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COMMON FIXED POINT THEOREMS FOR MULTIVALUED MAPS SATISFYING CONTACTIVE CONDITIONS OF AN INTEGRAL TYPE

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ABSTRACT. We prove the existence of common fixed points for multivalued maps satisfying a contractive condition of an integral type. Our results are extentions of results of Feng and Liu[Y. Feng, S. Liu, Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings, J. Math. Anal. Appl. 317(2006), 103-112] and also, extentions of results of Daffer and Kaneko[P. Z. Daffer, H. Kaneko, Fixed points of generalized contractive multi-valued mappings, J. Math. Anal. Appl. 192(1995), 655-666]. A main result in Feng and Liu[Y. Feng, S. Liu, Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings, J. Math. Anal. Appl. 317(2006), 103-112] is proved under necessary additional conditions.

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1. Introduction and preliminaries

Let (X, d) be a metric space. We denote by $C(X)$ the family of nonempty closed subsets of X and by $CB(X)$ the family of nonempty closed bounded subsets of X. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $C(X)$. That is, for $A, B \in C(X)$,

$$H(A, B) = \max\left\{sup_{a\in A}d(a, B), sup_{b\in B}d(b, A)\right\}$$

where $d(a, B) = \inf\left\{d(a, b) : b \in B\right\}$ is the distance from the point a to the subset B.

For $k \in (0, 1)$, let $\Psi(k)$ be the family of Lebesgue measurable functions $\varphi : [0, \infty) \to [0, \infty)$ such that

(φ_1) $\varphi > 0$ almost everywhere,

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(φ_2) $\int_0^b \varphi(t)dt < \infty$ for each $b \in (0, \infty)$,

(φ_3) for any sequence $\{t_n\}$ in $(0, \infty)$, if $\left\{k^{-n} \int_0^{t_n} \varphi(t)dt\right\}$ is bounded then $\sum_{n=0}^{\infty} t_n < \infty$.

We denote by Ψ the family of Lebesgue measurable functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying (φ_1), (φ_2) and

(φ_4) for any sequence $\{t_n\}$ in $(0, \infty)$, if $\sum_{n=0}^{\infty} \int_0^{t_n} \varphi(t)dt < \infty$ then $\sum_{n=0}^{\infty} t_n < \infty$.

We know that $\Psi \subset \Psi(k)$ for all $k \in (0, 1)$. For a multivalued map $F : X \to C(X)$, $b \in (0, 1]$ and $x \in X$, we denote

$$\varphi_b^x(F) = \left\{y \in Fx : b \int_0^{d(x,y)} \varphi(t)dt \leq \int_0^{d(x,Fx)} \varphi(t)dt\right\}$$

and

$$I_b^x(F) = \Big\{y \in Fx : bd(x, y) \leq d(x, Fx)\Big\}.$$

For a multivalued map $F : X \to C(X)$, let $f_F : X \to \mathbb{R}$ be a function defined by $f_F(x) = d(x, Fx)$.

The Banach fixed point theorem which was first stated by Banach in 1922 is an important tool in the theory of nonlinear analysis. The theorem guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces and provides a constructive method to find those fixed points.

Also, the theorem has been used to show the existence of solutions of nonlinear Volterra integral equations, nonlinear integro-differential equations in Banach spaces and to show the convergence of algorithms in computational mathematics. Because of its importance for mathematical theory, many authors has been studied the numerous generalizations of the Banach fixed point theorem for single valued maps and also, extended in many different directions[1,2,3,5,9,11,12]. Nadler initially analyzied the existence of fixed points for multivalued contraction maps in metric spaces. He proved the following important theorem in [9].

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to BC(X)$ be a multivalued map. If there exists $q \in [0, 1)$ such that for any $x, y \in X$*

$$H(Tx, Ty) \leq qd(x, y),$$

then T has a fixed point in X.

In [8,10], the authors extended the Nadler's theorem(Theorem 1.1). Recentely, in [7], the authors gave an extention of the Nadler's theorem in an anothor direction than [8] and [10]. They proved the next two theorems.

Theorem 1.2[7]. *Let (X,d) be a complete metric space and $T : X \to C(X)$ be a multivalued map. If there exists $q \in (0,1)$ such that for any $x \in X$ there exists $y \in I_b^x(T)$ satisfying*

$$d(y,Ty) \leq q d(x,y),$$

then T has a fixed point in X provided that $q < b$ and f_T is lower semicontinuous.

