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ASYMPTOTIC BEHAVIOR OF HIGHER ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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ABSTRACT. The asymptotic behavior of solutions of higher order differential equations with deviating argument

$$(py^{(n-1)}(t))' + \sum_{i=1}^{n-1} c_i(t)y^{(i-1)}(t) = f\Big[t, y(t), y'(t), ..., y^{(n-1)}(t), y(\phi(t)), y'(\phi(t)), ..., y^{(n-1)}(\phi(t))\Big] \quad (1)$$

$t \in [0, \infty)$ is studied. Our technique depends on an integral inequality containing a deviating argument. From this we obtain some sufficient conditions under which all solutions of Eq.(1) have some asymptotic behavior.

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1. Introduction

We consider the differential equation

$$(py^{(n-1)}(t))' + \sum_{i=1}^{n-1} c_i(t)y^{(i-1)}(t) = f\Big[t, y(t), y'(t), ..., y^{(n-1)}(t), y(\phi(t)), y'(\phi(t)), ..., y^{(n-1)}(\phi(t))\Big] \quad (1)$$

where $p = p(t)$ is a positive and differentiable function on $R_+ = [0, \infty)$ such that $p(0) = 1$; $c_i = c_i(t)$ and ϕ are continuous functions on R_+ for $i = 1, 2, ...n-1$; f is a continuous function on $R_+ \times R^{2n}$.

We recall that the initial value problem of Eq.(1)is defined as follows [2].

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Let $\inf_{t\in[0,\infty)}\phi(t) = r$ and θ be a given differential function defined on $[r,0]$.The initial value problem of Eq.(1) in the interval $[0,\beta)$ is to find a function $y(t)$, such that

(*i*) $y(0^+) = \theta(0^-)$, $y'(0^+) = \theta'(0^-), ..., y^{(n-1)}(0^+) = \theta^{(n-1)}(0^-)$;

(*ii*) $y^{(j)}(t) = \theta^{(j)}(t)$, $t \in [r,0)$, $j = 0,1,...,n-1$;

(*iii*) $(py^{n-1}(t))'$ exist on $[0,\beta)$ and Eq.(1) is satisfied.

We also assume that the initial value problem of Eq.(1)is local existence.

As in [1], here we suppose that the general solution of the differential equation

$$(pz^{(n-1)})' + \sum_{i=1}^{n-1} c_i(t)z^{(i-1)} = 0, \qquad t \in R_+, \tag{2}$$

is known, and let $\{z_i\}_{i=1}^n$be a fundamental system of solutions of (2)which satisfies for each $i = 1,2,...n$,

$$(z_i(0), z_i'(0), ..., z_i^{(n-1)}(0)) = \bar{e}_i, \tag{3}$$

where $\bar{e}_i = (0,0,...,1,0,...,0)$ (1 in the i-th place)is the standard base of R^n.

We will prove that all solutions y of (1)are defined on all of $R^+$and these satisfy the equation as $t \to \infty$.

$$y^{(k)} = \sum_{i=1}^{n} (\delta_i + o(1))z_i^{(k)}, \tag{4}$$

where $k = 0,1,2,...,n-1$ and $\delta_i : i = 1,2,...n$ are constants.

In 1996, S.M.Aziz and A.H.Nasr [1] studied the asymptotic behavior and oscillations of solutions of second order differential equation with deviating argument of the form

$$(a(t)x')' + b(t)x' + c(t)x = f[t, x(t), x'(t), x(\phi(t)), x'(\phi(t))].$$

The purpose of this paper is to establish a new integral inequality containing a deviating argument and to study the asymptotic behavior of (1)which extend the result in [1].

2. Preliminary

In the subsequent discussion we shall require the following integral inequality.

