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INTUITIONISTIC FUZZY IDEALS IN ORDERED SEMIGROUPS

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ABSTRACT. We prove that a regular ordered semigroup S is left simple if and only if every intuitionistic fuzzy left ideal of S is a constant function. We also show that an ordered semigroup S is left (resp. right) regular if and only if for every intuitionistic fuzzy left(resp. right) ideal $A = \langle \mu_A, \gamma_A \rangle$ of S we have $\mu_A(a) = \mu_A(a^2), \gamma_A(a) = \gamma_A(a^2)$ for every $a \in S$. Further, we characterize some semilattices of ordered semigroups in terms of intuitionistic fuzzy left(resp. right) ideals. In this respect, we prove that an ordered semigroup S is a semilattice of left (resp. right) simple semigroups if and only if for every intuitionistic fuzzy left (resp. right) ideal $A = \langle \mu_A, \gamma_A \rangle$ of S we have $\mu_A(a) = \mu_A(a^2)$, $\gamma_A(a) = \gamma_A(a^2)$ and $\mu_A(ab) = \mu_A(ba)$, $\gamma_A(ab) = \gamma_A(ba)$ for all $a, b \in S$.

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1. Introduction

The theory of fuzzy sets proposed by Zadeh in 1965, (see [24]) has achieved a great success in various fields. Also several higher order fuzzy sets, introduced by Atanassov (see [2], [3] and [4]) have been found to be highly useful to deal with vagueness. Gau and Buehre in [9], presented the concept of vague sets. But, Burillo and Bustine in [5], have shown that the notion of vague sets coincides with that of intuitionistic fuzzy sets. Szmidt and Kacprzyk [23] proposed a non-probabilistic type entropy measures for intuitionistic fuzzy sets. De et al. [7] studied the Sanchez's approach for medical diagnosis and extended this concept with the notion of intuitionistic fuzzy set theory. Dengfeng and Chunfian [8] introduced the concept of the degree of similarity between intuitionistic

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fuzzy sets, which may be finite or continuous, and gave corresponding proofs of these similarity measure and discussed applications of the similarity measures between intuitionistic fuzzy sets to pattern recognition problems. Intuitionistic fuzzy sets have many applications in mathematics, Davvaz et al. [6], applied this concept in H_v-modules. They introduced the notion of an intuitionistic fuzzy H_v-submodule of an H_v-module and studied the related properties. Jun in [10], introduced the concept of an intuitionistic fuzzy bi-ideal in ordered semigroups and characterized the basic properties of ordered semigroups in terms of intuitionistic fuzzy bi-ideals. In [15], [16], Kim and Jun introduced the concept of intuitionistic fuzzy (interior) ideals of semigroups. In [20], Shabir and Khan gave the concept of an intuitionistic fuzzy interior ideal of ordered semigroups and characterized different classes of ordered semigroups in terms of intuitionistic fuzzy interior ideals. They also gave the concept of an intuitionistic fuzzy generalized bi-ideal in [21] and discussed different calsses of ordered semigroups in terms of intituitionistic fuzzy generalized bi-ideals.

In this paper, we characterize regular, left and right simple ordered semigroups and completely regular ordered semigroups in terms of intuitionistic fuzzy left (resp. right) ideals. In this respect, we prove that: A regular ordered semigroup S left simple if and only if every intuitionistic fuzzy left ideal $A = \langle \mu_A, \gamma_A \rangle$ of S is a constant mapping. We also prove that S is left regular if and only if for every intuitionistic fuzzy left ideal $A = \langle \mu_A, \gamma_A \rangle$ of S we have $\mu_A(a) = \mu_A(a^2)$ and $\gamma_A(a) = \gamma_A(a^2)$ for every $a \in S$. Next we characterize some semilattices of left simple ordered semigroups in terms of intuitionistic fuzzy left ideals of S. We prove that an ordered semigroup S is a semilattice of left simple semigroups if and only if for every intuitionistic fuzzy left ideal $A = \langle \mu_A, \gamma_A \rangle$ of S we have, $\mu_A(a) = \mu_A(a^2)$, $\gamma_A(a) = \gamma_A(a^2)$ and $\mu_A(ab) = \mu_A(ba)$, $\gamma_A(ab) = \gamma_A(ba)$ for all $a, b \in S$.

2. Preliminaries

By an ordered semigroup (or po-semigroup) we mean a structure $(S, \cdot, \leq)$ in which

$(OS1)$ $(S, \cdot)$ is a semigroup,

$(OS2)$ $(S, \leq)$ is a poset,

$(OS3)$ $(\forall a, b, x \in S)(a \leq b \Longrightarrow ax \leq bx$ and $xa \leq xb)$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $A \subseteq S$, we denote $(A] := \{t \in S | t \leq h$ for some $h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $A, B \subseteq S$, we denote, $AB := \{ab | a \in A, b \in B\}$. Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *left*(resp. *right) ideal* of S (see [11]) if

(i) $SA \subseteq A$(resp. $AS \subseteq A$) and

(ii) $(\forall a \in A)(\forall b \in S)(b \leq a \Longrightarrow b \in A)$.

