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T–VAGUE n-ARY SUBGROUPS

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ABSTRACT. In this paper,we introduced the notion of T–vague n-ary subgroups on n-ary subgroups (G, f) and have studied their related properties.

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1. Introduction

Many authors from time to time have introduced a different number of generalization of Zadeh's fuzzy set theory [32] and have been applied to many branches in mathematics. The notion of vague theory first introduced by Gau and Buehrer [23]. Later vague theory of the "group" concept into "vague group" was made by Biswas [2]. This work was the first vagueness of any algebraic structure and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests.

The study of n-ary systems was initiated by Kasner [26] in 1904,but the important study on n-ary groups was done by Dörnte [4].The theory of n-ary systems have many applications.For example,in the theory of automata [24] n-ary semigroup and n-ary groups are used.The n-ary groupoids are applied in the theory of quantum groups [29].Also the ternary structures in physics are described by Kerner in [25].The first fuzzification of n-ary system was introduced by Dudek [11].Further,the concept of fuzzy n-ary subgroups was introduced by Davvaz and Dudek [3].The first vagueness of n-ary system was introduced by Prince Williams and Said Al-Jelihaw [30]. The aim of this paper is to introduce the notion of T–vague n-ary subgroups in n-ary group (G, f) and investigate their related properties.

2. Preliminaries

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A non-empty set G together with one n-ary operation $f : G^n \to G$,where $n \geq 2$, is called an *n-ary groupoid* and is denoted by (G, f) .Accroding to the general convention used in the theory of n-ary groupoids the sequence of elements $x_i, x_{i+1}, ..., x_j$ is denoted by x_i^j.In the case $j < i$,it denoted the empty symbol.If $x_{i+1} = x_{i+2} = ... = x_{i+t} = x$, then instead of x_{i+1}^{i+t} and we write $\overset{(t)}{x}$. In this convention

$$f(x_1, ..., x_n) = f(x_1^n)$$

and

$$f(x_1, ..., x_i, \underbrace{x, ..., x}_{t}, x_{i+t+1}, ..., x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

An n-ary groupoid (G, f) is called an *(i, j)-associative* if

$$f\left(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}\right)$$

hold for all $x_1, ..., x_{2n-1} \in G$.If this identity holds for all $1 \leq i \leq j \leq n$,then we say that the operation f is associative and (G, f) is called an *n-ary semigroup*.It is clear that an n-ary groupoid is associative if and only if it is $(1, j)$-associative for all $j = 2, .., n$.In the binary case (i.e. n=2)it is usual semigroup.If for all $x_0, x_1, ..., x_n \in G$ and fixed $i \in \{1, ..., n\}$ there exists an element $z \in G$ such that

$$f\left(x_1^{i-1}, z, x_{i+1}^n\right) = x_0 \tag{1}$$

then we say that this equation is *i-solvable* or *solvable at the place i*.If the solution is unique,then we say that (1) is *uniquely i-solvable*.An n-ary groupoid (G, f) uniquely solvable for all $i = 1, ..., n$ is called an *n-ary quasigroup* .An associative n-ary quasigroup is called an *n-ary group* .

Fixing an an n-ary operation f,where $n \geq 3$, the elements a_2^{n-2} we obtain the new binary operation $x \diamond y = f(x, a_2^{n-2}, y)$.If (G, f) is an n-ary group then $(G, \diamond)$ is a group.Choosing different elements a_2^{n-2} we obtain different groups.All these groups are isomorphic[8].So,we can consider only group of the form

$$ret_a(G, f) = (G, \circ), \text{ where } x \circ y = f(x, \overset{(n-2)}{a}, y).$$

In this group $e = \overline{a}, x^{-1} = f(\overline{a}, \overset{(n-3)}{a}, \overline{x}, \overline{a})$.

In the theory of n-ary groups,the following Theorem plays an important role.

Theorem 2.1.[14] *For any n-ary group (G, f) there exist a group $(G, \circ)$,its automorphism φ and an element $b \in G$ such that*

$$f(x_1^n) = x_1 \circ \varphi(x_2) \circ \phi^2(x_3) \circ ... \circ \phi^{n-1}(x_n) \circ b \tag{2}$$

holds for all $x_1^n \in G$.

To study more about n-ary system see [5-11,13,15-22].

In what follows,G is a non-empty set and (G, f) is a n-ary group unless otherwise specified.In what follows,G is a non-empty set and (G, f) is a n-ary group unless otherwise specified.

Definition 2.2.[1] By a t-norm ,a function $T : [0,1] \times [0,1] \to [0,1]$satisfying the following conditions is meant:

(T1) $T(x, 1) = x$;

(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$;

(T3) $T(x, y) = T(y, x)$;

(T4) $T(x, T(y, z)) = T(T(x, y), z)$;

for all $x, y, z \in [0, 1]$.

Now we generalize the domain of T to $\prod_{i=1}^{n} [0,1]$ as follows:

Definition 2.3. The function $T_n = \prod_{i=1}^{n} [0,1] \to [0,1]$ is defined by:

$$T_n(\alpha_1^n) = T_n(\alpha_1, \alpha_2, ..., \alpha_n) = T_n(\alpha_i, T_{n-1}(\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n)) \quad (3)$$

for all $\alpha_1^n \in [0,1]$ and $1 \leq i \leq n$ where $n \leq 2$,$T_2 = T$, and $T_1 = id$(identity). For a t-norm T_n on $\prod_{i=1}^{n} [0,1]$,it is denoted by

$$\triangle_t = \{\alpha \in [0,1] \mid T_n(\alpha, \alpha, ..., \alpha) = \alpha\}.$$

It is clear that every t-norm has the following property:

$$T_n(\alpha_1^n) \leq min\{\alpha_1, \alpha_2, ..., \alpha_n\}$$

for all $\alpha_1^n \in [0,1]$.

Remark 2.4. If T_n is of the form (3),then we say T_n is a function induced by t-norm T.

Definition 2.5. [2,27] A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

(V1) A true membership function $t_A : U \to [0,1]$,and

(V2) A false membership function $f_A : U \to [0,1]$,

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the "evidence for u", $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and $t_A(u) + f_A(u) \leq 1$.

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0,1]$.This indicates that if the actual grade of membership u is $\mu(u)$,then $t_A(u) \leq \mu(u) \leq 1 - f_A(u)$.

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle | u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A, denoted by $V_A(u)$.

