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DISCRETE-TIME BULK-SERVICE QUEUE WITH MARKOVIAN SERVICE INTERRUPTION AND PROBABILISTIC BULK SIZE

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ABSTRACT. This paper analyzes a discrete-time bulk-service queue with probabilistic bulk size, where the service process is interrupted by a Markov chain. We study the joint probability generating function of system occupancy and the state of the Markov chain. We derive several performance measures of interest, including average system occupancy and delay distribution.

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1. Introduction

Recently, interests in discrete-time queues have increased due to their numerous applications in the analysis of communication networks and other related areas [8],[9],[13],[14],[15],[16],[17]. One of the reasons for this is that discrete-time queues fit the slotted nature of communication networks better than the continuous-time counterparts, and hence they give more accurate performance measures of these networks [3].

In many communication networks, resources are assigned on a statistical basis [1],[2]. Lee [12] analyzed a discrete-time single server queue where the service process is interrupted by a semi-Markov process. Fiems et al. [6] investigated a discrete-time single-server queue, where the service process is interrupted by an on/off process. Mokhtar and Azizoglu [17] analyzed discrete-time queues, where the behavior of a server at a given time depends on the number of customers in

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the system at that time. All these models assumed non-bulk service in common, implying that a server can serve at most one customer at a time.

This paper considers a discrete-time bulk-service queue with Markovian service interruption and probabilistic bulk size, where the service process is interrupted by a Markov chain. We are interested in the joint probability distribution of the system occupancy and the state of the Markov chain at an arbitrary slot boundary, the expected value of the system occupancy, as well as the delay distribution in the steady-state. While this queueing system may be of interest from the viewpoint of a queueing theory, it also finds applications in the performance analysis of communication networks. Our motivation is the study of the queueing behavior of a secondary user dynamically sharing the spectrum in the time-domain by exploiting whitespace between the bursty transmissions of a set of primary users of a wireless network [7],[15],[16]. The primary users' occupancy on the channel at a given time can be collected in the binary random process, which indicates either that a primary user is actively transmitting on the channel or that the channel is free for secondary transmissions. The secondary user monitors the channel every time slot to determine whether or not it is in use by a primary user. When the channel is temporally unoccupied, the secondary user uses the channel for secondary transmission. If the channel is detected to busy, the secondary user does not transmit. In this environment, when we model the occupancy process with a Markov chain, the queueing system of a secondary user can be modeled by a two-state Markovian service interruption framework. Furthermore, to model a random backoff procedure for the competition among secondary users [5],[15],[16], we consider the probabilistic bulk size for service.

2. Queueing Model

This paper considers a discrete-time bulk-service queue with Markovian service interruption and probabilistic bulk size, in which the time axis is divided into fixed-length contiguous intervals, referred to as slots. It is assumed that the service times equal to exactly one slot [12].

Customers arrive to the system in accordance with a batch geometric process [3] and are accommodated in the buffer with infinite waiting-room. Let a_k be the number of customers that arrive during slot k. The numbers of customers entering the system during consecutive slots are assumed to be i.i.d. non-negative discrete random variables with an arbitrary probability distribution, and are characterized by the probability generating function $A(z) \equiv E[z^{a_k}]$ with finite mean.

The service process is interrupted by a discrete-time Markov chain $\{x_k, k \geq 1\}$ with two states, 0(available) and 1(unavailable). A value of $x_k = 0$ indicates that the server is available for service during slot k, while a value of $x_k = 1$ indicates that the server is not available during slot k. The server transits from state i to state j with positive probability p_{ij} for $i, j = 0, 1$. When the server is available, the customers are served in bulks of variable size. The maximum bulk size is assumed to be a finite integer value C. If c_k is the bulk size for slot k, the server can serve up to c_k customers during slot k. We define $u_j \equiv P\Big\{c_k = C - j\Big\}$ for $j = 0, 1, \cdots, C$, and $U(z) \equiv \sum_{j=0}^{C} u_j z^j$. It is assumed that the bulk size is independent of the underlying Markov chain $\Big\{x_k, k \geq 1\Big\}$ and the arrival process $\{a_k, k \geq 1\}$. In this paper, it is also assumed that arriving customer cannot be accepted into the bulk already undergoing service even if the capacity of server is available, but has to wait until the next service instant.