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is called a *subsemigroup* of S [13] if $A^2 \subseteq A$. Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup A of S is called a *bi-ideal* (see [13]) of S if

(i) $ASA \subseteq A$ and

(ii) $(\forall a \in A)(\forall b \in S)(b \leq a \Longrightarrow b \in A)$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup A of S is called a $(1,2)$-*ideal* (see [22]) of S if

(i) $ASA^2 \subseteq A$ and

(ii) $(\forall a \in A)(\forall b \in S)(b \leq a \Longrightarrow b \in A)$.

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then the set $(A \cup SA]$ is the left ideal of S generated by A. In particular, if $A = \{x\}(x \in S)$, then we write $(x \cup Sx]$, instead of $(\{x\} \cup S\{x\}]$.

Let S be an ordered semigroup. By a *fuzzy subset* f of S we mean a *mapping* $f : S \longrightarrow [0,1]$. Let $(S, \cdot, \leq)$ be an ordered semigroups and f a fuzzy subset of S. Then f is called a *fuzzy left(resp. right) ideal* (see [12]) of S if

(1) $(\forall x, y \in S)(x \leq y \Longrightarrow f(x) \geq f(y))$.

(2) $(\forall x, y \in S)(f(xy) \geq f(y)(\text{resp. } f(xy) \geq f(x))$.

A fuzzy left and right ideal f of S is called a *fuzzy two-sided ideal* of S.

3. Intuitionistic fuzzy ideals

As an important generalization of the notion of fuzzy sets in S, Atanassov [2], introduced the concept of an intuitionistic fuzzy set (IFS for short) defined on a non-empty set S as objects having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in S\}$, where the functions $\mu_A : S \longmapsto [0,1]$ and $\gamma_A : S \longmapsto [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in S$ to the set A, respectively and $0 \leq \mu(x) + \gamma(x) \leq 1$, for each $x \in S$. For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \gamma_A \rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in S\}$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. An IFS $A = \langle \mu_A, \gamma_A \rangle$ in S is called an *intuitionistic fuzzy subsemigroup* (see [10, 21]) of S if:

(1) $(\forall x, y \in S)(\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y))$,

(2) $(\forall x, y \in S)(\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y))$.

Definition 1. An intuitionistic fuzzy set $A = \langle \mu_A, \gamma_A \rangle$ in S is called an intuitionistic fuzzy left(resp. right) ideal (see [19]) of S if

(1) $(\forall x, y \in S)(x \leq y \Longrightarrow \mu_A(x) \geq \mu_A(y), \gamma_A(x) \leq \gamma_A(y))$,

(2) $(\forall x, y \in S)(\mu_A(xy) \geq \mu_A(y)(\text{resp. } \mu_A(xy) \geq \mu_A(x)))$,

(3) $(\forall x, y \in S)(\gamma_A(xy) \leq \gamma_A(y)(\text{resp. } \gamma_A(xy) \leq \gamma_A(x)))$.

Definition 2. An intuitionistic fuzzy subsemigroup $A = \langle \mu_A, \gamma_A \rangle$ of S is called an intuitionistic fuzzy $(1,2)$-ideal of S if

(1) $(\forall x, y \in S)(x \leq y \Longrightarrow \mu_A(x) \geq \mu_A(y), \gamma_A(x) \leq \gamma_A(y))$,

(2) $(\forall x, y, z, w \in S)(\mu_A(xw(yz)) \geq \min\{\mu_A(x), \mu_A(y), \mu_A(z)\})$,

(3) $(\forall x, y, z, w \in S)(\gamma_A(xw(yz)) \leq \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\})$.

Definition 3. An *intuitionistic fuzzy subsemigroup* $A = \langle \mu_A, \gamma_A \rangle$ of S is called an *intuitionistic fuzzy bi-ideal* (see [10, 21]) of S if

(1) $(\forall x, y \in S)(x \leq y \Longrightarrow \mu_A(x) \geq \mu_A(y), \gamma_A(x) \leq \gamma_A(y))$,
(2) $(\forall x, y, z \in S)(\mu_A(xyz) \geq \mu_A(x) \wedge \mu_A(z))$,
(3) $(\forall x, y, z \in S)(\gamma_A(xyz) \leq \gamma_A(x) \vee \gamma_A(z))$.

