J. Appl. Math. & Informatics Vol. 28(2010), No. 1 - 2, pp. 243 - 256
Website: http://www.kcam.biz

# EXISTENCE OF THREE POSITIVE PERIODIC SOLUTIONS OF NEUTRAL IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

YUJI LIU* AND JIANYE XIA

ABSTRACT. This paper is concerned with the neutral impulsive functional differential equations

$$\begin{cases} x'(t)+a(t)x(t)=f(t,x(t-\tau(t)),x'(t-\delta(t))), \ a.e.\ t\in R,\\ \Delta x(t_k)=b_kx(t_k),\ k\in Z. \end{cases}$$

Sufficient conditions for the existence of at least three positive $T-$periodic solution are established. Our results generalize and improve the known ones. Some examples are presented to illustrate the main results.

AMS Mathematics Subject Classification : 92D25, 34A37, 34K15
*Key words and phrases* : Positive periodic solution; neutral impulsive functional differential equation; fixed-point theorem

## 1. Introduction

In recent years, there has been a large number of papers concerned with the existence of periodic solutions of functional differential equations without impulses effects, see [1,2,9-21], and with impulse effects, see papers [3-8]. The methods used in these papers are upper and lower solutions methods and monotone iterative technique, fixed point theorems and the lower and upper solution methods.

In paper [1], the nonlinear differential equation

$$x'(t)=-\delta(t)x(t)+f(t,x(t)),\ t\in R \tag{1}$$

is considered, where $\delta(t)$ is a periodic function of periodic $T$, $f(t,x)$ is continuous and periodic in $t$. It is showed that equation (1) has at least two positive $T-$periodic solutions under certain growth conditions imposed on $f$. The methods used are fixed point theorem in cones in Banach spaces.

Received September 11, 2008. Accepted October 27, 2009. *Corresponding author.

Yu and Liu in [2] studied the existence of $T-$periodic solutions of the following neutral differential equation

$$x'(t) = f(t, x(t-\tau), x'(t-\tau)) + p(t), \ t \in R, \tag{2}$$

where $\tau > 0$ is a constant, $p$ is a $T-$periodic function, $f(t,x,y)$ is continuous and $T$-periodic in $t$. Sufficient conditions are established for the existence of at least one $T$-periodic solution of equation (2). The methods used are $k-$set contractive Operator principle and abstract continuous theorem in Banach spaces.

Impulsive delay differential equations can suitably model various evolutionary processes that exhibit both delay and impulse characteristics. In particular, they provide a natural description of the motion of several real world processes which, on one hand, depends on the processes history that often turns out to be the cause of phenomena substantially affecting the motion and, on other hand, is subject to short time perturbations whose duration is almost negligible. Such processes are often investigated in various fields of science and technology, such as physics, population dynamics, ecology, biological systems, optimal control, etc.

In a recent paper [3], Liu and Ge studied the existence of periodic solutions of the following first order differential equation with linear impulses effects

$$\begin{cases} x'(t) + a(t)x(t) + f(t, \ x(t-\tau(t))) = 0, & t \in R, \ t \neq t_k, \ k \in Z, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k = 1, 2, \cdots, \end{cases} \tag{3}$$

where $b_k > -1$ are constants, $a$ is $T-$periodic function with

$$\prod_{0<t_k<T} (1 + b_k) > \exp\left(\int_0^T a(u)du\right),$$

$f(t,x)$ is continuous, nonnegative and $T-$periodic in $t$. Denote

$$a^+(t) = \max\{0, a(t)\}, \ a^-(t) = \max\{0, -a(t)\}$$

and

$$b_k^+ = \max\{0, b_k\}, \ b_k^- = -\max\{0, -b_k\}.$$

Using fixed point theorems in cones in Banach spaces, they proved that equation (3) has at least one $T-$periodic solution under some assumptions and has at least three positive $T-$periodic solutions under the following assumptions that there exist positive constant $d < a$ such that

**(H1)** $f$ satisfies

$$\overline{\lim_{x \to +\infty}} \frac{f(t,x)}{x} < \frac{\prod_{0<t_k<T}(1+b_k) - \exp\left(\int_0^T a(u)du\right)}{T \exp\left(\int_0^T a^+(u)du\right) \prod_{0<t_k<T}(1+b_k^+)} = c$$

uniformly for $t \in [0,T]$;

**(H2)** it is satisfied that

$$f(t,x) < d\frac{\prod_{0<t_k<T}(1+b_k) - \exp\left(\int_0^T a(u)du\right)}{T\exp\left(\int_0^T a^+(u)du\right)\prod_{0<t_k<T}(1+b_k^+)} \text{ for all } x \in [0,d], t \in [0,T];$$

