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ON APPROXIMATED PROBLEMS FOR LOCALLY LIPSCHITZ OPTIMIZATION PROBLEMS

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ABSTRACT. In this paper, using nonsmooth analysis, we established equivalence results between a locally Lipschitz vector optimization problem and its associated approximated problem under the proper efficiency.

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1. Introduction and preliminaries

Multiobjective optimization problems consist of conflicting objective functions and constraint sets and are to optimize the objective functions over the constraint sets under some concepts of solution, for example, properly efficient solutions, efficient solutions and weakly efficient solutions. Many authors have studied sufficient and necessary optimality conditions, alternative theorems, multipliers rules, duality results, and etc.([6]-[16], [20]).

Considerable attention has been given recently to devising new methods which solve the original multiobjective mathematical programming problem and its dual by the help of some associated vector optimization problem. One of a such method is that proposed by Antczak [1]. He introduced a new approach with a modified objective function for solving a differentiable multiobjective optimization problems involved invex functions.

Recently, Antczak [2] considered η-approximated problem associated with a primal differentiable scalar optimization problem and established equivalence between the primal problem and its associated η-approximated optimization problem under invexity assumptions. Some authors [3, 14, 17] extended the results

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of Antczak [2] to scalar nonlinear optimization problems under invexity assumptions and vector nonlinear optimization problems. In particular, Antczak [3] gave an equivalence result between the differentiable vector optimization problem and the η-approximated vector optimization problem under the (weak) efficiency.

However the η-approximated problem in [2, 3] is a nonlinear and nonconvex optimization problem. From approaches of Antczak [2], it is clear that we can consider an approximated problem associated with a primal problem which is a convex optimization problem.

Mäkelä and Neittaanmäki [18] defined approximations of the locally Lipschitz function, which are expressed in terms of generalized Clarke directional derivatives [5] and considered approximated problems for locally Lipschitz optimization problems, and showed that such approximated problems provided strong tools for the original locally Lipschitz optimization problems. Their approximation is more efficient and more applicable to many optimization problems than the method of Antczak [2, 3].

Very recently, Kim [15] considered a differentiable vector optimization problem, and establish equivalence results between the problem and its associated approximated problem under the proper efficiency.

In this paper, applying the approximation method of Mäkelä and Neittaanmäki [18] to a locally Lipschitz vector optimization problem, we obtain equivalence results between the original problem and the approximated problem under the proper efficiency. Our equivalence results can be regarded as extensions of ones in [15].

Now we give notations and preliminary results that will be used later.

We consider the following vector optimization problem:

$$\text{(VP)} \qquad \text{Minimize} \quad (f_1(x), \cdots, f_p(x)) \qquad \text{subject to} \quad g_j(x) \leq 0,\ j = 1, \cdots, m,$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, p$ and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \cdots, m$ are locally Lipschitz functions. Further let, $S := \left\{x \in \mathbb{R}^n \mid g_j(x) \leq 0,\ j = 1, \cdots, m\right\}$.

Optimization of (VP) is of finding an efficient solution defined as follows:

Definition 1. (1) A point $\bar{x} \in S$ is said to be an efficient solution of (VP) if there exist no other feasible point $x \in S$ such that $f_i(x) \leq f_i(\bar{x})$, for all $i = 1, \cdots, p$, but $f_j(x) < f_j(\bar{x})$ for some $j \neq i$.

(2) [8] A point $\bar{x} \in S$ is called a properly efficient solution of (VP) if it is efficient for (VP) and if there exists a scalar $M > 0$ such that for each $i = 1, \cdots, p$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M$$

for some $j \neq i$ such that $f_j(x) > f_j(\bar{x})$ whenever $x \in S$ and $f_i(x) < f_i(\bar{x})$.

We denote the set of all properly efficient solutions of (VP) by $PrEff$(VP).

The following basic definitions can be found in [19].