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then the *intuitionistic characteristic function* $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is defined as

$$\mu_{\chi_A} : S \longmapsto [0,1], x \longmapsto \mu_{\chi_A}(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A \end{cases}$$

and

$$\gamma_{\chi_A} : S \longmapsto [0,1], x \longmapsto \gamma_{\chi_A}(x) := \begin{cases} 0 \text{ if } x \in A, \\ 1 \text{ if } x \notin A \end{cases}.$$

For *IFSs* $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ of S define the order relation "$\subseteq$" as follows:

$A \subseteq B$ if and only if $\mu_A \preceq \mu_B, \gamma_A \succeq \gamma_B$ if and only if $\mu_A(x) \leq \mu_B(x)$, $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in S$. Let $A = \langle \mu_A, \gamma_A \rangle$ and $B = \langle \mu_B, \gamma_B \rangle$ be any two *IFSs* in an ordered semigroup S. Then

(1) $A = B \Longleftrightarrow A \subseteq B$ and $B \subseteq A$
(2) $A^c = \langle \gamma_A, \mu_A \rangle$
(3) $A \cap B = \langle \mu_{A \wedge B}, \gamma_{A \vee B} \rangle$ and
(4) $0_\sim = \langle 0, 1 \rangle, 1_\sim = \langle 1, 0 \rangle$.

Lemma 3.1. (cf. [19,21]) *Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then the following are equivalent:*

(i) A is a left(resp. right and bi-) ideal of S.

(ii) The intuitionistic characteristic mapping $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy left (resp. right and bi-) ideal of S.

A subset T of an ordered semigroup S is called semiprime (cf. [13]) if for every $a \in S$ such that $a^2 \in T$, we have $a \in T$. Equivalent definition:

For each subset A of S such that $A^2 \subseteq T$, we have $A \subseteq T$.

Lemma 3.2. (cf. [13]) *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:*

(i) $(x)_\mathcal{N}$ is a left simple ubsemigroup of S, for every $x \in S$.

(ii) Every left ideal of S is a right ideal of S and semiprime.

4. Characterizations of left simple and left regular ordered semigroups in terms of intuitionistic fuzzy left ideals

An ordered semigroup S is *regular* (see [11]) if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$. Equivalent definitions:

(1) $(\forall a \in S)(a \in (aSa])$.
(2) $(\forall A \subseteq S)(A \subseteq (ASA])$.

An ordered semigroup S is *left*(resp. *right*) *simple* (see [11]) if for every left (resp. right) ideal A of S, we have $A = S$. S is called *simple* if it is left simple and right simple.

Theorem 4.1. *For a regular ordered semigroup S, the following conditions are equivalent:*

(i) S is left simple.

(ii) Every intuitionistic fuzzy left ideal of S is a constant mapping.

Proof. (i)$\Longrightarrow$(ii). Let S be a left simple ordered semigroup, $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy left ideal of S and $a \in S$. We consider the set,

$$E_\Omega := \{e \in S | e^2 \geq e\}.$$

Then $E_\Omega \neq \emptyset$. In fact: Since S is regular and $a \in S$, there exists $x \in S$ such that $a \leq axa$. It follows from $(OS3)$ that

$$(ax)^2 = (axa)x \geq ax,$$

and so $ax \in E_\Omega$ and hence $E_\Omega \neq \emptyset$.

(1) $A = \langle \mu_A, \gamma_A \rangle$ is a constant mapping on E_Ω. Let $t \in E_\Omega$, then $\mu_A(e) = \mu_A(t)$ and $\gamma_A(e) = \gamma_A(t)$ for every $e \in E_\Omega$. In fact: Since S is left simple and $t \in S$ we have $(St] = S$. Since $e \in S$, then $e \in (St]$ and there exists $z \in S$ such that $e \leq zt$. Hence $e^2 \leq (zt)(zt) = (ztz)t$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left ideal of S, we have

$$\begin{aligned} \mu_A(e^2) &\geq \mu_A((ztz)t) \geq \mu_A(t), \\ \text{and } \gamma_A(e^2) &\leq \gamma_A((ztz)t) \leq \gamma_A(t). \end{aligned}$$

Since $e \in E_S$, we have $e^2 \geq e$. Then $\mu_A(e) \geq \mu_A(e^2)$ and $\gamma_A(e) \leq \gamma_A(e^2)$ and we have $\mu_A(e) \geq \mu_A(t)$ and $\gamma_A(e) \leq \gamma_A(t)$. Besides, since S is left simple and $e \in S$, we have $(Se] = S$. Since $t \in E_\Omega$, exactly on the previous case−by symmetry− we get $\mu_A(t) \geq \mu_A(e)$ and $\gamma_A(t) \leq \gamma_A(e)$. Hence $\mu_A(t) = \mu_A(e)$ and $\gamma_A(t) = \gamma_A(e)$.