Definition 2.6. [30] Let (G, f) be a n-ary group.A vague set A of G is called a *vague n-ary subgroup* of (G, f) if the following axioms holds:

(VnS1)$(\forall x_1^n \in G), (V_A(f(x_1^n) \succeq imin\{V_A(x_1), ..., V_A(x_n)\}),$

(VnS2)$(\forall x \in G), (V_A(\overline{x}) \succeq V_A(x))$. that is,

$$\begin{aligned} t_A(f(x_1^n)) &\geq min\{t_A(x_1),...,t_A(x_n)\}) \\ 1-f_A(f(x_1^n)) &\geq min\{1-f_A(x_1),...,f_A(x_n)\} \\ t_A(\overline{x}) &\geq t_A(x) \\ 1-f_A(\overline{x}) &\geq 1-f_A(x)). \end{aligned}$$

Example 2.7.[30] Let $(\mathbb{Z}_4,f)$ be a 4-ary subgroup derived from additive group $\mathbb{Z}_4$.Let A be the vague set in $\mathbb{Z}_4$ defined as follows:

$$A = \{\langle 0, [0.8, 0.02], \langle 1, [0.8, 0.02], \langle 2, [0.8, 0.02], \langle 3, [0.2, 0.07]\}.$$

By routine calculations, it is clear that A is a vague 4-ary subgroup of $(\mathbb{Z}_4, f)$.

2. T-Vague n-ary subgroups

In this section, we define the notion of $T-$vague n-ary subgroups.For our discussion, we shall use the following notations on interval arithmetic:

Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$.We define the term "*tmax*" to mean the maximum of n intervals as:

$$tmax(I_1, I_2, ..., I_n) := T_n[max(a_1, b_1), max(a_2, b_2), ..., max(a_n, b_n)],$$

where $I_1 = [a_1, b_1], I_2 = [a_2, b_2], ..., I_n = [a_n, b_n]$. Similarly, we define "*tmin*".The concept of "*tmax*" and "*tmin*" could be extended to define "*tsup*" and "*tinf*" of infinite number of elements of $[0,1]$.

It is obvious that $L = \{I[0,1], tsup, tinf, \succeq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$.

Definition 3.1. Let (G, f) be a n-ary group.A vague set A of G is called a $T-$*vague n-ary subgroup* of (G, f) if the following axioms holds:

(TVnS1)$(\forall x_1^n \in G), (V_A(f(x_1^n) \succeq tmin\{V_A(x_1), ..., V_A(x_n)\}),$

(TVnS2)$(\forall x \in G), (V_A(\overline{x}) \succeq V_A(x))$. that is,

$$\begin{aligned} t_A(f(x_1^n)) &\geq T_n\{t_A(x_1),...,t_A(x_n)\}) \\ 1-f_A(f(x_1^n)) &\geq T_n\{1-f_A(x_1),...,f_A(x_n)\} \\ t_A(\overline{x}) &\geq t_A(x) \\ 1-f_A(\overline{x}) &\geq 1-f_A(x)). \end{aligned}$$

Example 3.2. Let $(\mathbb{Z}_4, f)$ be a 4-ary subgroup derived from additive group $\mathbb{Z}_4$.Let A be the vague set in Z_4 defined as follows:

$$A = \{\langle 0, [0.8, 0.02], \langle 1, [0.8, 0.02], \langle 2, [0.8, 0.02], \langle 3, [0.2, 0.07]\}$$

and we define $f(x_1^n) = x_1 + x_2 + x_3 + x_4$.

Let $T_m : \prod_{i=1}^{4} [0,1] \longrightarrow [0,1]$ be a function defined by as follows:

$$T_m(y_1^4) = max\,\{y_1 + y_2 + y_3 + y_4 - 1, 0\}$$

for all $y_1^4 \in [0,1]$.Then,T_m is a function induced by t-norm.By routine calculations, it is clear that A is a $T-$vague 4-ary subgroup of $(\mathbb{Z}_4, f)$.

Theorem 3.3. *If $\{A_i|i \in I\}$ is an arbitrary family of $T-$vague n-ary subgroup of (G,f) then $\bigcap A_i$ is a $T-$vague n-ary subgroup of (G,f),where $\bigcap A_i(x) = sup\{A_i(x)|i \in I)\}$, for all $x \in G$.*

*Proof.*The proof is trivial.

Recall that if $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0,1]$,we can define a relation between I_1 and I_2 by $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.For $\alpha, \beta \in [0,1]$.Now we define $(\alpha, \beta) - cut$ and $\alpha - cut$ of a vague set.

Definition 3.4. Let A be a vague set in G with true membership function t_A and the false membership function f_A.The $(\alpha,\beta) - cut$ of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set G given by

$$A_{(\alpha,\beta)} = \{x \in G\,|V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly,$A_{(0,0)} = G$.The $(\alpha,\beta)-$cuts of the vague set A are also called *vague set* of A.

Definition 3.5. Let $\alpha - cut$ be a vague set A is a crisp subset A_α of the set G given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = G$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha,\alpha)} = A_\alpha$.Equivalently, we can define the $\alpha-$cut as

$$A_{(\alpha)} = \{x \in G\,|t_A(x) \geq \alpha\}.$$

The following Theorem is a consequence of the Transfer Principle described in [28].

Theorem 3.6. *Let A be a vague set of G. Then $A_{(\alpha,\beta)}$ is a crisp subset of G, is a $(\alpha,\beta) - cut$ is a $T-$vague n-ary subgroup of (G,f) if and only if the $(\alpha,\beta)-cut$ of G is n-ary subgroup of (G,f) for every $\alpha, \beta \in [0,1]$,which is called $T - vague - cut$ subgroup of (G,f).*

Proof. Let A be a vague set of G.Suppose the crisp subset $A_{(\alpha,\beta)}$ of G, is a $(\alpha,\beta) - cut$ is a $T-$vague n-ary subgroup of (G,f). If $x_1^n \in A_{(\alpha,\beta)}$and $\alpha,\beta \in [0,1]$, then $t_A(x_i) \geq \alpha$ and $1 - f_A(x_i) \geq \beta$for all $i = 1, 2, ..., n$. Thus

$$t_A(f(x_1^n) \geq T_n\{t_A(x_1), ..., t_A(x_n)\} \geq \alpha,$$

and

$$1 - f_A(f(x_1^n) \geq T_n\{1 - f_A(x_1), ..., 1 - f_A(x_n)\} \geq \beta.$$

which implies $f(x_1^n) \in A_{(\alpha,\beta)}$.