Definition 2. The subgradient of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ is the set

$$\partial f(x) = \Big\{\xi \in \mathbb{R}^n \mid f(y) \geq f(x) + \xi^T(y - x) \ \text{ for all } y \in \mathbb{R}^n\Big\}.$$

Definition 3. (1) A function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at $x \in \mathbb{R}^n$ if there exist $K > 0$ and $\delta > 0$ such that for any $y, z \in B_\delta(x)$,

$$|f(y) - f(z)| \leq K\|y - z\|,$$

where $B_\delta(x) = \{z \in \mathbb{R}^n \mid \|z - x\| < \delta\}$.

(2) Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $x \in \mathbb{R}^n$. The generalized directional derivative of f at x in the direction of $v \in \mathbb{R}^n$ is defined by

$$f^o(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

The Clarke generalized subgradient of a locally Lipschitz function f at x is denoted by

$$\partial^c f(x) = \Big\{\xi \in \mathbb{R}^n \mid f^o(x; v) \geq \xi^T v \ \text{ for all } v \in \mathbb{R}^n\Big\}.$$

It is well known [5] that
(i) $\partial^c f(x)$ is a nonempty, convex, compact set,
(ii) the function $v \mapsto f^o(x; v)$ is sublinear.

The following definitions can be found in [18].

Definition 4. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at $x \in \mathbb{R}^n$ and let $\xi \in \partial f(x)$ be an arbitrary subgradient. Then the ξ-linearization of f at x is defined as the function $\overline{f}_\xi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\overline{f}_\xi(y) = f(x) + \xi^T(y - x) \ \text{ for all } y \in \mathbb{R}^n$$

and the linearization (approximation) of f at x is the function $\widehat{f}_x : \mathbb{R}^n \to \mathbb{R}$ such that

$$\widehat{f}_x(y) = \max\Big\{\overline{f}_\xi(y) \mid \xi \in \partial f(x)\Big\} \ \text{ for all } y \in \mathbb{R}^n.$$

We can check that
(i) $\widehat{f}_x(x) = f(x)$ for any $x \in \mathbb{R}^n$,
(ii) $\widehat{f}_x(y) = f(x) + f^o(x; y - x)$ for all $y \in \mathbb{R}^n$,
(iii) $\widehat{f}_x$ is convex.

Definition 5. Let $g : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz function at $x \in \mathbb{R}^n$.

(1) g is called pseudo-convex at x if for any $y \in \mathbb{R}^n$,

$$g^o(x; y-x) \geq 0 \text{ implies } g(y) \geq g(x).$$

Equivalently, for any $y \in \mathbb{R}^n$ and for any $\xi \in \partial^c g(x)$, $g(y) < g(x)$ implies $\langle \xi, y-x \rangle < 0$.

(2) g is called quasi-convex at x if for any $y \in \mathbb{R}^n$,

$$g(y) \leq g(x) \text{ implies } g^o(x; y-x) \leq 0.$$

Equivalently, for any $y \in \mathbb{R}^n$ and for any $\xi \in \partial^c g(x)$, $\langle \xi, y-x \rangle > 0$ implies $g(y) > g(x)$.

2. Equivalence results

In this section, we show that the equivalence between a solution of a locally Lipschitz vector optimization problem and a solution of its approximated problem defined below.

Let x_0 be a feasible solution in (VP) and assume that $f_i, \ i = 1, \cdots, p$ and $g_j, \ j = 1, \cdots, m$ are locally Lipschitz at x_0. With the (VP) we also consider the following approximated problem (VP$_L$) given by

$$(\text{VP}_L) \quad \text{Minimize} \quad \Big(f_1(x_0) + f_1^o(x_0; x-x_0), \cdots, f_p(x_0) + f_p^o(x_0; x-x_0)\Big)$$
$$\text{subject to} \quad g_j(x_0) + g_j^o(x_0; x-x_0) \leq 0, \ j = 1, \cdots, m.$$