(2) $A = \langle \mu_A, \gamma_A \rangle$ is a constant mapping on S. Let $a \in S$, then $\mu_A(a) = \mu_A(t)$ and $\gamma_A(a) = \gamma_A(t)$ for every $t \in S$. Indeed: Since S is regular there exists $x \in S$ such that $a \leq axa$. We consider the element $xa \in S$. Then by $(OS3)$ it follows that,

$$(xa)^2 = x(axa) \geq xa,$$

then $xa \in E_\Omega$ and by (1), we have $\mu_A(xa) = \mu_A(t)$ and $\gamma_A(xa) = \gamma_A(t)$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left ideal of S, we have $\mu_A(xa) \geq \mu_A(a)$ and $\gamma_A(xa) \leq \gamma_A(a)$. Then $\mu_A(t) \geq \mu_A(a)$ and $\gamma_A(t) \leq \gamma_A(a)$. On the other hand, since S is left simple and $t \in S$ then $S = (St]$. Since $a \in S$, we have $a \leq st$ for some $s \in S$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left ideal of S, we have $\mu_A(a) \geq \mu_A(st) \geq \mu_A(t)$ and $\gamma_A(a) \leq \gamma_A(st) \leq \gamma_A(t)$. Thus $\mu_A(t) = \mu_A(a)$ and $\gamma_A(t) = \gamma_A(a)$.

(ii)$\Longrightarrow$(i). Let $a \in S$. Then the set $(Sa]$ is a left ideal of S. In fact: $S(Sa] = (S](Sa] \subseteq (SSa] \subseteq (Sa]$. If $x \in (Sa]$ and $S \ni y \leq x$, then $y \in ((Sa]] = (Sa]$. Since $(Sa]$ is a left ideal of S. By Lemma 3.1, the intuitionistic characteristic mapping $\chi_{(Sa]} = \langle \mu_{\chi_{(Sa]}}, \gamma_{\chi_{(Sa]}} \rangle$ of $(Sa]$ defined by:

$$\mu_{\chi_{(Sa]}} : S \longmapsto \{0,1\}, x \longmapsto \mu_{\chi_{(Sa]}}(x) := \begin{cases} 1 \text{ if } x \in (Sa], \\ 0 \text{ if } x \notin (Sa], \end{cases}$$

and

$$\gamma_{\chi_{(Sa]}} : S \longmapsto \{0,1\}, x \longmapsto \gamma_{\chi_{(Sa]}}(x) := \begin{cases} 0 \text{ if } x \in (Sa], \\ 1 \text{ if } x \notin (Sa]. \end{cases}$$

is an intuitionistic fuzzy left ideal of S. By hypothesis $\chi_{(Sa]}$ is a constant mapping that is, there exists $c \in \{0,1\}$ such that

$$\mu_{\chi_{(Sa]}}(x) = c \text{ and } \gamma_{\chi_{(Sa]}}(x) = c \text{ for every } x \in S.$$

Let $(Sa] \subset S$ and let $t \in S$ be such that $t \notin (Sa]$ then $\mu_{\chi_{(Sa]}}(t) = 0$ and $\gamma_{\chi_{(Sa]}}(x) = 1$. On the other hand since $a^2 \in (Sa]$, then we have $\mu_{\chi_{(Sa]}}(a^2) = 1$ and $\gamma_{\chi_{(Sa]}}(a^2) = 0$, a contradiction to the fact that $\chi_{(Sa]}$ is a constant mapping. Hence $S = (Sa]$.

From left−right dual of Theorem 4.1, we have the following:

Theorem 4.2. *For a regular ordered semigroup. The following are equivalent:*
(1) *S is right simple.*
(2) *Every intuitionistic fuzzy right ideal of S is a constant mapping.*

An ordered semigroup $(S, \cdot, \leq)$ is *left* (resp. *right*) *regular* (see [13]), if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$). Equivalent definitions:
(1) $(\forall a \in S)(a \in (Sa^2]$(resp. $a \in (a^2S]))$.
(2) $(\forall A \subseteq S)(A \subseteq (SA^2]$(resp. $A \subseteq (A^2S]))$.
An ordered semigroup S is called *completely regular* (see [13]) if it is regular, left regular and right regular.