Lemma 1. *Let* $u(t)$ *and* $a(t)$: $I = [0,\infty) \to R_+ = [0,\infty)$ *are continuous functions,the functions* $f_i(t,s)$ $(i = 1,2...m) : D = [(t,s) : 0 \le s \le t < \infty] \to R_+$*are continuous. Suppose that the inequality*

$$u(t) \le a(t) + \sum_{i=1}^{m} \int_0^t f_i(t,s)[u(s)]^{r_i} ds, \tag{5}$$

holds for all $t \in I$, *where* r_i *are numbers from* $(0,1]$. *Then we have*

$$u(t) \le a(t) + F(t)exp\sum_{i=1}^{m} \int_0^t f_i(t,s)r_ik^{r_i-1}ds, \quad t \in I \tag{6}$$

for any $k > 0$*, where*

$$F(t) = \sum_{i=1}^{m} \int_0^t f_i(t,s)[(1-r_i)k^{r_i} + r_i k^{r_i-1} a(s)]ds. \tag{7}$$

Proof. Define a function $z(t)$ by

$$z(t) = \sum_{i=1}^{m} \int_0^t f_i(t,s)[u(s)]^{r_i} ds. \tag{8}$$

Then (5) can be restated as

$$u(t) \leq a(t) + z(t). \tag{9}$$

By (8) we have

$$z(t) \leq \sum_{i=1}^{m} \int_0^t f_i(t,s)[a(s) + z(s)]^{r_i} ds. \tag{10}$$

By [2, lemma 1], we have

$$z(t) \leq \sum_{i=1}^{m} \int_0^t f_i(t,s) \left[r_i k^{r_i-1}(a(s) + z(s)) + (1-r_i)k^{r_i} \right] ds,$$

for any $k > 0$, and then $z(t) \leq F(t) + \sum_{i=1}^{m} \int_0^t f_i(t,s) r_i k^{r_i-1} z(s) ds$. It is easy to see that $F(t)$ is nonnegative continuous and nondecreasing for $t \in I$.
Thus we have

$$z(t) \leq F(t) exp \sum_{i=1}^{m} \int_0^t f_i(t,s) r_i k^{r_i-1} ds. \tag{11}$$

The desired inequality (6) follows from (9) and (11).

Lemma 2. *Let the following conditions be satisfied:*

1. $u(t)$ *and* $a(t) : I = [0,\infty) \to R_+ = [0,\infty)$ *are continuous functions,* $\phi(t)$ *is a continuously differentiable function satisfying that* $\phi(t) \leq t$, $\phi'(t) > 0$ *and* $\phi(t)$ *is eventually positive.*

2. The functions $f_i(t,s)$ $(i = 1, 2...m)$ *and* $g_i(t,s)$ $(i = 1, 2...n)$: $D = [(t,s) : 0 \leq s \leq t < \infty] \to R_+$*are continuous and nondecreaing in* t *for* $s \in I$ *fixed.*

3. The following integral inequality

$$u(t) \leq a(t) + \sum_{i=1}^{m} \int_0^t f_i(t,s)[u(s)]^{r_i} ds + \sum_{i=1}^{n} \int_0^t g_i(t,s)[u(\phi(s))]^{p_i} ds, \tag{12}$$

holds for all $t \in I$*, where* r_i $(1 \leq i \leq m)$, p_i $(1 \leq i \leq n)$*, are numbers from* $(0,1]$*. Then we have*

$$u(t) \leq a_1(t) + \bar{F}(\phi^{-1}(t))$$

$$\times exp \sum_{i=1}^{m} \int_0^t f_i(t,s) r_i k^{r_i-1} ds exp \sum_{i=1}^{n} \int_{\phi^{-1}(0)}^{\phi^{-1}(t)} g_i(\phi^{-1}(t), s) p_i k^{p_i-1} ds \tag{13}$$

for any $k > 0$, where

$$\begin{aligned}\bar{F}(t) &= \textstyle\sum_{i=1}^{m} \int_{\phi^{-1}(0)}^{t} f_i(t,\phi(s))\phi'(s)[(1-r_i)k^{r_i} + r_i k^{r_i-1} a_1(\phi(s))]ds \\ &+ \textstyle\sum_{i=1}^{n} \int_{\phi^{-1}(0)}^{t} g_i(t,s)[(1-p_i)k^{p_i} + p_i k^{p_i-1} a_1(\phi(s))]ds,\end{aligned} \tag{14}$$

$$a_1(t) = a(t) + \sum_{i=1}^{n} \int_0^{\phi^{-1}(0)} g_i(t,s)[u(\phi(s))]^{p_i} ds. \tag{15}$$