Theorem 1. *Let $x_0 \in PrEff(VP)$ and suppose that the constraint qualification satisfies at x_0, i.e., $0 \notin co\Big\{\partial^c g_j(x_0) \mid j \in I(x_0)\Big\}$, where $I(x_0) = \{j \mid g_j(x_0) = 0\}$. Then there exist $\lambda_i > 0, \ i = 1, \cdots, p, \ \mu_j \geq 0, \ j = 1, \cdots, m$ such that*

$$0 \in \sum_{i=1}^p \lambda_i \partial^c f_i(x_0) + \sum_{j=1}^m \mu_j \partial^c g_j(x_0), \qquad \sum_{j=1}^m \mu_j g_j(x_0) = 0.$$

Proof. Let $x_0 \in PrEff(\text{VP})$. By Theorem 3.1 in [11], there exist $\gamma > 0$ such that x_0 is a solution of the following optimization problem:

$$\text{Minimize} \quad \sum_{i=1}^p f_i(x) + \gamma \max\{f_1(x) - f_1(x_0), \cdots, f_p(x) - f_p(x_0), 0\}$$
$$\text{subject to} \quad x \in S.$$

So under the constraint qualification, using the optimality theorem, we can check that there exist $\mu_j \geq 0, \ j = 1, \cdots, m$ such that

$$0 \in \sum_{i=1}^p \partial^c f_i(x_0) + \gamma co\Big[\bigcup_{i=1}^p \partial^c f_i(x_0) \cup \{0\}\Big] + \sum_{j=1}^m \mu_j \partial^c g_j(x_0),$$

$$\sum_{j=1}^m \mu_j g_j(x_0) = 0,$$

where coA is the convex hull of the set A. Therefore, there exist $\lambda_i > 0,\ i = 1,\cdots,p,\ \mu_j \geq 0,\ j = 1,\cdots,m$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial^c f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial^c g_j(x_0),$$
$$\sum_{j=1}^{m} \mu_j g_j(x_0) = 0.$$

The following is the definition of a KKT point of (VP):

Definition 6. x_0 is said to be a KKT point of (VP) if $g_j(x_0) \leq 0,\ j = 1,\cdots,m$ and there exist $\lambda_i > 0,\ i = 1,\cdots,p,\ \ \mu_j \geq 0,\ j = 1,\cdots,m$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial^c f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial^c g_j(x_0), \tag{1}$$
$$\sum_{j=1}^{m} \mu_j g_j(x_0) = 0.$$

Theorem 2. *If x_0 is a KKT point of (VP), then $x_0 \in PrEff(VP_L)$.*

Proof. Let x_0 be a KKT point of (VP). Then $g_j(x_0) \leq 0,\ j = 1,\cdots,m$ and there exist $\lambda_i > 0,\ i = 1,\cdots,p,\ \mu_j \geq 0,\ j = 1,\cdots,m,\ \xi_i \in \partial^c f_i(x_0),\ \overline{\xi}_j \in \partial^c g_j(x_0)$ such that

$$\sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \overline{\xi}_j = 0 \quad \text{and} \quad \sum_{j=1}^{m} \mu_j g_j(x_0) = 0.$$

Hence

$$\sum_{i=1}^{p} \lambda_i \xi_i^t (x - x_0) + \sum_{j=1}^{m} \mu_j \overline{\xi}_j^t (x - x_0) = 0$$

for any $x \in \mathbb{R}^n$ and $\sum_{j=1}^{m} \mu_j g_j(x_0) = 0$. Then

$$\sum_{i=1}^{p} \lambda_i \max_{\xi_i \in \partial^c f_i(x_0)} \xi_i^t (x - x_0) + \sum_{j=1}^{m} \mu_j \max_{\overline{\xi}_j \in \partial^c g_j(x_0)} \overline{\xi}_j^t (x - x_0) \geq 0$$

for any $x \in \mathbb{R}^n$ and $\sum_{j=1}^{m} \mu_j g_j(x_0) = 0$. By Proposition 2.1.2 in [5],

$$\sum_{i=1}^{p} \lambda_i f_i^o(x_0; x - x_0) + \sum_{j=1}^{m} \mu_j g_j^o(x_0; x - x_0) \geq 0$$