Theorem 1.3[7]. *Let (X,d) be a complete metric space and let $T : X \to C(X)$ be a multivalued map. Assume there exists $q \in (0,1)$ such that for any $x \in X$ and $y \in Tx$, there is $z \in Ty$ satisfying*

$$\int_0^{d(y,z)} \varphi(t)dt \leq q \int_0^{d(x,y)} \varphi(t)dt,$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesque-integable mapping which is summable on each compact subset of $[0,\infty)$, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$.

Then T has a fixed point in X provided f_T is lower semicontinuous.

Throught this paper we denote

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}\{d(y,Tx) + d(x,Sy)\}\right\}$$

for given two multivalued maps $S,T : X \to C(X)$, where X is a metric space.

In this paper, we give a contractive condition of an integral type for multivalued maps in metric spaces and prove a common fixed point theorem for these maps. Our results are essentially generalizations of Theorem 1.1 and Theorem 1.2. We give an example which satisfies the contractive condition of integral type(Theorem 2.1) but does not satisfy the contractive condition(Corollary 2.3). And we give an example which satisfies the contractive condition of Corollary 2.3 but does not satisfy a general contractive condition:

(GC) there exists $k \in [0,1)$ such that for all $x,y \in X$,

$$H(Tx,Sy) \leq k m(x,y)$$

where $S,T : X \to CB(X)$.

Also, we give a counterexample for Theorem 1.3 and modify this theorem (Theorem 2.10).

2. Fixed point theorems for multivalued maps

In this section, we prove a common fixed point theorem for a pair of multivalued maps satisfying a contractive condition of integral type. Recall that a function $g : X \to \mathbb{R}$ is *lower semicontinuous* if for any sequence $\{x_n\}$ in X and $x \in X$, $g(x) \leq \underline{\lim}_{n\to\infty} g(x_n)$ whenever $\lim_{n\to\infty} x_n = x$.

Theorem 2.1. *Let (X,d) be a nonempty complete metric space and $T,S: X \to C(X)$ be multivalued maps. Let $k \in (0,1)$ and $\varphi \in \Psi(k)$. Let b and c be real numbers with $0 < c < b < 1$ with $\frac{c}{b} \leq k$ such that for any $x \in X$ there exists $y \in \varphi_b^x(T)$ such that*

$$\int_0^{d(y,Sy)} \varphi(t)dt \leq c \int_0^{m(x,y)} \varphi(t)dt \tag{2.1.1}$$

and there exists $z \in \varphi_b^y(S)$ such that

$$\int_0^{d(z,Tz)} \varphi(t)dt \leq c \int_0^{m(z,y)} \varphi(t)dt. \tag{2.1.2}$$

Then T and S have a common fixed point in X provided that f_T and f_S are lower semicontinuous.

Proof. Let $x_0 \in X$ be arbitrary fixed. From (2.1.1) and (2.1.2), there exists $x_1 \in \varphi_b^{x_0}(T)$ such that

$$\int_0^{d(x_1,Sx_1)} \varphi(t)dt \leq c \int_0^{m(x_0,x_1)} \varphi(t)dt \tag{2.1.3}$$

and there exists $x_2 \in \varphi_b^{x_1}(S)$ such that

$$\int_0^{d(x_2,Tx_2)} \varphi(t)dt \leq c \int_0^{m(x_2,x_1)} \varphi(t)dt. \tag{2.1.4}$$

Then by (φ_1) $d(x_1,Sx_1) < m(x_0,x_1)$ if $d(x_1,Sx_1) \neq 0$, and $d(x_2,Tx_2) < m(x_2,x_1)$ if $d(x_2,Tx_2) \neq 0$. Hence we have

$$\begin{aligned}
&m(x_0,x_1)\\
&= max\left\{d(x_0,x_1), d(x_0,Tx_0), d(x_1,Sx_1), \frac{1}{2}\{d(x_1,Tx_0)+d(x_0,Sx_1)\}\right\}\\
&\leq max\left\{d(x_0,x_1), d(x_0,x_1), d(x_1,Sx_1), \frac{1}{2}\{d(x_0,x_1)+d(x_1,Sx_1)\}\right\}\\
&= d(x_0,x_1)
\end{aligned}$$

and

$$\begin{aligned}
&m(x_2,x_1)\\
&= max\left\{d(x_2,x_1), d(x_2,Tx_2), d(x_1,Sx_1), \frac{1}{2}\{d(x_1,Tx_2)+d(x_2,Sx_1)\}\right\}\\
&\leq max\left\{d(x_1,x_2), d(x_2,Tx_2), d(x_1,x_2), \frac{1}{2}\{d(x_1,x_2)+d(x_2,Tx_2)\}\right\}\\
&= d(x_1,x_2).
\end{aligned}$$

