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# MINIMUM DEGREE AND INDEPENDENCE NUMBER FOR THE EXISTENCE OF HAMILTONIAN $[a, b]$-FACTORS

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ABSTRACT. Let $a$ and $b$ be nonnegative integers with $2 \le a < b$, and let $G$ be a Hamiltonian graph of order $n$ with $n > \frac{(a+b-5)(a+b-3)}{b-2}$. An $[a, b]$-factor $F$ of $G$ is called a Hamiltonian $[a, b]$-factor if $F$ contains a Hamiltonian cycle. In this paper, it is proved that $G$ has a Hamiltonian $[a, b]$-factor if $\delta(G) \ge \frac{(a-1)n+a+b-3}{a+b-3}$ and $\delta(G) > \frac{(a-2)n+2\alpha(G)-1}{a+b-4}$.

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## 1. Introduction

All graphs considered in this paper will be finite and undirected graphs without loops or multiple edges. In particular, a graph is said to be a Hamiltonian graph if it contains a Hamiltonian cycle. Let $G$ be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges, respectively. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of $x$ in $G$ and by $N_G(x)$ the set of vertices adjacent to $x$ in $G$. For $S \subseteq V(G)$, we define $N_G(S) = \cup_{x \in S} N_G(x)$, and $G[S]$ is the subgraph of $G$ induced by $S$. We denote by $G - S$ the subgraph obtained from $G$ by deleting vertices in $S$ together with the edges incident to vertices in $S$. Denote by $\alpha(G)$ the independence number of a graph $G$ and by $\delta(G)$ the minimum degree of vertices in $G$. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges.

Let $a$ and $b$ be integers with $0 \le a \le b$. An $[a, b]$-factor of a graph $G$ is defined as a spanning subgraph $F$ of $G$ such that $a \le d_F(x) \le b$ for each $x \in V(G)$ (where of course $d_F$ denotes the degree in $F$). And if $a = b = k$, then an $[a, b]$-factor is

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called a $k$-factor. An $[a, b]$-factor $F$ of $G$ is called a Hamiltonian $[a, b]$-factor if $F$ contains a Hamiltonian cycle. And if $a = b = k$, then a Hamiltonian $[a, b]$-factor is simply called a Hamiltonian $k$-factor. The other terminologies and notations not given here can be found in [1].

Many authors have investigated factors [2–8]. Y. Gao, G. Li and X. Li [9] gave a degree condition for a graph to have a Hamiltonian $k$-factor. H. Matsuda [10] showed a degree condition for graphs to have Hamiltonian $[a, b]$-factors. S. Zhou and B. Pu [11] obtained a neighborhood condition for a graph to have a Hamiltonian $[a, b]$-factor.

The following results on $k$-factors, Hamiltonian $k$-factors and Hamiltonian $[a, b]$-factors are known.

**Theorem 1.** [8] *Let $k \geq 2$ be an integer and let $G$ be a graph with $n$ vertices. If $k$ is odd, then suppose that $n$ is even and $G$ is connected. Let $G$ satisfy*

$$n > 4k + 1 - 4\sqrt{k+2},$$

$$\delta(G) \geq \frac{k-1}{2k-1}(n+2) \quad \text{and}$$

$$\delta(G) > \frac{1}{2k-2}((k-2)n + 2\alpha(G) - 2).$$

*Then $G$ has a $k$-factor.*

**Theorem 2.** [9] *Let $k \geq 2$ be an integer and let $G$ be a graph of order $n > 12(k-2)^2 + 2(5-\alpha)(k-2) - \alpha$. Suppose that $kn$ is even, $\delta(G) \geq k$ and*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n+\alpha}{2}$$

*for each pair of nonadjacent vertices $x$ and $y$ in $G$, where $\alpha = 3$ for odd $k$ and $\alpha = 4$ for even $k$. Then $G$ has a Hamiltonian $k$-factor if for a given Hamiltonian cycle $C$, $G - E(C)$ is connected.*

**Theorem 3.** [10] *Let $a$ and $b$ be integers with $2 \leq a < b$, and let $G$ be a Hamiltonian graph of order $n \geq \frac{(a+b-4)(2a+b-6)}{b-2}$. Suppose that $\delta(G) \geq a$ and*

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a-2)n}{a+b-4} + 2$$

*for each pair of nonadjacent vertices $x$ and $y$ of $V(G)$. Then $G$ has a Hamiltonian $[a, b]$-factor.*

**Theorem 4.** [11] *Let $a$ and $b$ be nonnegative integers with $2 \leq a < b$, and let $G$ be a Hamiltonian graph of order $n$ with $n \geq \frac{(a+b-3)(2a+b-6)-a+2}{b-2}$. Suppose for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geq \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor; \quad \text{or}$$

$$|N_G(X)| \geq \frac{a+b-3}{b-2}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor.$$

*Then $G$ has a Hamiltonian $[a, b]$-factor.*

G. Liu and L. Zhang [12] proposed the following problem.

**Problem.** *Find sufficient conditions for graphs to have connected $[a, b]$-factors related to other parameters in graphs such as binding number, independence number, neighborhood and connectivity.*

We now show our main theorem which partially solves the above problem.