Lemma 4.3. (cf. [13]) *An ordered semigroup S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Equivalently, if $a \in (a^2Sa^2]$ for every $a \in S$.*

Theorem 4.4. *An ordered semigroup $(S, \cdot, \leq)$ is a left regular if and only if for each intuitionistic fuzzy left ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, we have*

$$\mu_A(a) = \mu_A(a^2) \text{ and } \gamma_A(a) = \gamma_A(a^2) \text{ for all } a \in S.$$

Proof. $\Longrightarrow$. Suppose that $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left ideal of S and let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \leq xa^2$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left ideal of S, we have

$$\begin{aligned} \mu_A(a) &\geq \mu_A(xa^2) \geq \mu_A(a^2) \geq \mu_A(a), \\ \text{and } \gamma_A(a) &\leq \gamma_A(xa^2) \leq \gamma_A(a^2) \leq \gamma_A(a). \end{aligned}$$

$\Longleftarrow$. Let $a \in S$. We consider the left ideal $L(a^2) = (a^2 \cup Sa^2]$ of S, generated by a^2. Then by Lemma 3.1, the intuitionistic characteristic mapping $\chi_{L(a^2)} =$

$\langle \mu_{\chi_{L(a^2)}}, \gamma_{\chi_{L(a^2)}} \rangle$ is an intuitionistic fuzzy left ideal of S. By hypothesis we have

$$\mu_{\chi_{L(a^2)}}(a) = \mu_{\chi_{L(a^2)}}(a^2) \text{ and } \gamma_{\chi_{L(a^2)}}(a) = \gamma_{\chi_{L(a^2)}}(a^2).$$

Since $a^2 \in L(a^2)$, we have $\mu_{\chi_{L(a^2)}}(a^2) = 1$ and $\gamma_{\chi_{L(a^2)}}(a^2) = 0$ and hence $\mu_{\chi_{L(a^2)}}(a) = 1$ and $\gamma_{\chi_{L(a^2)}}(a) = 0$. Then $a \in L(a^2) = (a^2 \cup Sa^2]$ and $a \leq y$ for some $y \in a^2 \cup Sa^2$. If $y = a^2$, then $a \leq y = a^2 = aa = aa^2 \in Sa^2$ and $a \in (Sa^2]$. If $y = xa^2$ for some $x \in S$, then $a \leq y = xa^2 \in Sa^2$, and $a \in (Sa^2]$.

From left–right dual of Theorem 4.4, we have the following:

Theorem 4.5. *An ordered semigroup $(S, \cdot, \leq)$ is right regular if and only if for each intuitionistic fuzzy right ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, we have*

$$\mu_A(a) = \mu_A(a^2) \text{ and } \gamma_A(a) = \gamma_A(a^2) \text{ for all } a \in S.$$

From ([13, Theorem 3]) and Theorems 4.4 and 4.5, we have the following characterization Theorem for completely regular ordered semigroups.

Theorem 4.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:*

(i) S is completely regular.

(ii) For each intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S we have

$$\mu_A(a) = \mu_A(a^2) \text{ and } \gamma_A(a) = \gamma_A(a^2) \text{ for all } a \in S.$$

(iii) For each intuitionistic fuzzy left ideal $B = \langle \mu_B, \gamma_B \rangle$ and each intuitionistic fuzzy right ideal $C = \langle \mu_C, \gamma_C \rangle$ of S we have

$$\begin{aligned} \mu_B(a) &= \mu_B(a^2),\ \gamma_B(a) = \gamma_B(a^2) \text{ and} \\ \mu_C(a) &= \mu_C(a^2),\ \gamma_C(a) = \gamma_C(a^2) \text{ for all } a \in S. \end{aligned}$$

An ordered semigroup $(S, \cdot, \leq)$ is called *left* (resp. *right) duo* (see [13]) if every left (resp. right) ideal of S is a two-sided ideal of S. An ordered semigroup S is called *duo* if it is both left and right duo.

Definition 4.7. An ordered semigroup $(S, \cdot, \leq)$ is called *intuitionistic fuzzy left* (resp. *right) duo* if every *intuitionistic* fuzzy left (resp. right) ideal of S is an intuitionistic fuzzy two-sided ideal of S. An ordered semigroup S is called an *intuitionistic fuzzy duo* if it is both intuitionistic fuzzy left and intuitionistic fuzzy right duo.

Theorem 4.8. *Let $(S, \cdot, \leq)$ be a regular ordered semigroup. Then the following are equivalent:*

(i) S is left duo.

(ii) S is intuitionistic fuzzy left duo.

Proof. (i)$\Longrightarrow$(ii). Let S be left duo and $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy left ideal of S. Let $a, b \in S$. Then the set $(Sa]$ is a left ideal of S. In fact: $S(Sa] = (S](Sa] \subseteq (SSa] \subseteq (Sa]$ and if $x \in (Sa]$ and $S \ni y \leq x$ then $y \in ((Sa]] = (Sa]$.

Since S is left duo, then $(Sa]$ is a two-sided ideal of S. Since S is regular there exists $x \in S$ such that $a \leq axa$ then

$$ab \leq (axa)b \in (aSa)b \subseteq (Sa)S \subseteq (Sa]S \subseteq (Sa].$$

Then $ab \in ((Sa]] = (Sa]$ and $ab \leq xa$ for some $x \in S$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left ideal of S, we have

$$\mu_A(ab) \geq \mu_A(xa) \geq \mu_A(a) \text{ and } \gamma_A(ab) \leq \gamma_A(xa) \leq \gamma_A(a).$$

Let $x, y \in S$ be such that $x \leq y$. Then $\mu_A(x) \geq \mu_A(y)$, and $\gamma_A(x) \leq \gamma_A(y)$ because $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left ideal of S. Thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right deal of S and S is intuitionistic fuzzy left duo.