From (2.1.3) and (2.1.4) we have

$$\int_0^{d(x_1,Sx_1)} \varphi(t)dt \leq c\int_0^{d(x_0,x_1)} \varphi(t)dt \tag{2.1.5}$$

and

$$\int_0^{d(x_2,Tx_2)} \varphi(t)dt \leq c\int_0^{d(x_1,x_2)} \varphi(t)dt. \tag{2.1.6}$$

On the other hand, since $x_2 \in \varphi_b^{x_1}(S)$, we have that

$$b\int_0^{d(x_1,x_2)} \varphi(t)dt \leq \int_0^{d(x_1,Sx_1)} \varphi(t)dt. \tag{2.1.7}$$

From (2.1.5) and (2.1.7), we have

$$\int_0^{d(x_1,x_2)} \varphi(t)dt \leq \frac{c}{b}\int_0^{d(x_0,x_1)} \varphi(t)dt. \tag{2.1.8}$$

Also, from (2.1.6) and (2.1.8), we have

$$\int_0^{d(x_2,Tx_2)} \varphi(t)dt \leq \frac{c^2}{b}\int_0^{d(x_0,x_1)} \varphi(t)dt. \tag{2.1.9}$$

Repeat the above process, we have $x_3 \in \varphi_b^{x_2}(T)$ and $x_4 \in \varphi_b^{x_3}(S)$ such that

$$\int_0^{d(x_3,Sx_3)} \varphi(t)dt \leq c\int_0^{m(x_2,x_3)} \varphi(t)dt$$

and

$$\int_0^{d(x_4,Tx_4)} \varphi(t)dt \leq c\int_0^{m(x_4,x_3)} \varphi(t)dt.$$

Then we also have

$$\int_0^{d(x_3,x_4)} \varphi(t)dt \leq \frac{c}{b}\int_0^{d(x_2,x_3)} \varphi(t)dt \tag{2.1.10}$$

and

$$\int_0^{d(x_4,Tx_4)} \varphi(t)dt \leq \frac{c^2}{b}\int_0^{d(x_2,x_3)} \varphi(t)dt. \tag{2.1.11}$$

Since $x_3 \in \varphi_b^{x_2}(T)$, we have

$$b\int_0^{d(x_2,x_3)} \varphi(t)dt \leq \int_0^{d(x_2,Tx_2)} \varphi(t)dt. \tag{2.1.12}$$

From (2.1.9) and (2.1.12) we have

$$\int_0^{d(x_2,x_3)} \varphi(t)dt \le (\frac{c}{b})^2 \int_0^{d(x_0,x_1)} \varphi(t)dt \tag{2.1.13}$$

Inductively, we can construct a sequence $\{x_n\}$ in X such that for all $n = 0, 1, 2, \cdots$

$$x_{2n+1} \in \varphi_b^{x_{2n}}(T), x_{2n+2} \in \varphi_b^{x_{2n+1}}(S),$$

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \le \frac{c}{b} \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt,$$

and

$$\int_0^{d(x_{2n+2},x_{2n+3})} \varphi(t)dt \le (\frac{c}{b})^2 \int_0^{d(x_{2n},Tx_{2n+1})} \varphi(t)dt.$$

Then we have for each $n = 1, 2, \cdots$,

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le (\frac{c}{b})^n \int_0^{d(x_0,x_1)} \varphi(t)dt \le k^n \int_0^{d(x_0,x_1)} \varphi(t)dt.$$

Then since $\left\{k^{-n} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right\}$ is bounded above by $\int_0^{d(x_0,x_1)} \varphi(t)dt$, we have

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty \text{ by } (\varphi_3).$$

Therefore, $\{x_n\}$ is a Cauchy sequence, and let $\lim_{n\to\infty} x_n = x$ for some $x \in X$. Since $f_S(x_{2n+1}) = d(x_{2n+1}, Sx_{2n+1}) \le d(x_{2n+1}, x_{2n+2})$ and $f_T(x_{2n}) = d(x_{2n}, Tx_{2n}) \le d(x_{2n}, x_{2n+1})$, $\lim_{n\to\infty} f_S(x_{2n+1}) = 0$ and $\lim_{n\to\infty} f_T(x_{2n+1}) = 0$. Since f_S and f_T are lower semicontinuous, $f_S(x) = 0$ and $f_T(x) = 0$. Thus $x \in Sx$ and $x \in Tx$.