**(H3)** it is satisfied that

$$f(t,x) > a\frac{\prod_{0<t_k<T}(1+b_k) - \exp\left(\int_0^T a(u)du\right)}{T\exp\left(\int_0^T a^-(u)du\right)\prod_{0<t_k<T}(1+b_k^-)} \text{ for all } x \in [0,d], t \in [0,T].$$

The existence of at least three positive periodic solutions of the following well known impulsive biological models can be obtained by theorems in [3]:

**(i)** the impulsive red blood cell model

$$\begin{cases} N'(t) = -\mu(t)N(t) + p(t)e^{-r(t)N(t-\tau(t))}, t \neq t_k, \\ N(t_k) = (1+b_k)N(t_k^-), k \in Z; \end{cases}$$

**(ii)** the impulsive hematopoiesis model

$$\begin{cases} N'(t) = -\mu(t)N(t) + p(t)\frac{1}{1+r(t)N^n(t-\tau(t))}, t \neq t_k, \\ N(t_k) = (1+b_k)N(t_k^-), k \in Z; \end{cases}$$

and

$$\begin{cases} N'(t) = -\mu(t)N(t) + p(t)\frac{N^l(t-\tau(t))}{1+r(t)N^n(t-\tau(t))}, t \neq t_k, \\ N(t_k) = (1+b_k)N(t_k^-), k \in Z; \end{cases}$$

**(iii)** the impulsive Nicholson's Blowfly model

$$\begin{cases} N'(t) = -\mu(t)N(t) + p(t)N(t-\tau(t))e^{-r(t)N(t-\tau(t))}, t \neq t_k, \\ N(t_k) = (1+b_k)N(t_k^-), k \in Z. \end{cases}$$

Using Leggett-Williams fixed point theorem, the authors in [6] proved the existence of three positive periodic solutions of the following neutral impulsive functional differential equation

$$\begin{cases} \frac{d}{dt}[x(t) - cx(t-\gamma)] = -a(t)g(x(h_1(t)))x(t) + \lambda b(t)f\big(x(h_2(t))\big), \quad t \neq t_j; \\ \Delta\big[x(t) - cx(t-\gamma)\big] = I_j\big(x(t)\big), \quad t = t_j,\ j \in Z. \end{cases} \tag{4}$$

To the best of our knowledge, there has been no paper concerned with the existence of at least three positive $T-$periodic solutions of the following neutral impulsive functional differential equations

$$\begin{cases} x'(t) + a(t)x(t) = h(t)f(t, x(t-\tau(t)), x'(t-\delta(t))),\ a.e.\ t \in R, \\ \Delta x(t_k) = b_k x(t_k),\ k \in Z, \end{cases} \tag{5}$$

where

- $T > 0$ is a constant;
- $Z$ is the set of all integers, $R$ the set of all real numbers;
- $\{t_k\}$ a sequence satisfying that there exists $l > 0$ such that $t_k + T = t_{k+l}$ for all $k \in Z$;

- $b_k > -1$ is constants for all $k \in Z$ with $\prod_{t<t_k<t+T}(1+b_k) = constant$ for all $t \neq t_k(k \in Z)$ and $t \neq t_k - T(k \in Z)$;
- $a : R \to [0, +\infty)$ is piecewise continuous periodic function with period $T$ and $\exp(\int_0^T a(s)ds) > \prod_{0<t_k<T}(1+b_k)$;
- $h : R \to [0, +\infty)$ is piecewise continuous periodic function with period $T$ with $\int_0^T h(s)ds > 0$;
- $f : R \times R \times R \to R$ and $f(\cdot, x, y)$ is piecewise continuous periodic function with period $T$ and $f(t, \cdot, \cdot)$ is continuous;
- $\delta : R \to R$ is a $T$-periodic function;
- $\tau : R \to R$ is a $T-$periodic function.

A function $x : R \to [0, +\infty)$ is called a positive $T-$periodic solution of equation (5) if

- $x$ is positive and $T-$periodic on $R$;
- $x$ is continuous and differentiable on $(t_k, t_{k+1})$, there exist the limits $\lim_{t\to t_k^-} x(t) = x(t_k^-)$, $\lim_{t\to t_k^+} x(t) = x(t_k^+) = x(t_k)$ and $x$ satisfies (5).

The purpose of this paper is to establish new existence criterion for at least three positive $T$-periodic solutions of equation (5).

This paper is organized as follows. In section 2, we give main results. In section 3, to illustrate the main results obtained in Section 2, we present an example.

## 2. Main Results and Examples

In this section, we first present some background definitions in Banach spaces and state important three fixed point theorems. Then some lemmas are given at the end of this section.

**Definition 2.1.** Let $X$ be a semi-ordered real Banach space. The nonempty convex closed subset $P$ of $X$ is called a cone in $X$ if $ax \in P$ and $x + y \in P$ for all $x, y \in P$ and $a \geq 0$ and $x \in X$ and $-x \in X$ imply $x = 0$.