For all $x \in A_{(\alpha,\beta)}$, then $t_A(x) \geq \alpha$ and $1 - f_A(x) \geq \beta$ we have $t_A(\overline{x}) \geq t_A(x) \geq \alpha$, and

$$1 - f_A(\overline{x} \geq 1 - f_A(x) \geq \beta.$$

which implies $\overline{x} \in A_{(\alpha,\beta)}$. Thus $A_{(\alpha,\beta)}$ is a n-ary subgroup of (G,f).

Conversely, assume that $A_{(\alpha,\beta)}$ is a n-ary subgroup of (G,f).Let us define $\alpha_0 = T_n\{t_A(x_1),...,t_A(x_n)\}$ and

$$\beta_0 = T_n\{1 - f_A(x_1),...,1 - f_A(x_n)\},$$

for some $x_1^n \in G$.Then obviously $x_1^n \in A_{(\alpha,\beta)}$, consequently $f(x_1^n) \in A_{(\alpha,\beta)}$. Thus

$$t_A(f(x_1^n)) \geq \alpha_0 = T_n\{t_A(x_1),...,t_A(x_n)\}$$

and

$$1 - f_A(f(x_1^n)) \geq \beta_0 = T_n\{1 - f_A(x_1),...,1 - f_A(x_n)\}$$

Now, let $x \in A_{(\alpha,\beta)}$. Then $t_A(x) = \alpha_0 \geq \alpha$. and $1 - f_A(x) = \beta_0 \geq \beta$. Thus $x \in A_{(\alpha,\beta)}$. Since , by the assumption ,$\overline{x} \in A_{(\alpha,\beta)}$, $t_A(x) = \alpha_0 \geq \alpha$ and $1 - f_A(x) = \beta_0 \geq \beta$. Whence $t_A(\overline{x}) \geq \alpha_0 = t_A(x)$ and $1 - f_A(\overline{x}) \geq \alpha_0 = 1 - f_A(x)$. This complete the proof.

Using the above theorem,we can prove the following characterization of T - vague n-ary subgroups.

Theorem 3.7. *A vague set A in G, is a $T-$vague n-ary subgroups of (G,f) if and only if the (α,β)-cut subset $A_{(\alpha,\beta)}$ of G is a n-ary subgroup of (G,f) for all $i = 1,2,...,n$ and all $x_1^n \in G$, A satisfies the following conditions:*
(i) $V_A(f(x_1^n) \succeq tmin\{V_A(x_1),...,V_A(x_n)\}$,
(ii) $V_A(x_i) \succeq tmin\{V_A(x_1),...,V_A(x_{i-1}),V_A(f(x_1^n)),V_A(x_{i-1}),...,V_A(x_n)\}$.

Proof. Assume that A is a vague n-ary subgroupsof (G,f) .Similarly as in the proof of Theorem 3.6, we can prove that each non-empty (α,β)-cut subset $A_{(\alpha,\beta)}$ is closed under the operation f,that is $x_1^n \in A_{(\alpha,\beta)}$ implies $f(x_1^n) \in A_{(\alpha,\beta)}$.
Now let $x_0, x_1^{i-1}, x_{i+1}^n$,where $x_0 = f(x_1^{i-1}, z, x_{i+1}^n)$ for some $i = 1,2,...,n$ and $z \in G$ which implies $x_0 \in A_{(\alpha,\beta)}$ .Then,according to (ii),we have $t_A(x_i) \geq \alpha$ and $1 - f_A(x_i) \geq \beta$.So,the the equation (1) has a solution $z \in A_{(\alpha,\beta)}$ .This mean that (α,β)-cut subset $A_{(\alpha,\beta)}$ is a n-ary subgroups.

Conversely, assume that (α,β)-cut subset $A_{(\alpha,\beta)}$ is a n-ary subgroups of (G,f) .Then it is easy to prove the condition (i).
For $x_1^n \in G$, we define

$$\alpha_0 = T_n\{t_A(x_1),...,t_A(x_{i-1}),t_A(f(x_1^n)),t_A(x_{i-1}),...,t_A(x_n)\}.$$

and

$$\beta_0 = T_n\{1-f_A(x_1),...,1-f_A(x_{i-1}),1-f_A(f(x_1^n)),1-f_A(x_{i-1}),...,1-f_A(x_n)\}.$$

Then $x_1^{i-1}, x_{i+1}^n, f(x_1^n) \in A_{(\alpha_0,\beta_0)}$.Whence, according to the definition of n-ary group,we conclude $x_i \in A_{(\alpha_0,\beta_0)}$.Thus $t_A(x_i) \geq \alpha_0$ and $1 - f_A(x_i) \geq \beta_0$.This proves the condition (ii).

Definition 3.8. Let (G,f) and (G',f) be a n-ary groups. A mapping $g : G \to G'$ is called a *n-ary homomorphism* if $g(f(x_1^n)) = f(g^n(x_1^n))$, where $g^n(x_1^n) = (g(x_1),...,g(x_n))$ for all $x_1^n \in G$.

For any vague set A in G', we define the *preimage* of A under g,denoted by $g^{-1}(A)$, is a vague set in G defined by $g^{-1}(t_A) = t_{A_{g^{-1}}}(x) = t_A(g(x))$ and

$$1 - g^{-1}(f_A) = 1 - f_{A_{g^{-1}}}(x) = 1 - f_A(g(x)), \forall x \in G.$$

For any vague set A in G, we define the *image* of A under g, denoted by $g(A)$,is a vague set in G' defined by

$$g\left(t_A\right)(y) = \begin{cases} \sup\limits_{x \in g^{-1}(y)} \quad t_A(x), & if \;\; g^{-1}(y) \neq \phi, \\ 0, & otherwise. \end{cases}$$

and

$$g\left(f_A\right)(y) = \begin{cases} \inf\limits_{x \in g^{-1}(y)} \quad f_A(x), & if \;\; g^{-1}(y) \neq \phi, \\ 0, & otherwise. \end{cases}$$

for all $x \in G$ and $y \in G'$.