3. Queueing analysis

In this section, the queue lengths at the beginning of slots are analyzed. Before proceeding to the analysis of the queue length, we define a random variable n_k indicating the number of customers in the queue at the beginning of slot k. Then $\{(n_k, x_k), k \geq 1\}$ constitutes a two dimensional Markov chain embedded at the beginning of each slot. If we denote by a_k the number of customers entering the system during slot k, then the system under consideration evolves as follows:

$$n_{k+1} = \begin{cases} (n_k - c_k)^+ + a_k & \text{if } x_k = 0, \\ n_k + a_k & \text{if } x_k = 1, \end{cases} \tag{1}$$

where $(y)^+ = \max(0, y)$.

Let $P_i^{(k)}(z) \equiv E\Big[I_{\{x_k = i\}} z^{n_k}\Big]$ for $i = 0, 1$. From (1) we have

$$\begin{aligned} & P_i^{(k+1)}(z) \\ = \; & E\Big[I_{\{x_{k+1}=i\}} z^{n_{k+1}}\Big] \\ = \; & p_{1i} A(z) P_1^{(k)}(z) + p_{0i} A(z) \sum_{j=0}^{C} u_j \Big[z^{j-C} P_0^{(k)}(z) + E\left[I_{\{x_k=0, n_k<C-j\}}\right] \\ & \qquad - z^{j-C} E\left[I_{\{x_k=0,n_k<C-j\}} z^{n_k}\right]\Big] \\ = \; & p_{0i} A(z) \frac{U(z)}{z^C} P_0^{(k)}(z) + p_{1i} A(z) P_1^{(k)}(z) + p_{0i} A(z) \left[Q^{(k)}(1) - \frac{Q^{(k)}(z)}{z^C}\right] \\ = \; & A(z) \left[\frac{p_{0i} U(z)}{z^C} P_0^{(k)}(z) + p_{1i} P_1^{(k)}(z) + \frac{p_{0i}}{z^C}\Big\{z^C Q^{(k)}(1) - Q^{(k)}(z)\Big\}\right] \end{aligned} \tag{2}$$

for $i = 0, 1$, where the function $Q^{(k)}(z)$ is defined as

$$\begin{aligned} Q^{(k)}(z) \; & \equiv \; \sum_{j=0}^{C-1} u_j z^j E\left[I_{\{x_k=0,n_k<C-j\}} z^{n_k}\right] \\ & = \; \sum_{c=0}^{C-1} P\{x_k = 0, n_k = c\} \sum_{j=0}^{C-c-1} u_j z^{j+c} \end{aligned} \tag{3}$$

Now, we will find the ergodic condition for the Markov chain $\Big\{(n_k, x_k), k \geq 1\Big\}$. We assume that the Markov chain is irreducible and aperiodic, which is not a strong assumption. Note that this is true if $P\{a_k = 0\}P\{a_k = 1\} > 0$. Obviously, $A'(1) < p_{10}[C - U'(1)]/(p_{01} + p_{10})$ is a necessary condition for the Markov chain $\Big\{(n_k, x_k), k \geq 1\Big\}$ to be positive recurrent. Note that the average service capacity is $p_{10}[C - U'(1)]/(p_{01} + p_{10})$, while during a service time $A'(1)$ customers will arrive on average. Now, we intend to show that $A'(1) < p_{10}\Big[C - U'(1)\Big]/(p_{01} + p_{10})$ is the sufficient condition for the Markov chain $\{(n_k, x_k), k \geq 1\}$ to be positive recurrent. We assume that $A'(1) < p_{10}[C - U'(1)]/(p_{01} + p_{10})$. Foster's criterion [11] will be is used in order to show that the Markov chain $\{(n_k, x_k), k \geq 1\}$ is positive recurrent. We choose a real number α such that $p_{01}/\Big[C - U'(1) - A'(1)\Big] < \alpha < p_{10}/A'(1)$ and choose the test function as follows: $f(i, j) \equiv \alpha i + j$. The mean drift of the test function is