Corollary 2.2. *Let (X, d) be a complete metric space and $T, S : X \to C(X)$ be multivalued maps and let $k \in (0, 1)$ and $\varphi \in \Psi(k)$ satysfying there exists $c \in (0, k)$ such that for any $x, y \in X$*

$$\int_0^{H(Tx,Sy)} \varphi(t)dt \le c \int_0^{m(x,y)} \varphi(t)dt.$$

Then T and S have a common fixed point in X provided that f_T and f_S are lower semicontinuous.

Proof. Let us take $b \in (c, 1)$ with $\frac{c}{b} \le k$. Then for any $x \in X$, we know $\varphi_b^x(T) \ne \emptyset$ and $\varphi_b^y(S) \ne \emptyset$. Since for any $y \in \varphi_b^x(T)$ and $z \in \varphi_b^y(S)$, $d(y, Sy) \le H(Tx, Sy)$

and $d(z, Tz) \leq H(Sy, Tz)$, we have

$$\int_0^{d(y,Sy)} \varphi(t)dt \leq c \int_0^{m(x,y)} \varphi(t)dt$$

and

$$\int_0^{d(z,Tz)} \varphi(t)dt \leq c \int_0^{m(z,y)} \varphi(t)dt.$$

Therefore, from Theorem 2.1, T and S have a common fixed point in X.

If we take $\varphi(t) = 1$ in Theorem 2.1 and Corollary 2.2, then we know $\varphi \in \Psi(k)$ for all $k \in (0, 1)$ and hence we have the next two corollaries.

Corollary 2.3. *Let (X, d) be a complete metric space and $T, S : X \to C(X)$ be multivalued maps satisfying there exist real numbers b and c with $0 < c < b < 1$ such that for any $x \in X$ there exists $y \in I_b^x(T)$ such that*

$$d(y, Sy) \leq c\, m(x, y)$$

and there exists $z \in I_b^y(S)$ such that

$$d(z, Tz) \leq c\, m(z, y).$$

Then T and S have a common fixed point in X provided f_T and f_S are lower semicontinuous.

Corollary 2.4. *Let (X, d) be a complete metric space and $T, S : X \to C(X)$ be multivalued maps satysfying there exists $c \in (0, 1)$ such that for any $x, y \in X$*

$$H(Tx, Sy) \leq c\, m(x, y).$$

Then T and S have a common fixed point in X provided f_T and f_S are lower semicontinuous.

Now we give an example which satisfies conditions in Theorem 2.1 but does not satisfy conditions in Corollary 2.3.

Example 2.5. Let $X = \left\{ \frac{1}{n} : n = 1, 2, \cdots \right\} \cup \{0\}$ with the Euclidean metric d. Then (X, d) is a complete metric space.

Let $\varphi(t) = \begin{cases} 1 & \left(t > \frac{1}{2} \right), \\ \frac{n(n+1)(n+2)}{2^{n+1}} & \left(\frac{1}{(n+1)(n+2)} < t \leq \frac{1}{n(n+1)}, n \geq 1 \right). \end{cases}$
Then (φ_1) and (φ_2) are satisfied.

Let $\{t_n\}$ be a sequence in $(0,\infty)$ such that $\left\{\left(\frac{1}{2}\right)^{-n}\int_0^{t_n}\varphi(t)dt\right\}$ is bounded. Then there exists $M>0$ such that $\int_0^{t_n}\varphi(t)dt \le \left(\frac{1}{2}\right)^n M$ for all $n\ge 1$. Take a natural number N such that $M\le 2^N$. Then we have for $n\ge N$

$$\int_0^{t_n}\varphi(t)dt \le \left(\frac{1}{2}\right)^{n-N}. \qquad (*)$$

Since $\left(\frac{1}{2}\right)^n = \int_0^{\frac{1}{(n+1)(n+2)}}\varphi(t)dt$, from $(*)$ we have $t_n \le \frac{1}{(n-N+1)(n-N+2)}$ for each $n\ge N$. Thus we have $\varphi\in\Psi\left(\frac{1}{2}\right)$.