**Theorem 5.** *Let $a$ and $b$ be nonnegative integers with $2 \le a < b$, and let $G$ be a Hamiltonian graph of order $n$ with $n > \frac{(a+b-5)(a+b-3)}{b-2}$. Suppose that $G$ satisfies*

$$\delta(G) \ge \frac{(a-1)n + a + b - 3}{a + b - 3} \quad \text{and}$$

$$\delta(G) > \frac{(a-2)n + 2\alpha(G) - 1}{a + b - 4}.$$

*Then $G$ has a Hamiltonian $[a, b]$-factor.*

## 2. The Proof of Theorem 5

The proof of our main Theorem relies heavily on the following lemma. Lemma 2.1 is a well-known necessary and sufficient for a graph to have a $(g, f)$-factor which was given by Lovasz. The following result is the special case which we use to prove our main theorem.

**Lemma 2.1.** [13] *Let $G$ be a graph, and let $a$ and $b$ be two nonnegative integers with $a < b$. Then $G$ has an $[a, b]$-factor if and only if*

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \ge 0$$

*for any disjoint subsets $S$ and $T$ of $V(G)$.*

*Proof of Theorem 5.* According to assumption, $G$ has a Hamiltonian cycle $C$. Let $G' = G - E(C)$. Note that $V(G') = V(G)$.

Obviously, $G$ has a Hamiltonian $[a, b]$-factor if and only if $G'$ has an $[a-2, b-2]$-factor. By way of contradiction, we assume that $G'$ has no $[a-2, b-2]$-factor. Then, by Lemma 2.1, there exist disjoint subsets $S$ and $T$ of $V(G')$ such that

$$\delta_{G'}(S, T) = (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \le -1. \tag{1}$$

We choose such subsets $S$ and $T$ so that $|T|$ is as small as possible.

If $T = \emptyset$, then by (1), $-1 \ge \delta_{G'}(S, T) = (b-2)|S| \ge |S| \ge 0$, which is a contradiction. Hence, $T \ne \emptyset$. Set

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

We choose $x_1 \in T$ satisfying $d_{G-S}(x_1) = h$. Clearly,

$$\delta(G) \le d_G(x_1) \le d_{G-S}(x_1) + |S| = h + |S|. \tag{2}$$

Now, we prove the following claims.

**Claim 1.** $d_{G'-S}(x) \le a-3$ for all $x \in T$.
*Proof.* If $d_{G'-S}(x) \ge a-2$ for some $x \in T$, then the subsets $S$ and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of $S$ and $T$.

**Claim 2.** $d_{G-S}(x) \le d_{G'-S}(x) + 2 \le a-1$ for all $x \in T$.
*Proof.* Note that $G' = G - E(C)$. Thus, we get from Claim 1

$$d_{G-S}(x) \le d_{G'-S}(x) + 2 \le a-1$$

for all $x \in T$.

According to the definition of $h$ and Claim 2, we obtain $0 \le h \le a-1$. The proof splits into two cases by the value of $h$.

**Case 1.** $h = 0$.

Let $X = \{x \in T : d_{G-S}(x) = 0\}$, $Y = \{x \in T : d_{G-S}(x) = 1\}$, $Y_1 = \{x \in Y : N_{G-S}(x) \subseteq T\}$ and $Y_2 = Y - Y_1$. Then the graph induced by $Y_1$ in $G-S$ has maximum degree at most 1. Let $Z$ be a maximum independent set of this graph. Obviously, $|Z| \ge \frac{1}{2}|Y_1|$. In terms of our definitions, $X \cup Z \cup Y_2$ is an independent set of $G$. Thus, we have

$$\alpha(G) \ge |X| + |Z| + |Y_2| \ge |X| + \frac{1}{2}|Y_1| + \frac{1}{2}|Y_2| = |X| + \frac{1}{2}|Y|. \qquad (3)$$

Using (1), (2), (3) and Claim 2, we obtain

$$\begin{aligned}
\alpha(G) - 1 &\ge |X| + \frac{1}{2}|Y| + \delta_{G'}(S,T) \\
&= |X| + \frac{1}{2}|Y| + (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \\
&\ge |X| + \frac{1}{2}|Y| + (b-2)|S| + d_{G-S}(T) - 2|T| - (a-2)|T| \\
&= |X| + \frac{1}{2}|Y| + (b-2)|S| + d_{G-S}(T) - a|T| \\
&= |X| + \frac{1}{2}|Y| + (b-2)|S| + d_{G-S}(T \setminus (X \cup Y)) + |Y| - a|T| \\
&= |X| + \frac{3}{2}|Y| + (b-2)|S| + d_{G-S}(T \setminus (X \cup Y)) - a|T| \\
&\ge |X| + \frac{3}{2}|Y| + (b-2)|S| + 2|T - (X \cup Y)| - a|T| \\
&= (b-2)|S| - (a-2)|T| - (|X| + \frac{1}{2}|Y|) \\
&\ge (b-2)\delta(G) - (a-2)|T| - \alpha(G),
\end{aligned}$$

the above inequality implies

$$(a-2)|T| \ge (b-2)\delta(G) - 2\alpha(G) + 1. \qquad (4)$$

**Subcase 1.1.** $a = 2$.

From (4), we have

$$\delta(G) \le \frac{(a-2)|T| + 2\alpha(G) - 1}{b-2} = \frac{2\alpha(G) - 1}{b-2}. \qquad (5)$$

