})\right\| \le \xi\|w_n-w_{n-1}\|$$
$$\le \xi\left(1+\frac{1}{n}\right)\tilde{H}(\tilde{C}(x_n),\tilde{C}(x_{n-1}))$$
$$\le \xi\lambda_C\left(1+\frac{1}{n}\right)\|x_n-x_{n-1}\|. \tag{3.14}$$

By the η-Lipschitz continuity of k and λ_D-Lipschitz continuity of $\tilde{D}$, we have

$$\|k(y_n)-k(y_{n-1})\| \le \eta\|y_n-y_{n-1}\|$$
$$\le \eta\left(1+\frac{1}{n}\right)\tilde{H}(\tilde{D}(x_n),\tilde{D}(x_{n-1}))$$
$$\le \eta\lambda_D\left(1+\frac{1}{n}\right)\|x_n-x_{n-1}\|. \tag{3.15}$$

Since $\tilde{E}$ is λ_E-Lipschitz continuous, we have

$$\|z_n-z_{n-1}\| \le \left(1+\frac{1}{n}\right)\tilde{H}(\tilde{E}(x_n),\tilde{E}(x_{n-1}))$$
$$\le \lambda_E\left(1+\frac{1}{n}\right)\|x_n-x_{n-1}\|. \tag{3.16}$$

From (3.10)-(3.16), we deduce that

$$\begin{aligned}
&\left\|x_{n+1}-x_n\right\| \\
&\le \Big[2\sqrt{1-2\sigma+\delta^2}+\rho\varepsilon\gamma\lambda_A\left(1+\frac{1}{n}\right)+\sqrt{1-2\rho\alpha+\rho^2\beta^2\lambda_B^2\left(1+\frac{1}{n}\right)^2} \\
&\quad +\rho\xi\lambda_C\left(1+\frac{1}{n}\right)+\rho\eta\lambda_D\left(1+\frac{1}{n}\right)+\mu\lambda_E\left(1+\frac{1}{n}\right)\Big]\|x_n-x_{n-1}\| \\
&=(k_n+t_n(\rho))\|x_n-x_{n-1}\| \\
&=\theta_n\|x_n-x_{n-1}\|,
\end{aligned} \tag{3.17}$$

where

$$k_n=2\sqrt{1-2\sigma+\delta^2}+\mu\lambda_E\left(1+\frac{1}{n}\right),$$

$$t_n(\rho)=\sqrt{1-2\rho\alpha+\rho^2\beta^2\lambda_B^2\left(1+\frac{1}{n}\right)^2}+\rho(\varepsilon\gamma\lambda_A+\xi\lambda_C+\eta\lambda_D)\left(1+\frac{1}{n}\right)$$

and

$$\theta_n=k_n+t_n(\rho).$$

Let

$$k=2\sqrt{1-2\sigma+\delta^2}+\mu\lambda_E,$$
$$t(\rho)=\sqrt{1-2\rho\alpha+\rho^2\beta^2\lambda_B^2}+\rho(\varepsilon\gamma\lambda_A+\xi\lambda_C+\eta\lambda_D)$$

and

$$\theta = k + t(\rho).$$

Then we obtain that $k_n \to k$, $t_n(\rho) \to t(\rho)$ and $\theta_n \to \theta$ as $n \to \infty$. The condition (3.9) implies $0 < \theta < 1$ and hence $\theta_n < 1$ for sufficiently large n. It follows from (3.17) that $\{x_n\}$ is a Cauchy sequence in H. Let $x_n \to x \in H$ as $n \to \infty$. By (3.7) and the Lipschitz continuities of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$,

$$\begin{aligned}\|u_{n+1} - u_n\| &\le \left(1 + \frac{1}{n+1}\right) \tilde{H}(\tilde{A}(x_{n+1}), \tilde{A}(x_n))\\ &\le \left(1 + \frac{1}{n+1}\right) \lambda_A \|x_{n+1} - x_n\|,\end{aligned}$$

$$\begin{aligned}\|v_{n+1} - v_n\| &\le \left(1 + \frac{1}{n+1}\right) \tilde{H}(\tilde{B}(x_{n+1}), \tilde{B}(x_n))\\ &\le \left(1 + \frac{1}{n+1}\right) \lambda_B \|x_{n+1} - x_n\|,\end{aligned}$$

$$\begin{aligned}\|w_{n+1} - w_n\| &\le \left(1 + \frac{1}{n+1}\right) \tilde{H}(\tilde{C}(x_{n+1}), \tilde{C}(x_n))\\ &\le \left(1 + \frac{1}{n+1}\right) \lambda_C \|x_{n+1} - x_n\|,\end{aligned}$$

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \left(1 + \frac{1}{n+1}\right) \tilde{H}(\tilde{D}(x_{n+1}), \tilde{D}(x_n))\\ &\le \left(1 + \frac{1}{n+1}\right) \lambda_D \|x_{n+1} - x_n\|,\end{aligned}$$

$$\begin{aligned}\|z_{n+1} - z_n\| &\le \left(1 + \frac{1}{n+1}\right) \tilde{H}(\tilde{E}(x_{n+1}), \tilde{E}(x_n))\\ &\le \left(1 + \frac{1}{n+1}\right) \lambda_E \|x_{n+1} - x_n\|.\end{aligned}$$

It follows that $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences in H. We can assume that $u_n \to u^*$, $v_n \to v^*$, $w_n \to w^*$, $y_n \to y^*$ and $z_n \to z^*$, respectively. Note that for $u_n \in \tilde{A}(x_n)$ we have

$$\begin{aligned}d\Big(u^*, \tilde{A}(x^*)\Big) &\le \|u^* - u_n\| + \tilde{H}(\tilde{A}(x_n), \tilde{A}(x^*))\\ &\le \|u^* - u_n\| + \lambda_A \|x_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Hence we must have $u^* \in \tilde{A}(x^*)$. Similarly, we can show that $v^* \in \tilde{B}(x^*)$, $w^* \in \tilde{C}(x^*)$, $y^* \in \tilde{D}(x^*)$ and $z^* \in \tilde{E}(x^*)$. Hence we have that

$$A_{x^*}(u^*) \ge a(x^*), B_{x^*}(v^*) \ge b(x^*), C_{x^*}(w^*) \ge c(x^*), D_{x^*}(y^*) \ge d(x^*)$$

and $E_{x^*}(z^*) \geq e(x^*)$. By Algorithm 3.1, we see that

$$x_{n+1} = x_n - g(x_n) + J_\rho^{\partial\varphi(\cdot,z_n)}\Big[g(x_n) - \rho(M(f(u_n)) - N(v_n, w_n) + k(y_n))\Big], \quad \forall n = 0, 1, 2, \cdots. \tag{3.18}$$

By the Lipschitz continuities of $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, f, g, k$ and M, letting $n \to \infty$ in (3.18), we obtain

$$g(x^*) = J_\rho^{\partial\varphi(\cdot,z^*)}\Big[g(x^*) - \rho(M(f(u^*)) - N(v^*, w^*) + k(y^*))\Big].$$

By Theorem 3.1, $(x^*, u^*, v^*, w^*, y^*, z^*)$ is a solution of the problem (2.1). This completes the proof.

Remark 3.2. Theorem 3.2 unifies and generalized many corresponding known results in recent literature [4,6,13-16].

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