**Definition 2.2.** Let $X$ be a semi-ordered real Banach space and $P$ a cone in $X$. A map $\psi : P \to [0, +\infty)$ is a nonnegative continuous concave or convex functional map provided $\psi$ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y),$$

or

$$\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y),$$

for all $x, y \in P$ and $t \in [0, 1]$.

**Definition 2.3.** Let $X$ be a semi-ordered real Banach space. An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Suppose that $X$ is a real semi-ordered Banach space with the norm $||\cdot||$ and $P$ a nonempty cone of $X$. Let $a_1, a_2, a_3, a_4 > 0$ be positive constants, $\alpha_1$ be a nonnegative continuous concave functionals on the cone $P$, $\beta_1, \beta_2$ be three nonnegative continuous convex functionals on the cone $P$, $\psi$ be a nonnegative continuous functional on $P$. Define the following convex sets

$$P(\beta_1; a_4) = \{x \in P : \beta_1(x) < a_4\},$$
$$P(\beta_1, \alpha_1; a_2, a_4) = \{x \in P : \alpha_1(x) \geq a_2,\ \beta_1(x) \leq a_4\},$$
$$P(\beta_1, \beta_2, \alpha_1; a_2, a_3, a_4) = \{x \in P : \alpha_1(x) \geq a_2,\ \beta_2(x) \leq a_3,\ \beta_1(x) \leq a_4\}$$

and a closed set

$$R(\beta_1, \psi; a_1, a_4) = \{x \in P : \psi(x) \geq a_1, \beta_1(x) \leq a_4\}.$$

**Theorem 2.1[20].** *Let $P$ be a cone in a real Banach space $X$ with the norm $||\cdot||$. Suppose that*

(1) *$T : P \to P$ is completely continuous;*

(2) *$\beta_1$ and $\beta_2$ be nonnegative continuous convex functionals on $P$, $\alpha_1$ be a nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$ satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for all $x \in P$ and $\lambda \in [0,1]$, and $\alpha_1(x) \leq \psi(x)$ and there exists a constant $M > 0$ such that $||x|| \leq M\beta_1(x)$ for all $x \in P$;*

(3) *there exist positive numbers $a_1 < a_2, a_3$ and $a_4$ such that*

(E1) *$T(\overline{P(\beta_1; a_4)}) \subseteq \overline{P(\beta_1; a_4)}$;*

(E2) *$\alpha_1(Tx) > a_2$ for all $x \in P(\beta_1, \alpha_1; a_2, a_4)$ with $\beta_2(Tx) > a_3$;*

(E3) *$\{x \in P(\beta_1, \beta_2, \alpha_1; a_2, a_3, a_4) : \alpha_1(x) > a_2\} \neq \emptyset$ and $\alpha_1(Tx) > b$ for all $x \in P(\beta_1, \beta_2, \alpha_1; a_2, a_3, a_4)$;*

(E4) *$0 \notin R(\beta_1, \psi; a_1, a_4)$ and $\psi(Tx) < a_1$ for all $x \in R(\beta_1, \psi; a_1, a_4)$ with $\psi(x) = a_1$;*

*Then $T$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\beta_1; a_4)}$ such that*

$$\begin{aligned} \beta_1(x_i) &\leq a_4,\ i = 1,2,3; \\ \alpha_1(x_1) &> a_2,\ \psi(x_2) > a_1, \\ \alpha_1(x_2) &< a_2,\ \psi(x_3) < a_1. \end{aligned}$$

Choose

$$\begin{aligned} X = \Big\{x : R \to R \Big|\ & x, x' \text{ is } T-\text{periodic, continuous on } (t_k, t_{k+1}) \\ & \text{there exist the limits } x(t_k^+) = \lim_{t \to t_k^+} x(t) = x(t_k), \\ & x'(t_k^+) = \lim_{t \to t_k^+} x'(t) = x'(t_k), \\ & x(t_k^-) = \lim_{t \to t_k^-} x(t), x'(t_k^-) = \lim_{t \to t_k^-} x'(t) \\ & \text{and } x(t_k) = (1 + b_k)x(t_k^-) \text{ for all } k \in Z \Big\}. \end{aligned}$$

For $x \in X$, let

$$||x|| = \max\left\{\sup_{t\in[0,T]} |x(t)|,\ \sup_{t\in[0,T]} |x'(t)|\right\}.$$

It is easy to show that $X$ is a Banach space. Then $X$ is a real Banach space.

Let

$$P=\left\{x\in X:\quad x(t)\geq\sigma\max_{t\in[0,T]}|x(t)| \text{ on } [0,T]\right\},$$

where $\sigma$ is defined by

$$\sigma=\exp\left(-\int_0^T |a(s)|ds\right).$$

Then $P$ is a cone of space $X$.