Proof. From (12) we deduce that for sufficiently large t,

$$\begin{aligned} u(\phi(t)) \le\ & a(\phi(t)) + \sum_{i=1}^{m} \int_0^{\phi(t)} f_i(\phi(t),s)[u(s)]^{r_i} ds \\ & + \sum_{i=1}^{n} \int_0^{\phi(t)} g_i(\phi(t),s)[u(\phi(s))]^{p_i} ds \\ \le\ & a(\phi(t)) + \sum_{i=1}^{m} \int_0^{\phi(t)} f_i(\phi(t),s)[u(s)]^{r_i} ds \\ & + \sum_{i=1}^{n} \int_0^{\phi(t)} g_i(t,s)[u(\phi(s))]^{p_i} ds. \end{aligned} \tag{16}$$

It is easy to see that

$$\sum_{i=1}^{m} \int_0^{\phi(t)} f_i(\phi(t),s)[u(s)]^{r_i} ds = \sum_{i=1}^{m} \int_{\phi^{-1}(0)}^{t} f_i(t,\phi(s))[u(\phi(s))]^{r_i} \phi'(s) ds,$$

substituing this in (16) and keeping in mind that

$$\sum_{i=1}^{n} \int_0^{\phi(t)} g_i(t,s)[u(\phi(s))]^{p_i} ds \le \sum_{i=1}^{n} \int_0^{t} g_i(t,s)[u(\phi(s))]^{p_i} ds$$

$$= \sum_{i=1}^{n} \int_{\phi^{-1}(0)}^{t} g_i(t,s)[u(\phi(s))]^{p_i} ds + \sum_{i=1}^{n} \int_0^{\phi^{-1}(0)} g_i(t,s)[u(\phi(s))]^{p_i} ds$$

we get

$$\begin{aligned} u(\phi(t)) \le a_1(\phi(t)) &+ \sum_{i=1}^{m} \int_{\phi^{-1}(0)}^{t} f_i(t,\phi(s))[u(\phi(s))]^{r_i} \phi'(s) ds \\ &+ \sum_{i=1}^{n} \int_{\phi^{-1}(0)}^{t} g_i(t,s)[u(\phi(s))]^{p_i} ds. \end{aligned}$$

Applying Lemma 1 to the function $u(\phi(t))$, we obtain

$$u(\phi(t)) \le a_1(\phi(t)) + \bar{F}(t) exp\Bigg[\sum_{i=1}^{m} \int_{\phi^{-1}(0)}^{t} f_i(t,\phi(s))\phi'(s) r_i k^{r_i-1} ds$$

$$+\sum_{i=1}^{n}\int_{\phi^{-1}(0)}^{t} g_i(t,s)p_i k^{p_i-1}ds\Big].$$

From this we have
$u(t) \le a_1(t) + \bar{F}(\phi^{-1}(t))$

$$\times exp\sum_{i=1}^{m}\int_{0}^{t} f_i(t,s)r_i k^{r_i-1}ds exp\sum_{i=1}^{n}\int_{\phi^{-1}(0)}^{\phi^{-1}(t)} g_i(\phi^{-1}(t),s)p_i k^{p_i-1}ds.$$

The proof of this lemma is now completed.

Remark 1. If we take $m = n = 1$, or $r_i \equiv 1$ then the inequality established in Lemma 2 improves the inequality established in [1].

3. Main result

In this section we will prove (4). In this section $V_k, k = 1,2,...,n$ will denote a determinant, which is obtained from Wronsky's determinant $W[z_1,z_2,...,z_n]$, replacing its k-th column by $e_n = (0,...,0,1) \in R^n$.

Theorem 1. *In addition to the previous assumptions on (1) and (2),we assume further that*

(i) for $t \in R_+,\ u_1,...,u_n,v_1,...,v_n \in R$ we have

$$|f(t,u_1,...,u_n,v_1,...,v_n)| \le \sum_{i=1}^{n} b_i(t)|u_i|^{r_i} + \sum_{i=1}^{n} h_i(t)|v_i|^{p_i} + e_1(t),$$

where e_1 and h_i and $b_i : R_+ \to R_+$ are continuous $r_i,\ p_i \in (0,1]$ are constants.