(ii)$\Longrightarrow$(i). Let S be fuzzy left duo and A a left ideal of S. Then the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy left ideal of S. By hypothesis χ_A is an intuitionistic fuzzy right ideal of S and by Lemma 3.1, A is a right ideal of S. Thus S is left duo.

By the left–right dual of Theorem 4.8, we have the following:

Theorem 4.9. *Let $(S, \cdot, \leq)$ be a regular ordered semigroup. Then the following are equivalent:*
(i) S is right duo.
(ii) S is intuitionistic fuzzy right duo.

Theorem 4.10. *Let $(S, \cdot, \leq)$ be a regular ordered semigroup. Then the following are equivalent:*
(i) Every bi-ideal of S is a right ideal of S.
(ii) Every intuitionsitic fuzzy bi-ideal of S is an intuitionistic fuzzy right ideal of S.

Proof. (i)$\Longrightarrow$(ii). Let $a, b \in S$ and $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy bi-ideal of S. Then $(aSa]$ is a bi-ideal of S. In fact: $(aSa]^2 \subseteq (aSa](aSa] \subseteq (aSa]$, $(aSa]S(aSa] = (aSa](S](aSa] \subseteq (aSa]$ and if $x \in (aSa]$ and $S \ni y \leq x \in (aSa]$ then $y \in ((aSa]] = (aSa]$. Since $(aSa]$ is a bi-ideal of S, by hypothesis $(aSa]$ is right ideal of S. Since $a \in S$ and S is regular there exists $x \in S$ such that $a \leq axa$ then

$$ab \leq (axa)b \in (aSa)S \subseteq (aSa]S \subseteq (aSa].$$

Then $ab \leq aza$ for some $z \in S$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S, we have

$$\begin{aligned} \mu_A(ab) &\geq \mu_A(aza) \geq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a) \\ \text{and } \gamma_A(ab) &\leq \gamma_A(aza) \leq \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a). \end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$ because $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S. Thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy right ideal of S.

(ii)$\Longrightarrow$(i). Let A be a bi-ideal of S. Then by Lemma 3.1, $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ is an intuitionistic fuzzy bi-ideal of S. By hypothesis χ_A is an intuitionistic fuzzy right ideal of S. By Lemma 3.1, A is a right ideal of S.

By left−right dual of the Theorem 4.10, we have the following:

Theorem 4.11. *Let $(S, \cdot, \leq)$ be a regular ordered semigroup. Then the following are equivalent:*

(i) Every bi-ideal of S is a left ideal of S.

(ii) Every intuitionistic fuzzy bi-ideal of S is an intuitionistic fuzzy left ideal of S.

Proposition 4.12. *Every intuitionistic fuzzy bi-ideal of an ordered semigroup S is an intuitionsitic fuzzy $(1, 2)$-ideal of S.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-ideal of S and let $x, y, z, a \in S$. Then

$$\begin{aligned} \mu_A(xa(yz)) &= \mu_A((xay)z) \\ &\geq \min\{\mu_A(xay), \mu_A(z)\} \\ &\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, \mu_A(z)\} \\ &= \min\{\mu_A(x), \mu_A(y), \mu_A(z)\} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xa(yz)) &= \gamma_A((xay)z) \\ &\leq \max\{\gamma_A(xay), \gamma_A(z)\} \\ &\geq \max\{\max\{\gamma_A(x), \gamma_A(y)\}, \gamma_A(z)\} \\ &= \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\}. \end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$, because $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S.

Corollary 4.13. *Every intuitinistic fuzzy left(resp. right) ideal $A = \langle \mu_A, \gamma_A \rangle$ of an ordered semigroup S is an intuitionsitic fuzzy $(1, 2)$-ideal of S.*

The converse of the Corollary 4.13, is not true in general. However, if S is a regular ordered semigroup then we have the following Proposition:

Proposition 4.14. *If S is a regular ordered semigroup, then every intuitionistic fuzzy $(1, 2)$-ideal of S is an intuitionistic fuzzy bi-ideal of S.*