$$
\begin{aligned}
z_{ij} &\equiv E\Big[f(n_{k+1}, x_{k+1}) - f(n_k, x_k)\,|(n_k, x_k) = (i, j)\,\Big] \\
&= \begin{cases} \alpha A'(1) + p_{01} - \alpha i + \alpha \displaystyle\sum_{k=C-i}^{C} (i - C + k)u_k & \text{if } 0 \leq i < C, \quad j = 0, \\ \alpha A'(1) + p_{01} - \alpha\Big[C - U'(1)\Big] & \text{if } i \geq C, \quad j = 0, \\ \alpha A'(1) - p_{10} & \text{if } j = 1. \end{cases}
\end{aligned} \tag{4}
$$

Then, $z_{ij} \leq \infty$ for all i and j, and

$$z_{ij} \leq \max\Big(\alpha A'(1) + p_{01} - \alpha\Big[C - U'(1)\Big], \alpha A'(1) - p_{10}\Big) < 0$$

for $i \geq C$. Let $\epsilon \equiv \max\Big(\alpha A'(1) + p_{01} - \alpha\Big[C - U'(1)\Big], \alpha A'(1) - p_{10}\Big)\Big/ 2 < 0$. Then, $z_{ij} < \epsilon$ for $i \geq C$. Hence, except finite subset $\Big\{(i, 0)\,|0 \leq i < C\,\Big\}$ of the state space of $\{(n_k, x_k)\}$, we have $z_{ij} < \epsilon < 0$. Therefore, by Foster's criterion, we see that $\Big\{(n_k, x_k), k \geq 1\Big\}$ is positive recurrent. Thus, assuming that $P\{a_k = 0\}P\{a_k = 1\} > 0$, the Markov chain is ergodic if and only if

$$A'(1) < \frac{p_{10}}{p_{01} + p_{10}}[C - U'(1)]. \tag{5}$$

Assume that the Markov chain $\Big\{(n_k, x_k), k \geq 1\Big\}$ is ergodic. Then, there exists unique stationary distribution. Let $P_i(z) \equiv \lim_{k\to\infty} P_i^{(k)}(z)$, $i = 0, 1$, denote the stationary joint probability generating function of the Markov chain

$\Big\{(n_k, x_k), k \geq 1\Big\}$. Letting $k \to \infty$ in (2) we obtain

$$P_0(z) = A(z)\left[\frac{p_{00}U(z)}{z^C}P_0(z) + p_{10}P_1(z) + \frac{p_{00}}{z^C}\left\{z^C Q(1) - Q(z)\right\}\right], \qquad (6)$$

$$P_1(z) = A(z)\left[\frac{p_{01}U(z)}{z^C}P_0(z) + p_{11}P_1(z) + \frac{p_{01}}{z^C}\left\{z^C Q(1) - Q(z)\right\}\right], \qquad (7)$$

where the function $Q(z)$ is defined as $Q(z) \equiv \lim_{k\to\infty} Q^{(k)}(x)$. Solving for $P_0(z)$ in (6) and then substituting it in (7), we get the expression for $P_1(z)$:

$$P_1(z) = \frac{p_{01}A(z)\Big[z^C Q(1) - Q(z)\Big]}{z^C\Big[1 - p_{11}A(z)\Big] - p_{00}A(z)U(z)\Big[1 - p_{11}A(z)\Big] - p_{01}p_{10}A^2(z)U(z)}. \qquad (8)$$