Theorem 3.9. *Let g be a n-ary homomorphism mapping from G into G' with $g(\overline{x}) = g(x)$ for all $x \in G$ and A is a $T-$vague n-ary subgroup of (G', f).Then $g^{-1}(A)$ is a $T-$vague n-ary subgroup of (G, f).*

Proof. Let $x_1^n \in G$,we have

$$\begin{aligned} t_{A_{g^{-1}}}(f(x_1^n)) &= t_A(g(f(x_1^n)) = t_A(f(g^n(x_1^n))) \\ &\geq T_n\{t_A(g(x_1)), ..., t_A(g(x_n))\} \\ &= T_n\{t_{A_{g^{-1}}}(x_1), ..., t_{A_{g^{-1}}}(x_n)\}. \end{aligned}$$

and

$$\begin{aligned} 1 - f_{A_{g^{-1}}}(f(x_1^n)) &= 1 - f_A(g(f(x_1^n)) = 1 - f_A(f(g^n(x_1^n))) \\ &\geq T_n\{1 - f_A(g(x_1)), ..., 1 - f_A(g(x_n))\} \\ &= T_n\{1 - f_{A_{g^{-1}}}(x_1), ..., 1 - f_{A_{g^{-1}}}(x_n)\}. \end{aligned}$$

Also, for all $x \in G$ $t_{A_{g^{-1}}}(\overline{x}) = t_A(g(\overline{x})) \geq t_A(g(x)) = t_{A_{g^{-1}}}(x)$ and

$$1 - f_{A_{g^{-1}}}(\overline{x}) \;\; = \;\; 1 - f_A(g(\overline{x})) \geq 1 - f_A(g(x)) = 1 - f_{A_{g^{-1}}}(x).$$

This completes the proof.

If we strengthen the condition of g, then we can construct the converse of Theorem 3.9 as follows.

Theorem 3.10. *Let g be a n-ary homomorphism from G into G' and $g^{-1}(A)$ is a $T-$vague n-ary subgroup of (G, f).Then A is a $T-$vague n-ary subgroup of (G', f).*

Proof. For any $x_1, ..., x_n \in G'$,there exists $a_1, ..., a_n \in G$ such that $g(a_1) = x_1, ..., g(a_n) = x_n$. For any $f(x_1^n) \in (G', f)$, there exists $f(a_1^n) \in (G, f)$ such that $g(f(a_1^n)) = f(x_1^n)$. Then

$$\begin{aligned} t_A(f(x_1^n)) &= t_A(g(f(a_1^n)) = t_{A_{g^{-1}}}(f(a_1^n)) \\ &\geq T_n\{t_{A_{g^{-1}}}(a_1), t_{A_{g^{-1}}}(a_2), ..., t_{A_{g^{-1}}}(a_n)\} \\ &= T_n\{t_A(g(a_1), ..., t_A(g(a_n))\} \\ &= T_n\{t_A(x_1), ..., t_A(x_n)\}. \end{aligned}$$

and

$$\begin{aligned} 1 - f_A(f(x_1^n)) &= 1 - f_A(g(f(a_1^n))) = 1 - f_{A_{g^{-1}}}(f(a_1^n)) \\ &\geq T_n\{1 - f_{A_{g^{-1}}}(a_1), 1 - f_{A_{g^{-1}}}(a_2), ..., 1 - f_{A_{g^{-1}}}(a_n)\} \\ &= T_n\{1 - f_A(g(a_1), ..., 1 - f_A(g(a_n))\} \\ &= T_n\{1 - f_A(x_1), ..., 1 - f_A(x_n)\}. \end{aligned}$$

For any $\overline{x} \in G'$, there exists $\overline{a} \in G$ such that $g(\overline{a}) = \overline{x}$, we have

$$t_A(\overline{x}) = t_A(g(\overline{a})) = t_{A_{g^{-1}}}(\overline{a}) \geq t_{A_{g^{-1}}}(a) = t_A(a) = t_A(x).$$

and $1 - f_A(\overline{x}) = 1 - f_A(g(\overline{a})) = 1 - f_{A_{g^{-1}}}(\overline{a}) \geq 1 - f_{A_{g^{-1}}}(a) = 1 - f_A(a) = 1 - f_A(x)$. This completes the proof.

Theorem 3.11. *Let g be a mapping from G into G'. If A is a $T-$vague n-ary subgroup of (G, f) ,then $g(A)$ is a $T-$vague n-ary subgroup of (G', f).*

Proof. Let g be a mapping from G into G' and let $x_1^n \in G$, $y_1^n \in G'$.Noticing that $\{x_i(i = 1, 2, ..., n) | x_i \in g^{-1}(f(y_1^n))\} \subseteq \{f(x_1^n) \in G | x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), ..., x_n \in g^{-1}(y_n))\}$. we have

$$\begin{aligned} & g(t_A)(f(y_1^n)) \\ &= sup\{t_A(x_1^n) | x_i \in g^{-1}(f(y_1^n))\} \\ &\geq sup\{t_A(f(x_1^n) | x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), ..., x_n \in g^{-1}(y_n))\} \\ &\geq sup\{T_n\{t_A(x_1), t_A(x_2), ..., t_A(x_n)\} | x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), ..., \\ &\qquad x_n \in g^{-1}(y_n))\} \\ &= T_n\{sup\{t_A(x_1) | x_1 \in g^{-1}(y_1)\}, sup\{t_A(x_2) | x_1 \in g^{-1}(y_2)\}, ..., \\ &\qquad sup\{t_A(x_n) | x_1 \in g^{-1}(y_n)\}\} \\ &\geq T_n\{g(t_A)(y_1), g(t_A)(y_2), ..., g(t_A)(y_n)\}. \end{aligned}$$

and

$$\begin{aligned} & 1 - g(f_A)(f(y_1^n)) \\ &= sup\{1 - f_A(x_1^n) | x_i \in g^{-1}(f(y_1^n))\} \\ &\geq sup\{1 - f_A(f(x_1^n) | x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), ..., x_n \in g^{-1}(y_n))\} \\ &\geq sup\{T_n\{1 - f_A(x_1), 1 - f_A(x_2), ..., 1 - f_A(x_n)\} | x_1 \in g^{-1}(y_1), \\ &\qquad x_2 \in g^{-1}(y_2), ..., x_n \in g^{-1}(y_n))\} \\ &= T_n\{sup\{1 - f_A(x_1) | x_1 \in g^{-1}(y_1)\}, sup\{1 - f_A(x_2) | x_1 \in g^{-1}(y_2)\}, ..., \end{aligned}$$

$$sup\{1 - f_A(x_n)|x_1 \in g^{-1}(y_n)\}\}$$
$$\geq T_n\{1 - g(f_A)(y_1), 1 - g(f_A)(y_2), ..., 1 - g(f_A)(y_n)\}.$$

For all $x \in G$, we have

$$\begin{aligned} g(t_A)(\overline{x}) &= sup\{t_A(\overline{x})|\overline{x} \in g^{-1}(f(\overline{y}))\} \\ &\geq sup\{t_A(x)|x \in g^{-1}(f(y))\} \\ &= g(t_A)(x). \end{aligned}$$

and

$$\begin{aligned} 1 - g(f_A)(\overline{x}) &= sup\{1 - f_A(\overline{x})|\overline{x} \in g^{-1}(f(\overline{y}))\} \\ &\geq sup\{1 - f_A(x)|x \in g^{-1}(f(y))\} \\ &= 1 - g(f_A)(x). \end{aligned}$$

This completes the proof.