Proof. Assume that S is regular ordered semigroup and let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy $(1, 2)$-ideal of S. Let $x, y, a \in S$. Since S is regular and $(xSx]$ is a bi-ideal of S, so a right ideal of S, by Theorem 4.10. Thus

$$xa \leq (xSx)a \in (xSx)S \subseteq (xSx]S \subseteq (xSx],$$

then $xa \leq xyx$ for some $y \in S$. Thus $xay \leq (xyx)y$ and we have

$$\begin{aligned}\mu_A(xay) &\geq \mu_A((xyx)y)\\ &\geq \min\{\mu_A(xyx), \mu_A(y)\}\\ &\geq \min\{\min\{\mu_A(x), \mu_A(x)\}, \mu_A(y)\}\\ &= \min\{\mu_A(x), \mu_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\gamma_A(xay) &\leq \gamma_A((xyx)y)\\ &\geq \max\{\gamma_A(xyx), \gamma_A(y)\}\\ &\geq \max\{\max\{\gamma_A(x), \gamma_A(x)\}, \gamma_A(y)\}\\ &= \max\{\gamma_A(x), \gamma_A(y)\}\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $\mu_A(x) \geq \mu_A(y)$, and $\gamma_A(x) \leq \gamma_A(y)$ because $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionsitic fuzzy $(1,2)$-ideal of S. Thus $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-idealof S.

5. Some semilattices of left simple ordered semigroups in terms of intuitionistic fuzzy left ideals

Let $(S, \cdot \leq)$ be an ordered semigroup. A subsemigroup F of S is called *filter* of S if:

(1) $(\forall a, b \in S)(ab \in F \Longrightarrow a \in F$ and $b \in F)$.

(2) $(\forall c \in S)(c \geq a \in F \Longrightarrow c \in F)$.

For $x \in S$, we denote by $N(x)$ the filter of S generated by x (that is the least filter with respect to inclusion relation containing x). $\mathcal{N}$ denotes the equivalence relation on S defined by $\mathcal{N} := \{(x, y) \in S \times S | N(x) = N(y)\}$(see [13]).

Definition 5.1. [cf. 13] Let S be an ordered semigroup. An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for each $a, b \in S$. If σ is a semilattice congruence on S then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$.

An ordered semigroup S is called a semilattice of *left simple semigroups* if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left simple subsemigroup of S for every $x \in S$.

Equivalent definition:

There exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left simple subsemigroups of S such that

(i) $S_\alpha \cap S_\beta = \emptyset \;\; \forall \alpha, \beta \in Y, \;\; \alpha \neq \beta$,

(ii) $S = \bigcup_{\alpha \in Y} S_\alpha$,

(iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta} \; \forall \alpha, \beta \in Y$.

In ordered semigroups the semilattice congruences are defined exactly same as in the cse of semigroups −without order− so the two definitions are equivalent [10].

Lemma 5.2. (cf. [22]) *An ordered semigroup $(S,\cdot,\leq)$ is a semilattice of left simple semigroups if and only if for all left ideals A, B of S we have*

$$(A^2]=A \text{ and } (AB]=(BA].$$

Theorem 5.3. *An ordered semigroup $(S,\cdot,\leq)$ is a semilattice of left simple semigroups if and only if for every intuitionistic fuzzy left ideal $A=\langle\mu_A,\gamma_A\rangle$ of S, we have*

$$\begin{aligned}\mu_A(a^2) &= \mu_A(a),\ \gamma_A(a^2)=\gamma_A(a)\\ \text{and } \mu_A(ab) &= \mu_A(ba),\ \gamma_A(ab)=\gamma_A(ba) \text{ for all } a,b\in S.\end{aligned}$$

Proof. $\Longrightarrow$. (1). Let S be a semilattice of left simple semigroups. By hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of left simple subsemigroups of S such that:

(i) $S_\alpha\cap S_\beta=\emptyset\ \ \forall\alpha,\beta\in Y,\quad \alpha\neq\beta,$

(ii) $S=\bigcup\limits_{\alpha\in Y} S_\alpha,$

(iii) $S_\alpha S_\beta\subseteq S_{\alpha\beta}\ \forall\alpha,\beta\in Y.$

Let $A=\langle\mu_A,\gamma_A\rangle$ be an intuitionistic fuzzy left ideal of S and $a\in S$. Then $\mu_A(a)=\mu_A(a^2),\gamma_A(a)=\gamma_A(a^2)$. In fact: By Theorem 4.4, it is enough to prove that $a\in(Sa^2]$ for every $a\in S$. Let $a\in S$, then there exists $\alpha\in Y$ such that $a\in S_\alpha$. Since S_α is left simple, we have $S=(S_\alpha a]$ and hence

$$a\leq xa \text{ for some } x\in S_\alpha,$$

Since $x\in S_\alpha$, we have $x\in(S_\alpha a]$ and $x\leq ya$ for some $y\in S_\alpha$. Thus we have

$$a\leq xa\leq(ya)a=ya^2,$$

since $y\in S$, we have $a\in(Sa^2]$.