Let $b=1$, $c=\frac{1}{2}$ and $T,S: X\to C(X)$ be multivalued maps defined by

$$Tx=\begin{cases}\left\{\frac{1}{n+1},\frac{1}{n+2}\right\} & \left(x=\frac{1}{n}, n=1,2,\cdots\right),\\ \{0\} & (x=0)\end{cases}$$

and

$$Sx=\begin{cases}\left\{\frac{1}{n+1},\frac{1}{n+3}\right\} & \left(x=\frac{1}{n}, n=1,2,\cdots\right),\\ \{0\} & (x=0).\end{cases}$$

If $x=0$, then there exists $y=0\in T_b^x$ such that

$$\int_0^{d(y,Ty)}\varphi(t)dt \le c\int_0^{d(x,y)}\varphi(t)dt \le c\int_0^{m(x,y)}\varphi(t)dt$$

and there exists $z=0\in S_b^y$ such that

$$\int_0^{d(z,Sz)}\varphi(t)dt \le c\int_0^{d(y,z)}\varphi(t)dt \le c\int_0^{m(z,y)}\varphi(t)dt.$$

Suppose that $x=\frac{1}{n}, n=1,2,\cdots$. Then there exists $y=\frac{1}{n+1}\in\varphi_b^x(T)$ such that

$$\begin{aligned}\int_0^{d(y,Ty)}\varphi(t)dt &= \int_0^{d(\frac{1}{n+1},T\frac{1}{n+1})}\varphi(t)dt = \int_0^{d(\frac{1}{n+1},\frac{1}{n+2})}\varphi(t)dt\\ &= \left(\frac{1}{2}\right)^n = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}\\ &\le c\int_0^{d(\frac{1}{n},\frac{1}{n+1})}\varphi(t)dt = c\int_0^{d(x,y)}\varphi(t)dt\end{aligned}$$

and there exists $z=\frac{1}{n+2}\in\varphi_b^y(S)$ such that

$$\int_0^{d(z,Sz)} \varphi(t)dt = \int_0^{d(\frac{1}{n+2},S\frac{1}{n+2})} \varphi(t)dt = \int_0^{d(\frac{1}{n+2},\frac{1}{n+3})} \varphi(t)dt$$
$$= \left(\frac{1}{2}\right)^{n+1} = \frac{1}{2}\left(\frac{1}{2}\right)^n$$
$$\leq c\int_0^{d(\frac{1}{n+1},\frac{1}{n+2})} \varphi(t)dt = c\int_0^{d(z,y)} \varphi(t)dt.$$

Therefore, T and S satisfy conditions in Theorem 2.1 and $0 \in T0 \cap S0$. But T and S do not satisfy conditions in Corollary 2.3. In fact, suppose that T and S satisfy conditions in Corollary 2.3. Then for $x = \frac{1}{n}(n = 1, 2, \cdots)$ there exists $y = \frac{1}{n+1} \in I_b^x(T)$ such that

$$d(y,Ty) = d\left(\frac{1}{n+1}, \frac{1}{n+2}\right) \leq kd\left(\frac{1}{n}, \frac{1}{n+1}\right) = kd(x,y) \leq km(x,y)$$

and there exists $z = \frac{1}{n+2} \in I_b^y(S)$ such that

$$d(z,Sz) = d\left(\frac{1}{n+2}, \frac{1}{n+3}\right) \leq kd\left(\frac{1}{n+1}, \frac{1}{n+2}\right) = kd(y,z) \leq km(z,y)$$

Thus we have $k \geq \frac{n}{n+2}$ and $k \geq \frac{n+1}{n+3}$ for $n = 1, 2, \cdots$. Hence $k \geq 1$, which is a contradiction.

Thus T and S do not satisfy conditions in Corollary 2.3. Therefore, Theorem 2.1 is a generalization of Corollary 2.3.

The next example shows that there exist two maps S, T which satisfy conditions of Corollary 2.3 but not (GC).