Corollary 3.12. *A vague set A defined on group $(G, .)$ is a $T-$vague subgroup if and only if*
(1) $V_A(xy) \succeq tmin\{V_A(x), V_A(y)\}$,
(2) $V_A(x) \succeq tmin\{V_A(y), V_A(xy)\}$,
(3) $V_A(y) \succeq tmin\{V_A(x), V_A(xy)\}$
holds for all $x, y \in G$.

Theorem 3.13. *Let A be a $T-$vague n-ary subgroup of (G, f) .If there exists an element $a \in G$ such that $V_A(a) \succeq V_A(x)$ for every $x \in G$, then A is a $T-$vague n-ary subgroup of a group $ret_a(G, f)$.*

Proof. For all $x, y, a \in G$ we have

$$\begin{aligned} t_A(x \circ y) &= t_A(f(x, \overset{(n-2)}{a}, y) \\ &\geq T_n\{t_A(x), t_A(a), t_A(y)\} \\ &= T_n\{t_A(x), t_A(y)\}. \end{aligned}$$

and

$$\begin{aligned} 1 - f_A(x \circ y) &= 1 - f_A(f(x, \overset{(n-2)}{a}, y) \\ &\geq T_n\{1 - f_A(x), 1 - f_A(a), 1 - f_A(y)\} \\ &= T_n\{1 - f_A(x), 1 - f_A(y)\}. \end{aligned}$$

For all $x, a \in G$, we have

$$\begin{aligned} t_A(x^{-1}) &= t_A(f(\overline{a}, \overset{(n-3)}{x} \overline{x}, \overline{a})) \\ &\geq T_n\{t_A(x), t_A(\overline{x}), t_A(a), t_A(\overline{a})\} \\ &= t_A(x). \end{aligned}$$

and

$$1 - f_A(x^{-1}) = 1 - f_A(f(\overline{a}, \overset{(n-3)}{x} \overline{x}, \overline{a}))$$

$$\begin{aligned} &\geq T_n\{1-f_A(x), 1-f_A(\overline{x}), 1-f_A(a), 1-f_A(\overline{a})\} \\ &= 1-f_A(x). \end{aligned}$$

which complete the proof.

In Theorem 3.13, the assumption that $V_A(a) \succeq V_A(x)$ cannot be omitted.

Examples 3.14. Let $(\mathbb{Z}_4, f)$ be a ternary group from Example 3.2.
Define a vague set A as follows:

$$A = \{\langle 0, [0.8, 0.02], \langle 1, [0.3, 0.05], \langle 2, [0.3, 0.05], \langle 3, [0.3, 0.05]\}.$$

Clearly A is a vague ternary subgroup of $(\mathbb{Z}_4, f)$.For $ret_1(\mathbb{Z}_4, f)$, we have

$t_A(0 \circ 0) = t_A((f(0,1,0)) = t_A(1) = 0.3 \not\geq 0.8 = t_A(0) = T_n\{t_A(0), t_A(0)\}$.

Hence the assumption $V_A(a) \succeq V_A(x)$ cannot be omitted.

Theorem 3.15. *Let (G, f) be a n-ary group. If A is a $T-$vague n-ary subgroup of a group $ret_a(G, f)$ and $V_A(a) \succeq V_A(x)$ for all $a, x \in G$,then A is a $T-$vague n-ary subgroup of (G, f).*

Proof. According to Theorem 2.1, any n-ary group can be represented of the form (2) ,where $(G, \circ) = ret_a(G, f), \varphi(x) = f(\overline{a}, x, \overset{(n-2)}{x})$ and $b = f(\overline{a}, ..., \overline{a})$.Then we have

$$\begin{aligned} t_A(\varphi(x)) &= t_A(f(\overline{a}, x, \overset{(n-2)}{x})) \\ &\geq T_n\{t_A(\overline{a}), t_A(x), t_A(a)\} \\ &= t_A(x). \end{aligned}$$

and

$$\begin{aligned} t_A(\varphi^2(x)) &= t_A(f(\overline{a}, \varphi(x), \overset{(n-2)}{x})) \\ &\geq T_n\{t_A(\overline{a}), t_A(\varphi(x)), t_A(a)\} \\ &= t_A(\varphi(x)) \\ &\geq t_A(x). \end{aligned}$$

Consequently, $t_A(\varphi^k(x)) \geq t_A(x)$ for all $x \in G$ and $k \in \mathbb{N}$.
Similarly, we have

$$\begin{aligned} 1-f_A(\varphi(x)) &= 1-f_A(f(\overline{a}, x, \overset{(n-2)}{x})) \\ &\geq T_n\{1-f_A(\overline{a}), 1-f_A(x), 1-f_A(a)\} \\ &= 1-f_A(x). \end{aligned}$$

and

$$\begin{aligned} 1-f_A(\varphi^2(x)) &= 1-f_A(f(\overline{a}, \varphi(x), \overset{(n-2)}{x})) \\ &\geq T_n\{1-f_A(\overline{a}), 1-f_A(\varphi(x)), 1-f_A(a)\} \\ &= 1-f_A(\varphi(x)) \\ &\geq 1-f_A(x). \end{aligned}$$

Consequently,$1-f_A(\varphi^k(x)) \geq 1-f_A(x)$ for all $x \in G$ and $k \in \mathbb{N}$. For all $x \in G$, we have $t_A(b) = t_A(f(\overline{a},...,\overline{a})) \geq t_A(\overline{a}) \geq t_A(x)$ and

$$1 - f_A(b) \;=\; 1 - f_A(f(\overline{a},...,\overline{a})) \geq 1 - f_A(\overline{a}) \geq 1 - f_A(x).$$