(2) Let $a,b\in S$. Let $A=\langle\mu_A,\gamma_A\rangle$ be an intuitionistic fuzzy left ideal of S. Then by (1), we have

$$\begin{aligned}\mu_A(ab) &= \mu_A((ab)^2)=\mu_A(a(ba)b)\\ &\geq \mu_A(ba),\end{aligned}$$

and

$$\begin{aligned}\gamma_A(ab) &= \gamma_A((ab)^2)=\gamma_A(a(ba)b)\\ &\leq \gamma_A(ba).\end{aligned}$$

By symmetry we can prove that $\mu_A(ba)\geq\mu_A(ab),\gamma_A(ba)\leq\gamma_A(ab)$. Hence $\mu_A(ba)=\mu_A(ab),\gamma_A(ba)=\gamma_A(ab)$.

$\Longleftarrow$. Assume that for every intuitionistic fuzzy left ideal $A=\langle\mu_A,\gamma_A\rangle$ of S, we have

$$\begin{aligned}\mu_A(a^2) &= \mu_A(a),\ \gamma_A(a^2)=\gamma_A(a)\\ \text{and } \mu_A(ab) &= \mu_A(ba),\ \gamma_A(ab)=\gamma_A(ba) \text{ for all } a,b\in S.\end{aligned}$$

by condition (1) and by Theorem 4.4, we have that S is left regular. Let L be a left ideal of S and let $a \in L$. Then $a \in S$, since S is left regular there exists $x \in S$ such that

$$a \leq xa^2 = (xa)a \in (SL)L \subseteq LL = L^2,$$

then $a \in (L^2]$ and $L \subseteq (L]$. On the other hand, since L is a left ideal of S, we have $L^2 \subseteq SL \subseteq L$, then $(L^2] \subseteq (L] = L$. Let P and Q be left ideals of S and let $x \in (QP]$ then $x \leq ba$ for some $a \in P$ and $b \in Q$. We consider the left ideal $L(ab)$ generated by ab. That is, the set $L(ab) = (ab \cup Sab]$. Then by Lemma 3.1, the intuitionistic characteristic function $\chi_{L(ab)} = \langle \mu_{\chi_{L(ab)}}, \gamma_{\chi_{L(ab)}} \rangle$ of $L(ab)$ is an intuitionistic fuzzy left ideal of S. By hypothesis, we have $\mu_{\chi_{L(ab)}}(ab) = \mu_{\chi_{L(ab)}}(ba)$ and $\gamma_{\chi_{L(ab)}}(ab) = \gamma_{\chi_{L(ab)}}(ba)$. Since $ab \in L(ab)$, we have $\mu_{\chi_{L(ab)}}(ab) = 1$ and $\gamma_{\chi_{L(ab)}}(ab) = 0$ and hence $\mu_{\chi_{L(ab)}}(ba) = 1$, $\gamma_{\chi_{L(ab)}}(ba) = 0$ and thus $ba \in L(ab) = (ab \cup Sab]$. Then $ba \leq ab$ or $ba \leq yab$ for some $y \in S$. If $ba \leq ab$ then $x \leq ab \in AB$ and $x \in (AB]$. If $ba \leq yab$ then $x \leq yab \in (SA)B \subseteq AB$ and $x \in (AB]$. Thus $(BA] \subseteq (AB]$. By symmetry we can prove that $(AB] \subseteq (BA]$. Therefore $(AB] = (BA]$ and by Lemma 5.2, it follows that S is a semilattice of left simple semigroups.

From left−right dual of Theorem 5.3, we have the following:

Theorem 5.4. *An ordered semigroup $(S, \cdot, \leq)$ is a semilattice of right simple semigroups if and only if for every intuitionistic fuzzy right ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, we have*

$$\begin{aligned} \mu_A(a^2) &= \mu_A(a),\ \gamma_A(a^2) = \gamma_A(a) \\ \text{and } \mu_A(ab) &= \mu_A(ba),\ \gamma_A(ab) = \gamma_A(ba) \text{ for all } a, b \in S. \end{aligned}$$

Lemma 5.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $A = \langle \mu_A, \gamma_A \rangle$ an intuitionistic fuzzy left(resp. right) ideal of S, $a \in S$ such that $a \leq a^2$. Then $\mu_A(a) = \mu_A(a^2)$ and $\gamma_A(a) = \gamma_A(a^2)$.*

Proof. Since $a \leq a^2$ and $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy left ideal of S, we have

$$\begin{aligned} \mu_A(a) &\geq \mu_A(a^2) = \mu_A(aa) \geq \mu_A(a), \\ \gamma_A(a) &\leq \gamma_A(a^2) = \gamma_A(aa) \leq \gamma_A(a). \end{aligned}$$

and so $\mu_A(a) = \mu_A(a^2)$ and $\gamma_A(a) = \gamma_A(a^2)$.

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