Define the functionals on $P\to[0,+\infty)$ by

$$\begin{aligned}\beta_1(x) &= \sup_{t\in[0,T]}|x'(t)|,\ x\in P;\quad \psi(x)=\sup_{t\in[0,T]}|x(t)|,\ x\in P,\\ \beta_2(x) &= \sup_{t\in[0,T]}|x(t)|,\ x\in P;\quad \alpha_1(x)=\inf_{t\in[0,T]}|x(t)|,\ x\in P.\end{aligned}$$

It is easy to show that

- $\alpha_1$ is a nonnegative continuous concave functional on the cone $P$;
- $\beta_1,\beta_2$ three nonnegative continuous convex functionals on the cone $P$;
- $\psi$ a nonnegative continuous functional on the cone $P$;
- $\psi(\lambda x)\leq\lambda\psi(x)$ for all $x\in P$ and $\lambda\in[0,1]$;
- $\alpha_1(x)\leq\psi(x)$ for all $x\in P$.

Suppose that $t\neq t_k$ and $t\neq t_k-T$. One sees that

$$\left[x(t)\exp\left(\int_0^t a(u)du\right)\right]'=[x'(t)+a(t)x(t)]\exp\left(\int_0^t a(u)du\right). \tag{6}$$

Since $x(t)=x(t+T)$ and $x(t_k)=(1+b_k)x(t_k^-)$, integrating (6) from $t$ to $t+T$, it follows that

$$x(t)=\int_t^{t+T}G(t,s)[x'(s)+a(t)x(s)]ds,\ a.e.\ t\in R, \tag{7}$$

where

$$G(t,s)=\frac{\prod_{s<t_k<t+T}(1+b_k)\exp\left(\int_t^s a(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)},\ s\in[t,t+T]. \tag{8}$$

Then

$$\begin{aligned}|x(t)| &= \left|\int_t^{t+T}G(t,s)[x'(s)+a(t)x(s)]ds\right|\\ &\leq \int_t^{t+T}G(t,s)\left(|x'(s)|+|a(s)||x(s)|\right)ds\\ &\leq \int_t^{t+T}G(t,s)ds\sup_{t\in[0,T]}|x'(t)|+\int_t^{t+T}G(t,s)|a(s)|ds\sup_{t\in[0,T]}|x(t)|\end{aligned}$$

$$\leq \frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\sup_{t\in[0,T]}|x'(t)|$$
$$+\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\sup_{t\in[0,T]}|a(t)|\sup_{t\in[0,T]}|x(t)|.$$

If

$$\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\sup_{t\in[0,T]}|a(t)|<1, \tag{9}$$

then

$$\sup_{t\in[0,T]}|x(t)|\leq\frac{\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}}{1-\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\sup_{t\in[0,T]}|a(t)|}\sup_{t\in[0,T]}|x'(t)|.$$

Hence

- there exists a constant $M>0$ such that $||x||\leq M\beta_1(x)$ for all $x\in P$ if (9) holds, where

$M=$
$$\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)-\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)\sup_{t\in[0,T]}|a(t)|}.$$

Define the operator $T:\ X\to X$ by

$$(Tx)(t)\quad=\quad\int_t^{t+T}G(t,s)h(s)f(s,x(s-\tau(s)),x'(s-\delta(s)))ds \text{ for all } x\in X.$$

It is easy to show that $T$ is completely continuous; $TP\subset P$ and $x$ is a positive $T-$periodic solution of equation (5) if and only if $x$ is a fixed point of the operator $T$ in $P$.

**Theorem L.** *Suppose that (9) holds. Let $e_1, e_2, c$ be positive numbers and $Q, W$ and $E$ given by*

$$Q\quad=\quad\min\left\{c\frac{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}{\int_0^T h(s)ds\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)},\right.$$
$$\left.\frac{c}{\sup_{t\in[0,T]}|a(t)|\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\int_0^T h(s)ds+\sup_{t\in[0,T]}h(t)}\right\};$$

$$W = e_2\frac{\exp\left(\int_0^T a(u)du\right) - \prod_{0<t_k<T}(1+b_k)}{\sigma\int_0^T h(s)ds\prod_{0<t_k<T}(1+b_k^-)\exp\left(-\int_0^T a^-(u)du\right)};$$

$$E = e_1\frac{\exp\left(\int_0^T a(u)du\right) - \prod_{0<t_k<T}(1+b_k)}{\int_0^T h(s)ds\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}.$$

*If*

$$Mc > \frac{e_2}{\sigma} > e_2 > \frac{e_1}{\sigma} > e_1 > 0$$

*and*

*(A1)* $f(t,u,v) \le Q$ *for all* $t \in [0,T-1], u \in [0,Mc], v \in [-c,c]$;
*(A2)* $f(t,u,v) \ge W$ *for all* $t \in [0,T-1], u \in [e_2, e_2/\sigma], v \in [-c,c]$;
*(A3)* $f(t,u,v) \le E$ *for all* $t \in [0,T-1], u \in [0,e_1], v \in [-c,c]$;

*then equation has at least three positive* $T-$*periodic solutions* $x_1, x_2, x_3$ *such that*

$$\max_{t\in[0,T]} x_1(t) < e_1, \ \min_{t\in[0,T]} x_2(t) > e_2,$$

*and*

$$\max_{t\in[0,T]} x_3(t) > e_1, \ \min_{t\in[0,T]} x_3(t) < e_2.$$

*Proof.* To apply Theorem 2.1, we prove that all conditions in Theorem 2.1 are satisfied.