Thus

$$\begin{aligned} t_A(f(x_1^n)) &= t_A(x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ ... \circ \varphi^{n-2}(x_n) \circ b) \\ &\geq T_n\{t_A(x_1), t_A\varphi(x_2), t_A(\varphi^2(x_3)), ..., t_A(\varphi^{n-2}(x_n)), t_A(b)\} \\ &\geq T_n\{t_A(x_1), t_A(x_2), t_A(x_3), ..., t_A(x_n), t_A(b)\} \\ &\geq T_n\{t_A(x_1), t_A(x_2), t_A(x_3), ..., t_A(x_n)\}. \end{aligned}$$

and

$$\begin{aligned} 1 - f_A(f(x_1^n)) &= 1 - f_A(x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ ... \circ \varphi^{n-2}(x_n) \circ b) \\ &\geq T_n\{1 - f_A(x_1), 1 - f_A\varphi(x_2), 1 - f_A(\varphi^2(x_3)), ..., \\ &\qquad 1 - f_A(\varphi^{n-2}(x_n)), 1 - f_A(b)\} \\ &\geq T_n\{1 - f_A(x_1), 1 - f_A(x_2), 1 - f_A(x_3), ..., \\ &\qquad 1 - f_A(x_n), 1 - f_A(b)\} \\ &\geq T_n\{1 - f_A(x_1), 1 - f_A(x_2), 1 - f_A(x_3), ..., 1 - f_A(x_n)\}. \end{aligned}$$

From (4) and (7)of [2],we have

$$\overline{x} = \left(\varphi(x) \circ \varphi^2(x) \circ ... \circ \varphi^{n-2}(x) \circ b\right)^{-1}$$

Thus

$$\begin{aligned} t_A(\overline{x}) &= t_A\left(\left(\varphi(x) \circ \varphi^2(x) \circ ... \circ \varphi^{n-2}(x) \circ b\right)^{-1}\right) \\ &\geq t_A\left(\varphi(x) \circ \varphi^2(x) \circ ... \circ \varphi^{n-2}(x) \circ b\right) \\ &\geq T_n\{t_A(\varphi(x)), t_A(\varphi^2(x)), ..., t_A(\varphi^{n-2}(x)), t_A(b)\} \\ &\geq T_n\{t_A(x), t_A(b)\} = t_A(x). \end{aligned}$$

and

$$\begin{aligned} 1 - f_A(\overline{x}) &= 1 - f_A\left(\left(\varphi(x) \circ \varphi^2(x) \circ ... \circ \varphi^{n-2}(x) \circ b\right)^{-1}\right) \\ &\geq 1 - f_A\left(\varphi(x) \circ \varphi^2(x) \circ ... \circ \varphi^{n-2}(x) \circ b\right) \\ &\geq T_n\{1 - f_A(\varphi(x)), 1 - f_A(\varphi^2(x)), ..., 1 - f_A(\varphi^{n-2}(x)), 1 - f_A(b)\} \\ &\geq T_n\{1 - f_A(x), 1 - f_A(b)\} = 1 - f_A(x). \end{aligned}$$

This completes the proof.

Corollary 3.16. *If (G, f) is a ternary group,then any $T-$vague subgroup of $ret_a(G, f)$ is a $T-$vague ternary subgroup of (G, f).*

Proof. Since $\overline{a}$ is a neutral element of a group $ret_a(G, f)$ then $V_A(\overline{a}) \succeq V_A(x)$ for all $x \in G$.Thus $V_A(\overline{a}) \succeq V_A(a)$ .But in ternary group $\overline{\overline{a}} = a$ for any $a \in G$,whence $V_A(a) = V_A(\overline{\overline{a}}) \succeq V_A(\overline{a}) \succeq V_A(x)$ .So,$V_A(a) \succeq V_A(x)$ for all $x \in G$.This means that the assumption of Theorem 3.15 is satisfied.

Example 3.17. Consider the ternary group $(\mathbb{Z}_{12}, f)$,derived from the additive group $\mathbb{Z}_{12}$ Let A be a $T-$vague subgroup of the group of $ret_1(G, f)$ induced by subgroups $S_1 = \{11\}$,$S_2 = \{5, 11\}$ and $S_3 = \{1, 3, 5, 7, 9, 11\}$.
Define a vague set A as follows:

$$A(x) = \begin{cases} [0.8, 0.02] & if \;\; x = 11, \\ [0.6, 0.04] & if \;\; x = 5, \\ [0.4, 0.06] & if \;\; x = 1, 3, 7, 9, \\ [0.2, 0.08] & if \;\; x \notin S_3. \end{cases}$$

Then $t_A(\overline{5}) = t_A(7) = 0.4 \ngeq 0.6 = t_A(5)$. Hence A is not a $T-$vague ternary subgroup of $(\mathbb{Z}_{12}, f)$.

Observations. From the above Example 3.17 it follows that:

(1) There are $T-$vague subgroups of $ret_a(G, f)$ which are not $T-$vague n-ary subgroup of (G, f).

(2) In Theorem 3.15 the assumption $V_A(a) \succeq V_A(x)$ can not be omitted.In the above example we have $t_A(1) = 0.4 < 0.6 = t_A(5)$.

(3) The assumption $V_A(a) \succeq V_A(x)$ cannot be replaced by the natural assumption $V_A(\overline{a}) \succeq V_A(x)$. ($\overline{a}$ is the identity of $ret_a(G, f)$).

In the above example $\overline{1} = 11$, then $t_A(11) \geq t_A(x)$ and $1 - f_A(11) \geq 1 - f_A(x)$ for all $x \in \mathbb{Z}_{12}$.

Theorem 3.18. *Let (G, f) be a n-ary group of b-derived from the group $(G, \circ)$.Any vague set A of $(G, \circ)$ such that $V_A(b) \succeq V_A(x)$ for every $x \in G$ is a $T-$vague n-ary subgroup of (G, f).*

Proof. The condition (TVnS1) is obvious.To prove (TVnS2),we have n-ary group (G, f) b-derived from the group $(G, \circ)$,which implies $\overline{x} = (x^{n-2} \circ b)^{-1}$, where x^{n-2} is the power of x in $(G, \circ)$[4] .
Thus,for all $x \in G$

$$\begin{aligned} t_A(\overline{x}) &= t_A((x^{n-2} \circ b)^{-1}) \\ &\geq T_n\{t_A(x^{n-2}), t_A(b)\} \\ &= t_A(x). \end{aligned}$$

and

$$\begin{aligned} 1 - f_A(\overline{x}) &= 1 - f_A((x^{n-2} \circ b)^{-1}) \\ &\geq T_n\{1 - f_A(x^{n-2}), 1 - f_A(b)\} \\ &= 1 - f_A(x). \end{aligned}$$

This complete the proof.