Example 2.6. Let $X = \left\{\frac{1}{2^n} : n = 0, 1, 2 \cdots\right\} \cup \{0\}$ and let $S, T : X \to CB(X)$ be multivalued maps defined by

$$Sx = Tx = \begin{cases} \left\{\frac{1}{2^{n+1}}, 1\right\} & \left(x = \frac{1}{2^n}, n = 0, 1, 2, \cdots\right), \\ \left\{0, \frac{1}{2}\right\} & (x = 0). \end{cases}$$

Then we have

$$H\left(T\frac{1}{2^n}, S0\right) = \frac{1}{2}$$
$$\geq \frac{1}{2^n} = \max\left\{d\left(\frac{1}{2^n}, 0\right), d\left(\frac{1}{2^n}, T\frac{1}{2^n}\right), d(0, S0), \frac{1}{2}\left\{d(\frac{1}{2^n}, S0) + d(0, T\frac{1}{2^n})\right\}\right\}$$

for $n = 1, 2, \cdots$. Thus T and S do not satisfy (GC). It is easy to see that

$$f_S x = f_T x = d(x, Tx) = \begin{cases} \frac{1}{2^{n+1}} & \left(x = \frac{1}{2^n}, n = 1, 2, 3, \cdots\right), \\ 0 & (x = 0, 1). \end{cases}$$

Hence f_T and f_S are lower semicontinuous. Furthermore, for each $x \in X$ there exists $y \in I^x_{0.7}(T)$ such that $d(y, Ty) = \frac{1}{2}d(x, y)$ and there exists $z \in I^y_{0.7}(S)$ such that $d(z, Sz) = \frac{1}{2}d(y, z)$. Thus from Corollary 2.3, T and S have a common fixed point in X. Therefore, Corollary 2.3 is a generalization of theorem 3.3 of [6].

If we take $S = T$ in Theorem 2.1(resp., Corollary 2.3, Corollary 2.4), then we have the next Corollary 2.7(resp., Theorem 1.2, theorem 3.3[6])

Corollary 2.7. *Let (X, d) be a complete metric space and $T : X \to C(X)$ be multivalued maps. Let $k \in (0, 1)$ and $\varphi \in \Psi(k)$. Let b and c be real numbers with $0 < c < b < 1$ with $\frac{c}{b} \le k$ such that for any $x \in X$ there exists $y \in \varphi^x_b(T)$ such that*

$$\int_0^{d(y,Ty)} \varphi(t)dt \le c \int_0^{max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\{d(y,Tx)+d(x,Ty)\}\}} \varphi(t)dt.$$

Then T has a fixed point in X provided f_T is lower semicontinuous.

Corollary 2.8. *Let (X, d) be a complete metric space and $T : X \to C(X)$ be multivalued map satisfying there exists $c \in (0, b)$ such that for any $x \in X$ there exists $y \in I^x_b(T)$ such that*

$$d(y, Ty) \le c \ \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}\{d(y, Tx) + d(x, Ty)\}\right\}.$$

Then T has a fixed point in X provided f_T is lower semicontinuous.

Note that if $Sx = Tx = \begin{cases} \left\{\frac{1}{n+1}\right\} & \left(x = \frac{1}{n}, n = 1, 2, \cdots\right), \\ \{0\} & (x = 0) \end{cases}$ in Example 2.5, then the conditions of Corollary 2.7 are satisfied but the conditions of Corollary 2.8 are not satisfied.

Now, we give a counterexample for Theorem 1.3.

Example 2.9. Let $X = \left\{x_n : x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, n = 1, 2, \cdots\right\}$ with the Euclidean metric d. Let $\varphi(s) = \frac{2^{-1/s} ln2}{s^2}$ and $c = \frac{1}{2}$. Let $T : X \to C(X)$

be multivalued map defined by $Tx_n = \{x_{n+1}\}$. Then T is continuous and f_T is lower semicontinuous.

For any $x_n \in X$ and $x_{n+1} \in Tx_n$, there is $x_{n+2} \in Tx_{n+1}$ such that

$$\int_0^{d(x_{n+1},x_{n+2})} \varphi(s)ds = \int_0^{\frac{1}{n+2}} \varphi(s)ds = \frac{1}{2^{n+2}} \leq \frac{1}{2}\frac{1}{2^n} = \int_0^{d(x_n,x_{n+1})} \varphi(s)ds.$$

Hence T satisfies all conditions of Theorem 1.3. However T has no fixed points.

Note that φ does not satisfy (φ_3) for all $k \in (0,1)$. Therefore, we modify Theorem 1.3 as the next Theorem 2.10.

Let $\phi : [0,\infty) \to [0,\infty)$ be a function satisfying

(ϕ1) $\phi(0) = 0$ and $0 < \phi(t) < t$ for all $t > 0$,

(ϕ2) $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.