By the definitions, it is easy to see that $\alpha_1$ is a nonnegative continuous concave functional on the cone $P$, $\beta_1, \beta_2$ are three nonnegative continuous convex functionals on the cone $P$, $\psi$ a nonnegative continuous functional on the cone $P$. $x \in X$ is a positive $T-$periodic solution of equation (4) if and only if $x$ is a solution of the operator equation $x = Tx$ and $T$ is completely continuous.

It is easy to see that $\psi(\lambda x) \le \lambda\psi(x)$ for all $x \in P$ and $\lambda \in [0,1]$, and $\alpha_1(x) \le \psi(x)$, and there exists a constant $M > 0$ such that $||y|| \le M\beta_1(y)$ for all $y \in P$. It follows that (1) and (2) of Theorem 2.1 hold.

Corresponding to Theorem 2.1,

$$a_4 = c, \ a_3 = \frac{e_2}{\sigma}, \ a_2 = e_2, \ a_1 = e_1.$$

Now, we prove that (3) of Theorem 2.1 hold. One sees that $0 < a_1 < a_2, a_3 > 0, a_4 > 0$. The remainder is divided into four steps.

**Step 1.** Prove that $T(\overline{P(\beta_1;a_4)}) \subseteq \overline{P(\beta_1;a_4)}$; For $y \in \overline{P(\beta_1;a_4)}$, we have $\beta_1(y) = \sup_{t\in[0,T]}|y'(t)| \le c$. Then

$$0 \le y(t) \le M\beta_1(y) = \frac{\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}}{1-\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\sup_{t\in[0,T]}|a(t)|}\beta_1(y)c$$

for $t \in [0,T]$ and $-c \le y'(t) \le c$ for all $t \in [0,T]$. So (A1) implies that $f(t, y(t-\tau(t)), y'(t-\delta(t))) < Q,\ t \in [0,T]$. Hence

$$\begin{aligned}(Ty)(t) &= \int_t^{t+T} G(t,s)h(s)f(s,y(s-\tau(s)),y'(s-\delta(s)))ds \\ &\le Q\int_t^{t+T} G(t,s)h(s)ds \\ &\le Q\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\int_0^T h(s)ds \\ &\le c.\end{aligned}$$

On the other hand, we have

$$(Ty)'(t)+a(t)(Ty)(t) = h(t)f(t,y(t-\tau(t)),y'(t-\delta(t))),\quad a..e.\ t\in R$$

and

$$(Ty)(t_k) = (1+b_k)(Ty)(t_k^-),\ k \in Z.$$

Then

$$\begin{aligned}&|(Ty)'(t)| \\ &= \left|\left(\int_t^{t+T} G(t,s)h(s)f(s,y(s-\tau(s)),y'(s-\delta(s)))ds\right)'\right| \\ &= \left|-a(t)\int_t^{t+T} G(t,s)h(s)f(s,x(s-\tau(s)),x'(s-\delta(s)))ds\right. \\ &\quad \left.+\, h(t)f(t,x(t-\tau(t)),x'(t-\delta(t)))\right| \\ &\le Q\left(\sup_{t\in[0,T]}|a(t)|\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\int_0^T h(s)ds+\sup_{t\in[0,T]}h(t)\right) \\ &\le c.\end{aligned}$$

It follows that

$$||Ty|| = \max\left\{\max_{t\in[0,T]}|(Ty)(t)|,\ \max_{t\in[0,T]}|(Ty)'(t)|\right\} \le c.$$

Then $T(\overline{P(\beta_1;a_4)}) \subseteq \overline{P(\beta_1;a_4)}$. This completes the proof of (3)-(E1) in Theorem 2.1.

**Step 2.** Prove that $\alpha_1(Tx) > a_2$ for all $x \in P(\beta_1,\alpha_1;a_2,a_4)$ with $\beta_2(Tx) > a_3$;

For $y \in P(\beta_1,\alpha_1;a_2,a_4) = P(\beta_1,\alpha_1;e_2,c)$ with $\beta_2(Ty) > a_3 = \frac{e_2}{\sigma}$, we have that $\alpha_1(y) = \min_{t\in[0,T]} y(t) \ge e_2$ and

$$\beta_1(y) = \max_{t\in[0,T]}|y'(t)| \le c$$

and $\max_{t\in[0,T]}(Ty)(t) > \frac{e_2}{\sigma}$. Since $Ty \in P$, we get

$$\alpha_1(Ty) = \min_{t\in[0,T]}(Ty)(t) \geq \sigma\beta_2(Ty) > \sigma\frac{e_2}{\sigma} = e_2 = a_2.$$

This completes the proof of (3)-(E2) in Theorem 2.1.