(ii) the functions $b_i(t)(\sum_{k=1}^{n}|V_k(t)|)(\sum_{j=1}^{n}|z_j^{(i-1)}(t)|)^{r_i}$, $i = 1,2,...,n$, and $h_i(t)(\sum_{k=1}^{n}|V_k(t)|)$ $(\sum_{j=1}^{n}|z_j^{(i-1)}(\phi(t))|)^{p_i}$, $i = 1,2,...,n$, and $e_1(t)(\sum_{k=1}^{n}|V_k(t)|)$ are in the class $L_1(0,\infty)$;

Then, any solution y of (1) is defined on all of R_+ and satisfies (4).

Proof. let $y(t)$ be any solution of (1) existing on the interval $[r,0]\bigcup[0,\beta)$, and satisfying the initial condition $y(t) = \theta(t),\ t \in [r,0]$, here $0 < \beta \le \infty$. Put

$$y(t) = \sum_{i=1}^{n} A_i(t)z_i(t). \tag{17}$$

Since $A_i(t)$ $(i = 1,2,...,n)$ are unknown functions, we may require

$$\sum_{i=1}^{n} A_i'(t)z_i^{(j-1)}(t) = 0, \quad j = 1,2,...,n-1. \tag{18}$$

Then, after some straightforword computations, and using (18) many times, we have

$$y^{(k)}(t) = A_1(t)z_1^{(k)}(t) + ... + A_n(t)z_n^{(k)}(t), \quad k = 1,2,...,n-1, \tag{19}$$

and

$$(py^{(n-1)})' = p\left(\sum_{i=1}^{n} A_i' z_i^{(n-1)}\right) + \sum_{i=1}^{n} A_i (p z_i^{(n-1)})'.$$

Because $z_1, z_2, ... z_n$ are solutions of Eq.(2), using (1), the last equation can be reduced to

$$p(t)\left(\sum_{i=1}^{n} A_i'(t) z_i^{(n-1)}(t)\right) = H(t), \quad t \in [0, \beta) \tag{20}$$

where

$$H(t) = f\left[t, y(t), y'(t), ..., y^{(n-1)}(t), y(\phi(t)), y'(\phi(t)), ..., y^{(n-1)}(\phi(t))\right]$$

herein $y(t), y'(t), ..., y^{(n-1)}(t)$ being determined by (17) and (19),respectively. Having $p(t)w[z_1, z_2, ... z_n] = 1$, we solve Eqs (18) and (20) for $A_i'(t)$, and obtain

$$A_k'(t) = V_k(t) H(t) \quad k = 1, 2, ..., n. \tag{21}$$

Integrating (21) from 0 to t, we get $A_k(t) = A_k(0) + \int_0^t V_k(s) H(s) ds$. Setting $I(t) = \sum_{i=1}^{n} |A_i(t)|$, then for $t \in [0, \beta)$, we have

$$I(t) \le I(0) + \sum_{k=1}^{n} \int_0^t |V_k(s)||H(s)| ds.$$

Applying Conditions (i) to the above inequality we obtain

$$\begin{aligned}
I(t) &\le I(0) + \int_0^t \left(\sum_{k=1}^{n} |V_k(s)|\right) \\
&\quad \times \left| f\left[s, y(s), y'(s), ..., y^{(n-1)}(s), y(\phi(s)), y'(\phi(s)), ..., y^{(n-1)}(\phi(s))\right] \right| \\
&\le I(0) + \int_0^t \left(\sum_{k=1}^{n} |V_k(s)|\right) \Big[b_1(s)|y(s)|^{r_1} + ... + b_n(s)|y^{(n-1)}(s)|^{r_n} \\
&\quad + h_1(s)|y(\phi(s))|^{p_1} + ... + h_n(s)|y^{(n-1)}(\phi(s))|^{p_n} + e_1(s) \Big] \\
&\le I(0) + \int_0^t \left(\sum_{k=1}^{n} |V_k(s)|\right) \Big[b_1(s) \left(\left| \sum_{i=1}^{n} A_i(s) z_i(s) \right| \right)^{r_1} + ... \\
&\quad + b_n(s) \left(\left| \sum_{i=1}^{n} A_i(s) z_i^{(n-1)}(s) \right| \right)^{r_n} + h_1(s) \left(\left| \sum_{i=1}^{n} A_i(\phi(s)) z_i(\phi(s)) \right| \right)^{p_1} \\
&\quad + ... + h_n(s) \left(\left| \sum_{i=1}^{n} A_i(\phi(s)) z_i^{(n-1)}(\phi(s)) \right| \right)^{p_n} + e_1(s) \Big]
\end{aligned}$$