for any $x \in \mathbb{R}^n$ and $\sum_{j=1}^{m} \mu_j g_j(x_0) = 0$. If $g_j(x_0)+g_j^o(x_0; x-x_0) \leq 0$, $j = 1, \cdots, m$, then

$$\sum_{j=1}^{m} \mu_j g_j(x_0) + \sum_{j=1}^{m} \mu_j g_j^o(x_0; x - x_0) \leq 0$$

and hence $\sum_{i=1}^{p} \lambda_i f_i^o(x_0; x - x_0) \geq 0$, i.e.,

$$\sum_{i=1}^{p} \lambda_i f_i^o(x_0; x - x_0) \geq \sum_{i=1}^{p} \lambda_i f_i^o(x_0; x_0 - x_0).$$

Hence x_0 is a solution of the following scalar optimization problem:

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{p} \lambda_i f_i^o(x_0; x - x_0) \\ \text{subject to} & g_j(x_0) + g_j^o(x_0; x - x_0) \leq 0, \ j = 1, \cdots, m. \end{array}$$

So x_0 is a solution of the following scalar optimization problem:

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{p} \lambda_i f_i(x_0) + \sum_{i=1}^{p} \lambda_i f_i^o(x_0; x - x_0) \\ \text{subject to} & g_j(x_0) + g_j^o(x_0; x - x_0) \leq 0, \ j = 1, \cdots, m. \end{array}$$

Therefore, by Theorem 1 in [8], $x_0 \in PrEff(\text{VP}_L)$.

Theorem 3. *Let $x_0 \in PrEff(VP_L)$ and suppose that the constraint qualification satisfies at x_0, i.e., $0 \notin co\left\{\partial^c g_j(x_0) \mid j \in I(x_0)\right\}$. Then x_0 is a KKT point of (VP).*

Proof. Let $x_0 \in PrEff(\text{VP}_L)$. Note that $f_i(x_0) + f_i^o(x_0; x - x_0)$, $i = 1, \cdots, p$ and $g_j(x_0) + g_j^o(x_0; x - x_0)$, $j = 1, \cdots, m$ are convex functions with respect to x. Thus by Theorem 2 in [8], there exist $\lambda_i > 0$, $i = 1, \cdots, p$, such that x_0 is an optimal solution of the following optimization problem:

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{p} \lambda_i (f_i(x_0) + f_i^o(x_0; x - x_0)) \\ \text{subject to} & g_j(x_0) + g_j^o(x_0; x - x_0) \leq 0, \ j = 1, \cdots, m. \end{array}$$

From the optimality theorem for a scalar optimization problem, there exist $\lambda_i > 0$, $i = 1, \cdots, p$, $\mu_j \geq 0$, $j = 1, \cdots, m$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \partial^c f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial^c g_j(x_0), \qquad \sum_{j=1}^{m} \mu_j g_j(x_0) = 0.$$

Moreover, since x_0 is a feasible solution of (VP_L), $g_j(x_0) \leq 0$, $j = 1, \cdots, m$. Therefore x_0 is a KKT point of (VP).

Theorem 4. *Let $x_0 \in PrEff(VP_L)$ and suppose that the constraint qualification satisfies at x_0, i.e.,* $0 \notin co\Big\{\partial^c g_j(x_0) \mid j \in I(x_0)\Big\}$. *If* $\sum_{i=1}^{p} \lambda_i f_i$ *is pseudo-convex at x_0 and* $\sum_{j=1}^{m} \mu_j g_j$ *is quasi-convex at x_0, then $x_0 \in PrEff(VP)$.*