**Step 3.** Prove that $\{x \in P(\beta_1,\beta_2,\alpha_1;a_2,a_3,a_4) : \alpha_1(x) > a_2\} \neq \emptyset$ and $\alpha_1(Tx) > b$ for all $x \in P(\beta_1,\beta_2,\alpha_1;a_2,a_3,a_4)$;

It is easy to show that

$$\{x \in P(\beta_1,\beta_2,\alpha_1;a_2,a_3,a_4) : \alpha_1(x) > a_2\} \neq \emptyset.$$

For $y \in P(\beta_1,\beta_2,\alpha_1;a_2,a_3,a_4)$, one has that

$$\alpha_1(y) = \min_{t\in[0,T]} y(t) \geq e_2,\ \beta_2(y) = \max_{t\in[0,T]} y(t) \leq \frac{e_2}{\sigma},\ \beta_1(y) = \max_{t\in[0,T]} |y'(t)| \leq c.$$

Then

$$e_2 \leq y(t) \leq \frac{e_2}{\sigma},\ t \in [0,T],\ \ |y'(t)| \leq c.$$

Thus (A2) implies that

$$f(t, y(t-\tau(t)), y'(t-\delta(t))) \geq W,\ \ t \in [0,T].$$

Since

$$\alpha_1(Ty) = \min_{t\in[0,T]}(Ty)(t) \geq \sigma \max_{t\in[0,T]}(Ty)(t),$$

we get

$$\alpha_1(Ty) \geq \sigma \max_{t\in[0,T]}(Ty)(t).$$

Hence

$$\begin{aligned}
\alpha_1(Ty) &\geq \sigma \max_{t\in[0,T]}(Ty)(t) \\
&= \sigma \max_{t\in[0,T]} \int_t^{t+T} G(t,s)h(s)f(s,y(s-\tau(s)),y'(s-\delta(s))))ds \\
&\geq W\sigma \frac{\prod_{0<t_k<T}(1-b_k^-)\exp\left(-\int_0^T a^-(u)du\right)}{\exp\left(\int_0^T a(u)du\right) - \prod_{0<t_k<T}(1+b_k)} \int_0^T h(s)ds \\
&\geq e_2.
\end{aligned}$$

This completes the proof of (3)-(E3) in Theorem 2.1.

**Step 4.** Prove that $0 \notin R(\beta_1,\psi;a_1,a_4)$ and $\psi(Tx) < a_1$ for all $x \in R(\beta_1,\psi;a_1,a_4)$ with $\psi(x) = a_1$;

It is easy to see that $0 \notin R(\beta_1,\psi;a_1,a_4)$. For $y \in R(\beta_1,\psi;a_1,a_4)$ with $\psi(x) = a_1$, one has that

$$\psi(y) = \max_{t\in[0,T]} y(t) = a_1,\ \beta_1(y) = \max_{t\in[0,T]} |y'(t)| \leq a_4.$$

Hence we get that

$$0 \leq y(t) \leq a_1,\ t \in [0,T];\ \ -c \leq y'(t) \leq c,\ t \in [0,T].$$

Then (A3) implies that

$$f(t,y(t-\tau(t)),y'(t-\delta(t)))\le E,\ \ t\in[0,T].$$

So

$$\begin{aligned}\psi(Ty) &= \max_{t\in[0,T]}\int_t^{t+T} G(t,s)h(s)f(s,y(s-\tau(s)),y'(s-\delta(s)))\\ &\le E\sum_{s=n}^{n+T-1} G(n,s)h(s)\\ &\le E\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\int_0^T h(s)ds\\ &\le e_1.\end{aligned}$$

This completes the proof of (3)-(E4) in Theorem 2.1.

Then Theorem 2.1 implies that $T$ has at least three fixed points $x_1$, $x_2$ and $x_3$ such that

$$\beta(y_1)<e_1,\ \alpha(y_2)>e_2,\ \beta(y_3)>e_1,\ \alpha(y_3)<e_2.$$

Hence equation (5) has three positive $T$-periodic solutions $x_1, x_2$ and $x_3$ such that

$$\sup_{t\in[0,T]} x_1(t)<e_1,\ \ \inf_{t\in[0,T]} x_2(t)>e_2,$$

and

$$\sup_{t\in[0,T]} x_3(t)>e_1,\ \ \inf_{t\in[0,T]} x_3(t)<e_2.$$

The proof of Theorem L is completed.

## 3. An Example

Now, we present an example to illustrate the main results.