Theorem 2.10. *Let (X,d) be a complete metric space. If $T : X \to C(X)$ is a multivalued map and $\varphi \in \Psi$ satisfying*

for any $x \in X$ and $y \in Tx$, there exists $z \in Ty$ such that

$$\int_0^{d(y,z)} \varphi(t)dt \leq \phi\left(\int_0^{max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}\{d(y,Tx)+d(x,Ty)\}\}} \varphi(t)dt\right), \tag{2.10.1}$$

then T has a fixed point in X provided f_T is lower semicontinuous.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. we have a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and for $n = 1, 2, \cdots$

$$\begin{aligned}
&\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \\
&\leq \phi(\int_0^{max\{d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),\frac{1}{2}\{d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)\}\}} \varphi(t)dt) \\
&\leq \phi\left(\int_0^{max\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{1}{2}\{d(x_n,x_n)+d(x_{n-1},x_{n+1})\}\}} \varphi(t)dt\right) \\
&\leq \phi\left(\int_0^{max\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{1}{2}\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}\}} \varphi(t)dt\right) \\
&\leq \phi\left(\int_0^{d(x_{n-1},x_n)} \varphi(t)dt\right).
\end{aligned}$$

Thus we have

$$\sum_{n=0}^{\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq \sum_{n=0}^{\infty} \phi^n(\int_0^{d(x_0,x_1)} \varphi(t)dt) < \infty,$$

which implies $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$.

Hence $\{x_n\}$ is Cauchy. By the completeness of X, there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Since f_T is lower semicontinuous, we have

$$0 \le f_T(p) \le \lim_{n\to\infty} f_T(x_n) \le \lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

which implies $f_T(p) = d(p, Tp) = 0$. Since $Tp \in C(X)$, $p \in Tp$.

If we take $\phi(t) = ct$, $c \in (0,1)$ in Theorem 2.10, then we have the next theorem.

Theorem 2.11. *Let (X,d) be a complete metric space. If $T : X \to C(X)$ is a multivalued map and $\varphi \in \Psi$ satisfying there exists $c \in (0,1)$ such that for any $x \in X$ and $y \in Tx$, there exists $z \in Ty$ such that*

$$\int_0^{d(y,z)} \varphi(t)dt \le c\int_0^{max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}\{d(y,Tx)+d(x,Ty)\}\}} \varphi(t)dt,$$

then T has a fixed point in X provided f_T is lower semicontinuous.

Example 2.12. Let (X,d) be the metric space as in Example 2.5 and $\varphi(t) = 1$ for $0 \le t < \infty$. Let $\phi : [0,\infty) \to [0,\infty)$ be a function defined by

$$\phi(t) = \begin{cases} \dfrac{1}{(k+1)(k+2)} & \left(t = \dfrac{1}{k(k+1)}\right), \\ 0 & (x = 0), \\ \dfrac{1}{6} & \left(t \ge \dfrac{1}{2}\right), \\ linearly & (otherwise) \end{cases}$$

and let $T : X \to C(X)$ be a multivalued map defined by

$$Tx = \begin{cases} \left\{\dfrac{1}{n+1}\right\} & \left(x = \dfrac{1}{n}, n = 1, 2, \cdots\right), \\ \{0\} & (x = 0). \end{cases}$$

Then $\varphi \in \Psi$, and f_T is lower semicontinuous, and ϕ satisfies (ϕ1) and (ϕ2).

We now show that condition (2.10.1) of Theorem 2.10 is satisfied. If $x = 0$, then for $y = 0 \in Tx$, there exists $z = 0 \in Ty$ such that $\int_0^{d(y,z)} \varphi(t)dt \le \phi\left(\int_0^{d(x,y)} \varphi(t)dt\right)$. Suppose that $x = \frac{1}{n}(n = 1, 2, \cdots)$. Then for $y = \frac{1}{n+1} \in Tx$, there exists $z = \frac{1}{n+2} \in Ty$ such that

$$\int_0^{d(y,z)} \varphi(t)dt = \int_0^{\frac{1}{(n+1)(n+2)}} \varphi(t)dt = \frac{1}{(n+1)(n+2)}$$

$$\leq \phi(\frac{1}{n(n+1)}) = \phi(\int_0^{\frac{1}{n(n+1)}} \varphi(t)dt) = \phi\left(\int_0^{d(x,y)} \varphi(t)dt\right).$$

Therefore, all conditions of Theorem 2.10 are satisfied and $0 \in T0$.

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