Proof. If $x_0 \in PrEff(\mathrm{VP}_L)$ and suppose that the constraint qualification satisfies at x_0. Then by Theorem 3, x_0 is a KKT point of (VP), $\sum_{j=1}^{m} \mu_j g_j(x_0) = 0$. Also, for any $x \in S$, $\sum_{j=1}^{m} \mu_j g_j(x) \leq 0$. Therefore,

$$\sum_{j=1}^{m} \mu_j g_j(x) \leq \sum_{j=1}^{m} \mu_j g_j(x_0).$$

By the quasi-convexity of $\sum_{j=1}^{m} \mu_j g_j(x_0)$, $\sum_{j=1}^{m} \mu_j \left\langle \overline{\xi}_j, x - x_0 \right\rangle \leq 0$ for any $\overline{\xi}_j \in \partial^c g_j(x_0)$. From (1), we obtain $\sum_{i=1}^{p} \lambda_i \left\langle \xi_i, x - x_0 \right\rangle \geq 0$ for some $\xi_i \in \partial^c f_i(x_0)$. Thus, by the pseudo-convexity of $\sum_{i=1}^{p} \lambda_i f_i$,

$$\sum_{i=1}^{p} \lambda_i f_i(x) \geq \sum_{i=1}^{p} \lambda_i f_i(x_0)$$

for all $x \in S$. Thus x_0 is optimal solution of the following scalar optimization problem:

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{p} \lambda_i f_i(x) \\ \text{subject to} \quad & g_j(x) \leq 0, \ j = 1, \cdots, m. \end{aligned}$$

Therefore, by Theorem 1 in [8], $x_0 \in PrEff(\mathrm{VP})$.

References

1. T. Antczak, *A new approach to multiobjective programming with a modified objective function*, J. Global Optimization **27**(2003), 485–495.
2. T. Antczak, *An η-approximation approach for nonlinear mathematical programming problems involving invex functions*, Num. Functional Anal. Optimization **25**(2004), 423–438.
3. T. Antczak, *An η-approximation method in nonlinear vector optimization*, Nonlinear Analysis **63**(2005), 225–236.

4. S. Brumelle, *Duality for multiple objective convex programs*, Math. Oper. Res. **6**(1981), 159–172.
5. F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.
6. B.D. Craven, *Invex functions and constrained local minima*, Bull. Aust. Math. Soc. **24**(1981), 357–366.
7. B.D. Craven, *Quasimin and quasi saddle point for vector optimization*, Num. Functional Anal. Optimization **11**(1990), 45–54.
8. A. M. Geoffrion, *Proper efficiency and the theory of vector maximization*, J. Math. Anal. Appl. **22**(1968), 618–630.
9. G. Giorgi, A. Guerraggi, *Various types of nonsmooth invexity*, J. Inform. Optim. Sci. **17**(1996), 137–150.
10. M.A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80**(1981), 545–550.
11. X.X. Huang, X.Q. Yang, *On characterizations of proper efficiency for nonconvex multiobjective optimization*, J. Global Optimization **23**(2002), 213–231.
12. H. Isermann, *Proper efficiency and linear vector maximum problem*, Oper. Res. **22**(1974), 189–191.
13. V. Jeyakumar, B. Mond, *On generalised convex mathematical programming*, J. Austral. Math. Soc. Ser. B **34**(1992), 43–53.
14. D. S. Kim, *Multiobjective fractional programming with a modified objective function*, Commun. Korean Math. Soc. **20**(2005), 837–847.
15. M.H. Kim, *On linearized vector optimization problems with proper efficiency*, J. Appl. Math. & Informatics **27**(2009), 685–692.
16. G.M. Lee, *Optimality conditions in multiobejctive optimization problems*, J. Inform. Optim. Sci. **13**(1993), 39–41.
17. Lun Li and Jun Li, *Equivalence and existence of weak Pareto optima for multiobjective optimization problems with cone constraints*, Appl. Math. Lett. **21**(2008), 599–606.
18. Marko M. Mäkelä and Pekka Neittaanmäki, *Nonsmooth Optimization: Analysis and Algorithms with Applications to Optimal Control*, World Scientific Publishing Co. Pte. Ltd., 1992.
19. R.T. Rockafellar, *Convex Analysis*, Princeton, New Jersey, Princeton University Press, 1970.
20. T. Weir, B. Mond, B.D. Craven, *On duality for weakly minimized vector valued optimization problems*, Optimization **17**(1986), 711-721.

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