Thus, using

$$\left|\sum_{i=1}^{n} A_i z_i\right| \leq \left(\sum_{i=1}^{n} |A_i|\right)\left(\sum_{i=1}^{n} |z_i|\right)$$

the function $I(t)$ satisfy

$$I(t) \leq M(t) + \sum_{i=1}^{n} \int_0^t \left(\sum_{k=1}^{n} |V_k(s)|\right) b_i(s) \left(\sum_{j=1}^{n} \left|z_j^{(i-1)}(s)\right|\right)^{r_i} [I(s)]^{r_i}$$

$$+ \sum_{i=1}^{n} \int_0^t \left(\sum_{k=1}^{n} |V_k(s)|\right) h_i(s) \left(\sum_{j=1}^{n} \left|z_j^{(i-1)}(\phi(s))\right|\right)^{p_i} [I(\phi(s))]^{p_i} \qquad (22)$$

where

$$M(t) = I(0) + \int_0^t e_1(s) \left(\sum_{k=1}^{n} |V_k(s)|\right) ds$$

By our lemma, we obtain from (22),

$$I(t) \leq M_1(t) + \bar{F}(\phi^{-1}(t)) exp \sum_{i=1}^{n} \int_0^t \alpha_i(s) r_i k^{r_i - 1} ds exp \sum_{i=1}^{n} \int_{\phi^{-1}(0)}^{\phi^{-1}(t)} e_i(s) p_i k^{p_i - 1} ds, \qquad (23)$$

for any $k > 0$, where

$$\alpha_i(s) = \sum_{k=1}^{n} |V_k(s)| b_i(s) \left(\sum_{j=1}^{n} \left|z_j^{(i-1)}(s)\right|\right)^{r_i}$$

$$e_i(s) = \sum_{k=1}^{n} |V_k(s)| h_i(s) \left(\sum_{j=1}^{n} \left|z_j^{(i-1)}(\phi(s))\right|\right)^{p_i}$$

$$\begin{aligned} \bar{F}(t) = \ & \textstyle\sum_{i=1}^{n} \int_{\phi^{-1}(0)}^{t} \alpha_i(\phi(s))\phi'(s)[(1-r_i)k^{r_i} + r_i k^{r_i - 1} M_1(\phi(s))] ds \\ & + \textstyle\sum_{i=1}^{n} \int_{\phi^{-1}(0)}^{t} e_i(s))[(1-p_i)k^{p_i} + p_i k^{p_i - 1} M_1(\phi(s))] ds \end{aligned}$$

$$M_1(t) = M(t) + \sum_{i=1}^{n} \int_0^{\phi^{-1}(0)} e_i(s)[I(\phi(s))]^{p_i} ds$$

By the following well-known standard argument, using the conditions (ii) and in view of the local existence of Eq.(1), we can easily see that $\beta = \infty$ holds, and so the bounded of the functions $A_i(t)(i = 1, 2, ..., n)$ on R_+ follows from (23), and then it follows that the limits of these functions exist when $t \to \infty$. The proof of the Theorem 1 is complete.

4. The above result for other operators

Let us consider the integro-differential equations of order $n = m + l$ of the form

$$(Py^{(m)}(t))^{(l)} \pm qy(t)$$
$$= f\Big[t, y(t), y'(t), ..., y^{(m)}(t), (Py^{(m)}(t))', ..., (Py^{(m)}(t))^{(l-1)}, y(\phi(t)),$$
$$y'(\phi(t)), ..., y^{(m)}(\phi(t)), (Py^{(m)}(\phi(t))', ..., (Py^{(m)}(\phi(t))^{(l-1)}\Big], \tag{24}$$

where $P = P(t)$ is a function of class $C^l_{(R_+)}$ such that $P(0) = 1$; $q = q(t)$ is a continuous functions on R_+.