Equation (8) is of indeterminate form, that is, (8) has unknown term $Q(z)$, but the C unknowns $\lim_{k\to\infty} P\Big\{n_k = c, x_k = 0\Big\}$, $c = 0, 1, \cdots, C-1$, can be determined by consideration of the zeros of the denominator in (8) that lie in the closed unit disc $\{z : |z| \leq 1\}$. With Rouche's theorem, it can be shown that $z^C\Big[1 - p_{11}A(z)\Big] - p_{00}A(z)U(z)\Big[1 - p_{11}A(z)\Big] - p_{01}p_{10}A^2(z)U(z) = 0$ has exactly C zeros in the unit closed disc $\{z : |z| \leq 1\}$. A detailed explanation can be found in [10],[18]. Since $P_1(z)$ is a continuous function for $|z| \leq 1$, the numerator $p_{01}A(z)\Big[z^C Q(1) - Q(z)\Big]$ of $P_1(z)$ should vanish at each of the zeros, yielding C equations. One of the zeros equals 1, and leads to a trivial equation. However, the relation $P_1(1) = p_{01}/\Big[p_{01} + p_{10}\Big]$ provides an additional equation. Using l'Hospital's rule, this relation is found to be

$$\frac{p_{10}}{p_{01} + p_{10}}\Big[C - U'(1)\Big] - A'(1) = CQ(1) - Q'(1). \qquad (9)$$

The C roots of $z^C\Big[1 - p_{11}A(z)\Big] - p_{00}A(z)U(z)\Big[1 - p_{11}A(z)\Big] - p_{01}p_{10}A^2(z)U(z) = 0$ in the closed unit disc $\{z : |z| \leq 1\}$ are denoted by $z_0 = 1, z_1, \cdots, z_{C-1}$. If one of the roots is zero of $p_{01}A(z)$, then it should be 0, which can not be true. Thus, the C roots are zeros of $z^C Q(1) - Q(z)$, not of $p_{01}A(z)$. Hence, by writing $z^C Q(1) - Q(z)$ as $(z-1)G\prod_{c=1}^{C-1}(z - z_c)$ with G a constant, and using (9) to derive the value of G, it follows that

$$G = \left[\frac{p_{10}}{p_{01} + p_{10}}\{C - U'(1)\} - A'(1)\right]\prod_{c=1}^{C-1}\frac{1}{1 - z_c} \qquad (10)$$

and

$$z^C Q(1) - Q(z) = \left[\frac{p_{10}}{p_{01} + p_{10}}\{C - U'(1)\} - A'(1)\right](z-1)\prod_{c=1}^{C-1}\frac{z - z_c}{1 - z_c} \qquad (11)$$

so that (8) can be written as

$$P_1(z) = \frac{p_{01}A(z)}{z^C\{1-p_{11}A(z)\} - p_{00}A(z)U(z)\{1-p_{11}A(z)\} - p_{01}p_{10}A^2(z)U(z)} \times \left[\frac{p_{10}}{p_{01}+p_{10}}\{C-U'(1)\} - A'(1)\right](z-1)\prod_{c=1}^{C-1}\frac{z-z_c}{1-z_c} \tag{12}$$

for $|z| \le 1$. Using (11) and (12) in (6), we can determine

$$P_0(z) = \frac{A(z)\Big[p_{00}\Big\{1-p_{11}A(z)\Big\} + p_{01}p_{10}A(z)\Big]}{z^C\Big\{1-p_{11}A(z)\Big\} - p_{00}A(z)U(z)\Big\{1-p_{11}A(z)\Big\} - p_{01}p_{10}A^2(z)U(z)} \times \left[\frac{p_{10}}{p_{01}+p_{10}}\Big\{C-U'(1)\Big\} - A'(1)\right](z-1)\prod_{c=1}^{C-1}\frac{z-z_c}{1-z_c}. \tag{13}$$