On the other hand, by the assumption of the theorem, we get

$$\delta(G) > \frac{(a-2)n+2\alpha(G)-1}{a+b-4} = \frac{2\alpha(G)-1}{b-2},$$

which contradicts (5).

**Subcase 1.2.** $a \geq 3$.

Using (2), (4) and $|S|+|T| \leq n$, we obtain

$$\begin{aligned} 0 &\leq n-|S|-|T| \\ &\leq n-\delta(G)-\frac{(b-2)\delta(G)-2\alpha(G)+1}{a-2} \\ &= \frac{(a-2)n-(a+b-4)\delta(G)+2\alpha(G)-1}{a-2}, \end{aligned}$$

which implies

$$\delta(G) \leq \frac{(a-2)n+2\alpha(G)-1}{a+b-4}.$$

The above inequality contradicts $\delta(G) > \frac{(a-2)n+2\alpha(G)-1}{a+b-4}$.

**Case 2.** $1 \leq h \leq a-1$.

In terms of (1) and $|S|+|T| \leq n$, we get

$$\begin{aligned} -1 &\geq \delta_{G'}(S,T) = (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \\ &\geq (b-2)|S| + d_{G-S}(T) - 2|T| - (a-2)|T| \\ &= (b-2)|S| + d_{G-S}(T) - a|T| \\ &\geq (b-2)|S| + h|T| - a|T| \\ &= (b-2)|S| - (a-h)|T| \\ &\geq (b-2)|S| - (a-h)(n-|S|) \\ &= (a+b-h-2)|S| - (a-h)n, \end{aligned}$$

that is,

$$|S| \leq \frac{(a-h)n-1}{a+b-h-2}. \tag{6}$$

From (2) and (6), we have

$$\delta(G) \leq |S| + h \leq \frac{(a-h)n-1}{a+b-h-2} + h. \tag{7}$$

**Subcase 2.1.** $h = 1$.

According to (7), we obtain

$$\delta(G) \leq \frac{(a-1)n-1}{a+b-3} + 1 = \frac{(a-1)n+a+b-4}{a+b-3}.$$

That contradicts $\delta(G) \geq \frac{(a-1)n+a+b-3}{a+b-3}$.

**Subcase 2.2.** $2 \leq h \leq a-1$.

Set $f(h) = \frac{(a-h)n-1}{a+b-h-2}+h$. Then we get by $2 \le h \le a-1$ and $n > \frac{(a+b-5)(a+b-3)}{b-2}$

$$\begin{aligned} f'(h) &= \frac{-n(a+b-h-2)+(a-h)n-1}{(a+b-h-2)^2}+1 \\ &= \frac{-(b-2)n-1}{(a+b-h-2)^2}+1 \\ &\le \frac{-(b-2)n-1}{(a+b-4)^2}+1 \\ &< \frac{-(a+b-5)(a+b-3)-1}{(a+b-4)^2}+1 \\ &= 0. \end{aligned}$$

Hence, $f(h)$ attains its maximum value at $h=2$. Using (7) and $n > \frac{(a+b-5)(a+b-3)}{b-2}$,

$$\begin{aligned} \delta(G) &\le f(h) \le f(2) = \frac{(a-2)n-1}{a+b-4}+2 \\ &= \frac{(a-2)(a+b-3)n-(a+b-3)}{(a+b-4)(a+b-3)}+2 \\ &= \frac{((a-1)(a+b-4)-(b-2))n-(a+b-3)}{(a+b-4)(a+b-3)}+2 \\ &= \frac{(a-1)(a+b-4)n-(b-2)n-(a+b-3)}{(a+b-4)(a+b-3)}+2 \\ &< \frac{(a-1)(a+b-4)n-(a+b-5)(a+b-3)-(a+b-3)}{(a+b-4)(a+b-3)}+2 \\ &= \frac{(a-1)(a+b-4)n-(a+b-4)(a+b-3)}{(a+b-4)(a+b-3)}+2 \\ &= \frac{(a-1)n+a+b-3}{a+b-3}, \end{aligned}$$

which contradicts $\delta(G) \ge \frac{(a-1)n+a+b-3}{a+b-3}$. From the above contradictions we deduce that $G'$ has an $[a-2, b-2]$-factor. This completes the proof of Theorem 5.

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