Let $\{Z_i\}_{i=1}^{m+l}$ be a fundamental system of solutions of the linear differential equation

$$(PZ^m)^{(l)} \pm qZ = 0 \tag{25}$$

and satisfy (3).

we will demonstrate that every solution y of (24) is defined on all of $R^+$and satisfies

$$y^{(i)} = \sum_{j=1}^{n}(\delta_j + o(1))Z_j^{(i)} \quad (0 \le i \le m)$$
$$(Py^{(m)})^{(i)} = \sum_{j=1}^{n}(\delta_j + o(1))(PZ_j^{(m)})^{(i)}, \quad (0 \le i \le l-1) \tag{26}$$

where δ_i $i = 1, 2, ...n$ are constants.

In this section $\bar{V}_k, k = 1, 2, ..., n$ will denote a determinant, which is obtained from Wronsky's determinant $\bar{W} = \bar{W}[Z_1, ..., Z_k, ..., Z_n]$ replacing its k-th column by $e_n = (0, ..., 0, 1) \in R^n$. Here the $\bar{W} = \bar{W}[Z_1, ..., Z_k, ..., Z_n]$ of this system is formed by the k-column $\Big(Z_k, Z_k', ..., Z_k^{(m-1)}, (PZ_k^{(m)}), ..., (PZ_k^{(m)})^{(l-1)}(1 \le k \le n)\Big)$. This Wronsky's determinant $\bar{W}[Z_1, ..., Z_n]$ is everywhere equal to 1.

Theorem 2. *Let condition (i) of Theorem 1 hold, and also the following condition be satisfied:*

(ii)′ The functions $b_i(t)\bar{Z}(t), h_i(t)\hat{z}(t)$, $i = 1, 2, ..., n$, $e_1(t)v(t)$ are in the class $L_1(0, \infty)$;

where

$$\begin{cases} \bar{Z}_i(t) = v(t)(\sum_{k=1}^n |Z_k^{(i-1)}|)^{r_i} & (1 \le i \le m+1) \\ v(t)(\sum_{k=1}^n |(PZ_k^{(m)})^{(i-m-1)}|)^{r_i} & (m+2 \le i \le n). \end{cases}$$

$$\begin{cases} \hat{Z}_i(t) = v(t)(\sum_{k=1}^n |Z_k^{(i-1)}(\phi)|)^{p_i} & (1 \le i \le m+1) \\ v(t)(\sum_{k=1}^n |(PZ_k^{(m)})^{(i-m-1)}(\phi)|)^{p_i} & (m+2 \le i \le n). \end{cases}$$

$$v(t) = \sum_{k=1}^{n} |\bar{V}_k(t)|,$$

Then, any solution y of (24) is defined on all of R_+ and satisfies (26).

Proof. let $y(t)$ be any solution of (1) existing on the interval $[r,0]\bigcup[0,\beta)$, and satisfying the initial condition $y(t)=\theta(t),\ t\in[r,0]$, here $0<\beta\leq\infty$. Put

$$y(t)=\sum_{i=1}^{n}A_i(t)Z_i(t). \tag{27}$$

and we impose $n-1$ conditions on the unknown functions $A_i(t)$:

$$\sum_{i=1}^{m+l}A_i'Z_i^{(j)}=0,\quad (0\leq j\leq m{+}1),\ \ \sum_{i=1}^{m+l}A_i'(PZ_i^{(m)})^{(j)}=0,\quad (0\leq j\leq l{-}2). \tag{28}$$

Then $y^{(j)}=\sum_{i=1}^{n}A_iZ_i^{(j)},\quad (0\leq j\leq m)$ and

$$Py^{(m)}=\sum_{i=1}^{n}A_iPZ_i^{(m)}. \tag{29}$$

Hence, by (28)

$$(Py^{(m)})'=\sum_{i=1}^{m+l}([A_i'(PZ_i^{(m)})+A_i(PZ_i^{(m)})'])=\sum_{i=1}^{m+l}A_i(PZ_i^{(m)})'$$

and

$$(Py^{(m)})''=\sum_{i=1}^{m+l}([A_i'(PZ_i^{(m)})'+A_i(PZ_i^{(m)})''])=\sum_{i=1}^{m+l}A_i(PZ_i^{(m)})''.$$