Let $N(z)$ be the marginal probability generating function of the queue length at an arbitrary slot boundary. Clearly, $N(z)$ is given by

$$\begin{aligned} N(z) &\equiv \lim_{k\to\infty} E[z^{n_k}] \\ &= P_0(z) + P_1(z) \\ &= \frac{A(z)\Big[1+(1-p_{00}-p_{11})A(z)\Big]}{z^C\Big\{1-p_{11}A(z)\Big\} - p_{00}A(z)U(z)\Big\{1-p_{11}A(z)\Big\} - p_{01}p_{10}A^2(z)U(z)} \\ &\quad \times \left[\frac{p_{10}}{p_{01}+p_{10}}\{C-U'(1)\} - A'(1)\right](z-1)\prod_{c=1}^{C-1}\frac{z-z_c}{1-z_c}. \end{aligned} \tag{14}$$

Now we can calculate the mean queue length at an arbitrary slot boundary by using the differentiation of the probability generating function $N(z)$ for $z = 1$. We obtain

$$\begin{aligned} &N'(1) \\ &= \left[2 - \frac{1}{p_{01}+p_{10}}\right]A'(1) + \sum_{c=1}^{C-1}\frac{1}{1-z_c} \\ &- \frac{2(1-p_{01}-2p_{10})A'(1)U'(1) - p_{10}U''(1)}{2\Big[p_{10}\Big\{C-U'(1)\Big\} - (p_{01}+p_{10})A'(1)\Big]} \\ &- \frac{C(C-1)p_{10} - 2Cp_{11}A'(1) + 2(1-p_{01}-p_{10})\Big[A'(1)\Big]^2 - (p_{01}+p_{10})A''(1)}{2\Big[p_{10}\Big\{C-U'(1)\Big\} - (p_{01}+p_{10})A'(1)\Big]}. \end{aligned}$$

4. Delay analysis

In this section, we find the delay distribution of an arbitrary customer. To do this, we pick out any customer and call it tagged customer. Before we find the delay distribution, we need some conditional probabilities. Let $D_i(l|k)$ denote the conditional probability that the remaining delay of the tagged customer in the k-th position of the buffer at the beginning of a slot is l slots, given that the state of the server is i, $i = 0, 1$, at the beginning of the slot. Then, we get the following:

$$D_0(l|k) = \begin{cases} \sum_{j=0}^{C-k} u_j, & k \geq 1, l = 1, \\ \sum_{j=C-k+1}^{C} u_j \left[p_{00} D_0(l-1|k-C+j) \right. \\ \qquad \left. + p_{01} D_1(l-1|k-C+j)\right], & k \geq 1, l > 1, \end{cases}$$

$$D_1(l|k) = \begin{cases} 0 & k \geq 1, l = 1, \\ p_{10} D_0(l-1|k) + p_{11} D_1(l-1|k), & k \geq 1, l > 1 \end{cases}$$

For each k, the conditional probabilities $D_i(l|k)$, $i = 0, 1$, are calculated by applying above recursive relations on l repeatedly.

Let $\pi_{i,n} \equiv \lim_{k\to\infty} P\Big\{n_k = n, x_k = i\Big\}$ be the steady state distribution of the Markov chain $\{(n_k, x_k), k \geq 1\}$. Note that the probability of tagged customer being in the n-th position of its bulk is given by $P\{a \geq n\}/A'(1)$ [4], where a is the generic random variable for $\{a_k\}$. Then we obtain the probability distribution of the delay D of a customer as follows:

$$\begin{aligned} P\{D = l\} = \sum_{n=0}^{(l+1)C-1} \Bigg[& \pi_{1n} \sum_{q=1}^{lC-n} \frac{P\{a \geq q\}}{A'(1)} \{p_{10} D_0(l|n+q) + p_{11} D_1(l|n+q)\} \\ & + \pi_{0n} \sum_{j=0}^{C} u_j \sum_{q=1}^{lC-(n-C+j)^+} \frac{P\{a \geq q\}}{A'(1)} \{p_{00} D_0(l|(n-C+j)^+ + q) \\ & + p_{01} D_1(l|(n-C+j)^+ + q)\}\Bigg] \end{aligned}$$