**Example 4.1.** Consider the existence of positive periodic solutions of the following impulsive functional difference equation

$$\begin{cases} x'(t)+\frac{1}{8}\left(1-\frac{1}{2}\sin(2t)\right)x(t)=\frac{1}{8}f(t,x(t-\sin(2t)),x'(t-\cos(2t))),\\ a.e.\ t\in R,\\ \Delta x(k\pi+\pi/2)=-\left(\frac{3}{4}+e^{-\pi}\right)\frac{1}{2}x((k\pi+\pi/2)^-),\ k\in Z, \end{cases}\tag{10}$$

where

$$a(t)=\frac{1}{8}-\frac{1}{16}\sin(2t);\qquad h(t)=\frac{1}{8},$$

and

$$f(t,u,v)=f_0(u)+\frac{|v|(1+\sin t)}{25500000000000}$$

with

$$f_0(u)=\begin{cases}\frac{1}{64}u, & u\in[0,2],\\ \frac{20000-\frac{1}{32}}{1000-2}(u-1000)+20000, & u\in[2,1000],\\ 20000, & u\in[1000,64000000],\\ 20000e^{u-64000000}, & u\geq 64000000.\end{cases}$$

Corresponding to equation (5), one sees that

$$\begin{aligned} T &= \pi, \qquad a(t)=\frac{1}{8}-\frac{1}{16}\sin(2t),\\ h(t) &= \frac{1}{8}, \qquad \tau(t)=\sin(2t),\\ \delta(t) &= \cos(2t), \qquad b_k=-\left(\frac{3}{4}+e^{-\pi}\right). \end{aligned}$$

$f$ is nonnegative, $f(\cdot,x,y)$ is periodic with period $T$ and $f(t,\cdot,\cdot)$ is continuous. It is easy to see that

$$\exp\left(\int_0^T a(u)du\right)>\prod_{0<t_k<T}(1+b_k)$$

and

$$\begin{aligned} &\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\sup_{t\in[0,T]}|a(t)|\\ =\ &\frac{e^{\pi}}{e^{\pi}-\frac{1}{4}+e^{-\pi}}\frac{3}{16}<1. \end{aligned}$$

**Use Theorem L.** Choose $e_1=2, e_2=1000, c=2000000$, then

$$\sigma=\exp\left(-\int_0^{\pi}|a(s)|ds\right)=e^{-\pi};$$

$$M=\frac{\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}}{1-\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\sup_{t\in[0,T]}|a(t)|}=\frac{e^{\pi}}{\frac{13}{16}e^{\pi}+e^{-\pi}-\frac{1}{4}};$$

$$\begin{aligned} Q=c\min\Bigg\{&\frac{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}{\int_0^T h(s)ds\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)},\\ &\frac{1}{\sup_{t\in[0,T]}|a(t)|\frac{\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}\int_0^T h(s)ds+\sup_{t\in[0,T]}h(t)}\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= 2000000\frac{8e^{\pi}+8e^{-\pi}-2}{\pi e^{\pi}}; \\
W &= e_2\frac{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}{\sigma\int_0^T h(s)ds\prod_{0<t_k<T}(1-b_k^-)\exp\left(-\int_0^T a^-(u)du\right)} \\
&= 1000\frac{8e^{\pi}+8e^{-\pi}-2}{\pi e^{-\pi}\left(\frac{1}{4}-e^{-\pi}\right)}; \\
E &= e_1\frac{\exp\left(\int_0^T a(u)du\right)-\prod_{0<t_k<T}(1+b_k)}{\int_0^T h(s)ds\prod_{0<t_k<T}(1+b_k^+)\exp\left(\int_0^T a^+(u)du\right)} = \frac{16e^{\pi}+16e^{-\pi}-4}{\pi e^{\pi}}.
\end{aligned}
$$

It is easy to see that

$$Mc > e_2 > \frac{e_1}{\sigma_0} > e_1 > 0$$

and

$f(t,u,v) \leq 2000000\frac{8e^{\pi}+8e^{-\pi}-2}{\pi e^{\pi}}$ for all $t \in [0,\pi], u \in [0,64000000], v \in [-2000000, 2000000]$;

$f(t,u,v) \geq 1000\frac{8e^{\pi}+8e^{-\pi}-2}{\pi e^{-\pi}\left(\frac{1}{4}-e^{-\pi}\right)}$ for all $t \in [0,\pi], u \in [1000, 4000], v \in [-2000000, 2000000]$;