On proceeding inductively in this way, we have

$$(Py^{(m)})^{(j)}=\sum_{i=1}^{m+l}A_i(PZ_i^{(m)})^{(j)},\qquad (0\leq j\leq l-1). \tag{30}$$

Moreover, from (25) we get

$$(Py^{(m)})^{(l)}=\sum_{i=1}^{m+l}[A_i(PZ_i^{(m)})^{(l-1)}]'=\sum_{i=1}^{m+l}A_i'(PZ_i^{(m)})^{(l-1)}\mp q\sum_{i=1}^{m+l}A_iZ_i.$$

Then, by(24), we have

$$\sum_{i=1}^{m+l}A_i'(PZ_i^{(m)})^{(l-1)}=(Py^{(m)}(t))^{(l)}\pm qy(t)$$

$$=f\Big[t,y(t),y'(t),...,y^{(m)}(t),(Py^{(m)}(t))',...,(Py^{(m)}(t))^{(l-1)},y(\phi(t)),y'(\phi(t)),$$

$$...,y^{(m)}(\phi(t)),(Py^{(m)}(\phi(t))',...,(Py^{(m)}(\phi(t))^{(l-1)}\Big]=:\ H(t)$$

So,

$$\sum_{i=1}^{m+l}A_i'(PZ_i^{(m)})^{(l-1)}=H(t) \qquad (31).$$

Now, we solve Eq.(28) and (31) with respect to A_i' $(1 \le i \le n)$. Then we have

$$A_i(t) = A_i(0) + \int_0^t \bar{V}_i(s)H(s)ds; \quad (1 \le i \le n). \tag{32}$$

Furthermore, by(i),(29) and (30), we obtain

$$\begin{aligned}
|H(t)| \le & \sum_{i=1}^{m+1} b_i(t) \left(\left| \sum_{k=1}^{n} Z_k^{(i-1)} A_k \right| \right)^{r_i} + e_1(t) \\
& + \sum_{i=m+2}^{n} b_i(t) \left(\left| \sum_{k=1}^{n} (PZ_k^{(m)})^{(i-m-1)} A_k \right| \right)^{r_i} \\
& + \sum_{i=1}^{m+1} h_i(t) \left(\left| \sum_{k=1}^{n} Z_k^{(i-1)}(\phi) A_k(\phi) \right| \right)^{p_i} \\
& + \sum_{i=m+2}^{n} h_i(t) \left(\left| \sum_{k=1}^{n} (PZ_k^{(m)}(\phi))^{(i-m-1)} A_k(\phi) \right| \right)^{p_i} \\
\le & \sum_{i=1}^{m+1} b_i \left(\sum_{k=1}^{n} \left| Z_k^{(i-1)} \right| \right)^{r_i} \left(\sum_{k=1}^{n} |A_k| \right)^{r_i} + \sum_{i=m+2}^{n} b_i \left(\sum_{k=1}^{n} \left| (PZ_k^{(m)})^{(i-m-1)} \right| \right)^{r_i} \\
& \times \left(\sum_{k=1}^{n} |A_k| \right)^{r_i} + \sum_{i=1}^{m+1} h_i \left(\sum_{k=1}^{n} \left| Z_k^{(i-1)}(\phi) \right| \right)^{p_i} \left(\sum_{k=1}^{n} |A_k(\phi)| \right)^{p_i} + e_1(t) \\
& + \sum_{i=m+2}^{n} h_i \left(\sum_{k=1}^{n} \left| (PZ_k^{(m)}(\phi))^{(i-m-1)} \right| \right)^{p_i} \left(\sum_{k=1}^{n} |A_k(\phi)| \right)^{p_i} \Bigg].
\end{aligned}$$

Then proceeding as in the proof of Theorem 1, we have that $y(t)$ is defined on all of R_+ and satisfies (26).

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