Corollary 3.19. *Any $T-$vague group of a group $(G, \circ)$ is a $T-$vague n-ary subgroup of a n-ary group (G, f) derived from $(G, \circ)$.*

Proof. If n-ary group (G, f) is derived from the group $(G, \circ)$ then $b = e$.Thus $V_A(e) \succeq V_A(x)$ for all $x \in G$.

4. T_n-product of vague n-ary relations

Definition 4.1. A *vague n-ary relation* on any set G is a vague set

$$V : G^n = G \times G \times ... \times G \,(\text{n times}) \to [0, 1].$$

Definition 4.2. Let A be vague n-ary relation on any set G and B be a vague set on G.Then A is called *$T-$vague n-ary relation* on B if

$$V_A(x_1^n) \succeq tmax(V_B(x_1), V_B(x_2), ..., V_B(x_n)).$$

for all $x_1^n \in G$.

Definition 4.3. Let $A_1^n = A_1, A_2, ..., A_n$ be vague sets in G.Then *direct T_n-product* of A_1^n is defined by

$$(V_{A_1} \times V_{A_2} \times ... \times V_{A_n})(x_1^n) \approx tmax(V_{A_1}(x_1), V_{A_2}(x_2), ..., V_{A_n}(x_n)), \forall x_1^n \in G.$$

Lemma 4.4. *Let T_n be a function induced by t-norms and let A_1^n be vague sets in G. Then*
(*i*) $V_{A_1} \times V_{A_2} \times ... \times V_{A_n}$ is a $T-$vague n-ary relation on G,
(*ii*) $(A_1 \times A_2 \times ... \times A_n)_{(\alpha,\beta)} = (A_1)_{(\alpha,\beta)} \times (A_2)_{(\alpha,\beta)} \times ... \times (A_n)_{(\alpha,\beta)}$,
for all $t \in [0, 1]$.

Proof. The proof is obvious.

Proposition 4.5. *Let T_n be a function induced by t-norms and let $A_1, A_2, ..., A_n$ be $T-$vague n-ary subgroup of (G, f). Then,$A_1 \times A_2 \times ... \times A_n$ is a $T-$vague n-ary subgroup of (G^n, f).*

Proof. For $x_1^n \in G$ and $f(x_1^n) = (f_1(x_1^n), ..., f_n(x_1^n)) \in (G^n, f)$, we have

$$\begin{aligned}
&(t_{A_1} \times t_{A_2} \times, ..., \times t_{A_n})(f(x_1^n)) \\
&= (t_{A_1} \times t_{A_2} \times, ..., \times t_{A_n})(f_1(x_1^n), ..., f_n(x_1^n)) \\
&= T_n\{t_{A_1}(f_1(x_1^n)), t_{A_2}(f_2(x_1^n)) ..., t_{A_3}(f_n(x_1^n))\} \\
&\geq T_n\{T_n\{t_{A_1}(x_1), t_{A_1}(x_2), ..., t_{A_1}(x_n)\}, ..., \\
&\quad T_n\{t_{A_n}(x_1), t_{A_n}(x_2), ..., t_{A_n}(x_n)\}\} \\
&= T_n\{(t_{A_1} \times t_{A_2} \times ... \times t_{A_n})(x_1, ..., x_1), ..., \\
&\quad (t_{A_1} \times t_{A_2} \times ... \times t_{A_n})(x_n, ..., x_n)\} \\
&= T_n\{(t_{A_1} \times t_{A_2} \times ... \times t_{A_n})(x_1), ..., (t_{A_1} \times t_{A_2} \times ... \times t_{A_n})(x_n)\}.
\end{aligned}$$

and

$$\begin{aligned}
&1 - (f_{A_1} \times f_{A_2} \times, ..., \times f_{A_n})(f(x_1^n)) \\
&= (1 - f_{A_1} \times 1 - f_{A_2} \times, ..., \times 1 - f_{A_n})(f_1(x_1^n), ..., f_n(x_1^n))
\end{aligned}$$

$$= T_n\{1 - f_{A_1}(f_1(x_1^n)), 1 - f_{A_2}(f_2(x_1^n))..., 1 - f_{A_3}(f_n(x_1^n))\}$$
$$\geq T_n\{T_n\{1 - f_{A_1}(x_1), 1 - f_{A_1}(x_2), ..., 1 - f_{A_1}(x_n)\}, ...,$$
$$T_n\{1 - f_{A_n}(x_1), 1 - f_{A_n}(x_2), ..., 1 - f_{A_n}(x_n)\}\}$$
$$= T_n\{1 - (f_{A_1} \times f_{A_2} \times ... \times f_{A_n})(x_1, ..., x_1), ...,$$
$$1 - (f_{A_1} \times f_{A_2} \times ... \times f_{A_n})(x_n, ..., x_n)\}$$
$$= T_n\{1 - (f_{A_1} \times f_{A_2} \times ... \times f_{A_n})(x_1), ..., 1 - (f_{A_1} \times f_{A_2} \times ... \times f_{A_n})(x_n)\}.$$

For all $x = x_1^n, \overline{x} = \overline{x}_1^n \in G^n$ and , we have

$$\begin{aligned} (t_{A_1} \times t_{A_2} \times ... \times t_{A_n})(\overline{x}) &= (t_{A_1} \times t_{A_2} \times ... \times t_{A_n})(\overline{x}_1, ..., \overline{x}_n) \\ &= T_n\{t_{A_1}(\overline{x}_1), ..., t_{A_n}(\overline{x}_n)\} \\ &\geq T_n\{t_{A_1}(x_1), ..., t_{A_n}(x_n)\} \\ &= (t_{A_1} \times t_{A_2} \times ... \times t_{A_n})(x_1^n) \\ &= (t_{A_1} \times t_{A_2} \times ... \times t_{A_n})(x). \end{aligned}$$

This completes the proof.

The following corollary is the immediate consequence of Proposition 4.6.