$f(t,u,v) \leq \frac{16e^{\pi}+16e^{-\pi}-4}{\pi e^{\pi}}$ for all $t \in [0,\pi], u \in [0,2], v \in [-2000000, 2000000]$;

then equation (9) has at least three positive $T$−periodic solutions $x_1, x_2, x_3$ such that

$$\sup_{t\in[0,\pi]} x_1(t) < 2, \quad \inf_{t\in[0,\pi]} x_2(t) > 1000,$$

and

$$\sup_{t\in[0,\pi]} x_3(t) > 2, \quad \inf_{t\in[0,\pi]} x_3(t) < 1000.$$

## References


1. Y. Liu, *Positive periodic solutions of nonlinear differential equations*, Applied Mathematics A Journal of Chinese Universities, **B(4)** (2003), 1-5.
2. J. Yu, B. Liu, *Existence of periodic solutions for nonlinear neutral delay differential equations*, Applied Mathematics A Journal of Chinese Universities, **A(3)**(2001), 11-15.
3. Y. Liu, W. Ge, *Stability theorems and existence results for periodic solutions of nonlinear impulsive delay differential equations with variable coefficients*, Nonl. Anal., **57**(2004), 363-399.
4. X. Li, X. Lin, D. Jiang, X. Zhang, *Existence and multiplicity of positive periodic solutions to functional differential equations with impulse effects*, Nonl. Anal., **62**(2005), 683-701.
5. G. S. Ladde, V. Lakshmikantham, A. S. Vatsala, *Monotone iterative techniques for nonlinear differential equations*, Pitman Advanced Publishing Program, 1985.
6. X. Fan, Y. Li, *Positive periodic solutions of neutral functional differential equations with a parameter and impulse*, Electronic J. of Differential Equations, **38**(2008), 1-8.
7. J. Yan, *Existence of positive periodic solutions of impulsive functional differential equations with two parameters*, J. Math. Anal. Appl., **327**(2007), 854-868.

8. J. O. Alzabut, *A necessary and sufficient conditions for the existence of periodic solutions of linear impulsive differential equation with distributed delay*, Discrete and Continuous Dynamical Systems, Website: www.AIMSciences.org, **15** (2007), 35-43.
9. S. Cheng, G. Zhang, *Existence of positive periodic solutions for non-autonomous functional differential equations,* Electron J. Diff. Eqns., **59**(2001), 1-8.
10. S. Kang, G. Zhang, *Existence of nontrivial periodic solutions for first order functional differential equations*, Appl. Math. Letters, **18**(2005), 101-107.
11. S. Kang, G. Zhang, B. Shi, *Existence of three periodic positive solutions for a class of integral equations with parameters*, J. Math. Anal. Appl., **323**(2006), 654-665.
12. S. Padhi, S. Srivastava, *Multiple periodic solutions for a nonlinear first order functional differential equations with applications to population dynamics*, Appl. Math. Comput., **203**(2008), 1-6.
13. X.H. Tang, Z. Jiang, *Periodic solutions of first-order nonlinear functional differential equations*, Nonl. Anal., **68**(2008), 845-861.
14. X. Han, S. Ji, Z. Ma, *On the existence and multiplicity of positive periodic solutions for first-order vector differential equation*, J. Math. Anal. Appl., **329**(2007), 977-986.
15. D. Bai, Y. Xu, *Periodic solutions of first order functional differential equations with periodic deviations*, Comput. Math. Appl., **53**(2007), 1361-1366.
16. B. Liu, L. Huang, *Existence and uniqueness of periodic solutions for a kind of first order neutral functional differential equations*, J. Math. Anal. Appl., **322**(2006), 121-132.
17. G. Wang, *Existence theorem of periodic solutions for a delay nonlinear differential equation with piecewise constant arguments*, J. Math. Anal. Appl., **298**(2004), 298-307.
18. A. Cabada, J.B. Ferreiro, J.J. Nieto, *Green's function and comparison principles for first order periodic differential equations with piecewise constant arguments*, J. Math. Anal. Appl., **291**(2004), 690-697.
19. D. Li, Y. Xu, *Existence of periodic solutions for fully nonlinear first-order differential equations*, Nonl. Anal., **52**(2003), 1095-1109.
20. R. Hakl, A. Lomtatidze, B. Puza, *On periodic solutions of first order linear functional differential equations*, Nonl. Anal., **49**(2002), 929-945.
21. Ch. G. Philos, I. K. Purnaras, *Periodic first order linear neutral delay differential equations*, Appl. Math. Comput., **117**(2001), 203-222.

**Yuji Liu** received her BS from Hunan University and Ph.D at Beijing Institute of Technology under the direction of Professor W. Ge. Since 2006 he has been at Guangdong Universty of Business Studies. His research interests focus on the theory of functional differential equations.

Department of Mathematics, Guangdong University of Business Studies, Guangzhou 510320, P R China
e-mail: `liuyuji888@sohu.com`

**Jianye Xia** received her BS from Hunan Normal University and MD at Central South University. Since 1988 he has been at the of Guangdong University of Finance. His research interests focus on the theory of functional differential equations and number theory.

Department of Mathematics, Guangdong University of Finance, Guangzhou 510320, P R China
e-mail: `jianyexia@sohu.com`