5. Conclusions

This paper has presented the queueing analysis of a discrete-time bulk-service queue with probabilistic bulk size, where the service process is interrupted by a Markov chain. By means of the probability generating functions, we obtained key performance measures, such as joint probability distribution of system occupancy and the state of the Markov chain at an arbitrary slot boundary, average system occupancy, and delay distribution in steady-state.

REFERENCES

1. N. Amitay, *Distributed switching and control with fast resource assignment/handoff for personal communications systems*, IEEE Journal on Selected Areas in Communications **11(6)** (1993), 842-849.
2. P. Baht, I. Chlamtac and A.S. Farago, *Resource assignment for integrated services in wireless ATM networks*, International Journal of Communication Systems **11** (1998), 29-41.
3. H. Bruneel and B. Kim, *Discrete-time Models for Communication Systems Including ATM*, Kluwar Academic Publishers, 1993.
4. P.J. Burke, *Delays in single-server queues with batch input*, Operations Research **23(4)** (1975), 830-833.
5. M.Y. Chung, M.H. Chung, T.J. Lee and Y. Lee, *Performance analysis of HomePlug 1.0 MAC with CSMA/CA*, IEEE Journl on Selected Areas in Commnications **24(7)** (2006), 1411-1420.
6. D. Fiems, B. Steyaert and H. Bruneel, *Discrete-time queues with generally distributed service times and renewal-type server interruptions*, Performance Evaluation **55(3-4)** (2004), 277-298.
7. A.T. Hoang and Y.C. Liang, *A two-phase channel and power allocation scheme for cognitive radio networks*, Proceedings of IEEE 17th International Symposium on Personal, Indoor and Mobile Radio Communications, 2006.
8. G.U. Hwang and K. Sohraby, *Power tail asymptotic results of a discrete time queue with long range dependent input*, Journal of Korean Mathematical Society **40(1)** (2003), 87-107.
9. F. Ishizaki, *Loss probability in a finite queue with service interruptions and queue length distribution in the corresponding infinite queue*, Performance Evaluation **63** (2006), 682-699.
10. V. Klimenok, *On the modification of Rouche's theorem for the queueing theory problems*, Queueing Systems: Theory and Applications **38(4)** (2001), 431-434.
11. J. Lamperti, *Criteria for the recurrence or transience of stochastic process, I*, Journal of Mathematical Analysis and Applcations **1** (1960), 314-330.
12. D. Lee, *Analysis of a single server queue with semi-Markovian service interruption*, Queueing Systems: Theory and Applications **27(1-2)** (1997), 153-178.
13. Y. Lee, *Discrete-time* $Geo^X/G/1$ *queue with preemptive resume priority*, Mathematical and Computer Modeling **34** (2001), 243-250.
14. Y. Lee, *Discrete-time* $Geo^X/G/1$ *queue with place reeservation discipline*, Journal of Applied Mathematics and Computing **22(1-2)** (2006), 453-460.
15. Y. Lee, *Performance of secondary users in opportunistic spectrum access*, Electronics Letters **44(16)** (2008), 981-983.
16. Y. Lee, *Opportunistic spectrum access in cognitive networks*, Electronics Letters **44(17)** (2008), 1022-1024.
17. A. Mokhtar and M. Azizoglu, *Analysis of state-dependent probabilistic service interruptions in discrete-time queues*, IEEE Communication Letters **8(8)** (2004), 544-546.
18. T.L. Saaty, *Elements of Queueing Theory with Applications*, McGraw-Hill Book Company, 1961.

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