Corollary 4.6. *Let T_n be a function induced by t-norms and let $\prod_{i=1}^{n} (G_i, f)$ be the finite collection of n-ary subgroups and $G = \prod_{i=1}^{n} G_i$ the T_n-product of G_i. Let A_i be a $T-$vague n-ary subgroup of (G_i, f), where $1 \geq i \geq n$. Then,$A = \prod_{i=1}^{n} A_i$ defined by*

$$V_A(x_1^n) = \prod_{i=1}^{n} V_{A_i}(x_1^n) \approx tmax(V_A(x_1), V_A(x_2), ..., V_A(x_n)).$$

Then A is a $T-$vague n-ary subgroup of (G, f).

Definition 4.7. Let A_1^n be vague sets in G.Then, the *T_n-product* of A_1^n,written as

$$A_1^n(x) = \langle x, [t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n}(x), 1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n}(x)\rangle$$

is defined by:

$$[t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n}(x) = T_n(t_{A_1}(x), t_{A_2}(x), ..., t_{A_n}(x))$$

and

$$1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n}(x) = T_n(1 - f_{A_1}(x), 1 - f_{A_2}(x), ..., 1 - f_{A_n}(x)),$$

for all $x \in G$, respectively.

Theorem 4.8. *Let A_1^n be $T-$vague n-ary subgroups of (G, f) .If T_n^* is a function induced by t-norms dominates T_n,that is,*

$$T_n^*(T_n(x_1^n), T_n(y_1^n), ..., T_n(z_1^n)) \geq T_n(T_n^*(x_1, y_1, ..., z_1), ..., T_n^*(x_n, y_n, ..., z_n))$$

for all $x_1^n, y_1^n, ..., z_1^n \in [0,1]$. *Then* T_n^**-product of* A_1^n *is a* $T-$*vague* n*-ary subgroup of* (G, f).

Proof. Let $x_1^n \in G$, we have

$$\begin{aligned}
&[t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n^*}(f(x_1^n)) \\
&= T_n^*(t_{A_1}(f(x_1^n)), t_{A_2}(f(x_1^n)), ..., t_{A_n}(f(x_1^n))) \\
&\geq T_n^*(T_n(t_{A_1}(x_1), t_{A_1}(x_2), ..., t_{A_1}(x_n)), ..., T_n(t_{A_n}(x_1), t_{A_n}(x_2), ..., t_{A_n}(x_n))) \\
&\geq T_n(T_n^*(t_{A_1}(x_1), t_{A_2}(x_1), ..., t_{A_n}(x_1)), ..., T_n^*(t_{A_1}(x_n), t_{A_2}(x_n), ..., t_{A_n}(x_n))) \\
&= T_n([t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n^*}(x_1), ..., [t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n^*}(x_n)).
\end{aligned}$$

and

$$\begin{aligned}
&1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n^*}(f(x_1^n)) \\
&= T_n^*(1 - f_{A_1}(f(x_1^n)), 1 - f_{A_2}(f(x_1^n)), ..., 1 - f_{A_n}(f(x_1^n))) \\
&\geq T_n^*(T_n(1 - f_{A_1}(x_1), 1 - f_{A_1}(x_2), ..., 1 - f_{A_1}(x_n)), ..., \\
&\qquad\qquad T_n(1 - f_{A_n}(x_1), 1 - f_{A_n}(x_2), ..., 1 - f_{A_n}(x_n)) \\
&\geq T_n(T_n^*(1 - f_{A_1}(x_1), 1 - f_{A_2}(x_1), ..., 1 - f_{A_n}(x_1)), ..., \\
&\qquad\qquad T_n^*(1 - f_{A_1}(x_n), 1 - f_{A_2}(x_n), ..., 1 - f_{A_n}(x_n)) \\
&= T_n(1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n^*}(x_1), ..., 1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n^*}(x_n)).
\end{aligned}$$

For all $x \in G$, we have

$$\begin{aligned}
[t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n^*}(\overline{x}) &= T_n^*(t_{A_1}(\overline{x}), t_{A_2}(\overline{x}), ..., t_{A_n}(\overline{x})) \\
&\geq T_n^*(t_{A_1}(x), t_{A_2}(x), ..., t_{A_n}(x)) \\
&= [t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n^*}(x).
\end{aligned}$$

and

$$\begin{aligned}
1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n^*}(\overline{x}) &= T_n^*(1 - f_{A_1}(\overline{x}), 1 - f_{A_2}(\overline{x}), ..., 1 - f_{A_n}(\overline{x})) \\
&\geq T_n^*(1 - f_{A_1}(x), 1 - f_{A_2}(x), ..., 1 - f_{A_n}(x)) \\
&= 1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n^*}(x).
\end{aligned}$$

This completes the proof .

Let (G, f) and (G', f) be an n-ary groups.A mapping $g : G \to G'$ is an onto homomorphism.Let T_n and T_n^* be functions induced by t-norms such that T_n^* dominates T_n. If A_1^n are $T-$vague n-ary subgroup of (G, f), then the T_n^*-product of A_1^n is a $T-$vague n-ary subgroup. Since every onto homomorphic inverse image of a $T-$vague n-ary subgroup, the inverse images

$$g^{-1}(A_1), g^{-1}(A_2), ..., g^{-1}(A_n) \tag{4}$$

and

$$\langle g^{-1}\left([t_{A_1} \cdot t_{A_1} \cdot ... \cdot t_{A_1}]_{T_n^*}\right), \left(1 - [f_{A_1} \cdot f_{A_1} \cdot ... \cdot f_{A_1}]_{T_n^*}\right)\rangle \tag{5}$$

are $T-$vague n-ary subgroup (G, f).

The following theorem provides the relation between (4) and (5).

Theorem 4.9. *Let $g : G \rightarrow G'$be an onto n-ary homomorphism of n-ary groups. Let T_n^* be a function induced by t-norm such that T_n^* dominates T_n Let A_1^n be T−vague n-ary subgroup of (G, f). If $\langle [t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n^*}, 1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n^*}\rangle$ is a T_n^*- product of A_1^n, and $\langle [g^{-1}(t_{A_1}) \cdot g^{-1}(t_{A_2}) \cdot ... \cdot g^{-1}(t_{A_n})]_{T_n^*}, 1 - [g^{-1}(f_{A_1}) \cdot g^{-1}(f_{A_2}) \cdot ... \cdot g^{-1}(f_{A_n})]_{T_n^*}\rangle$ is the T_n^*-product of $g^{-1}(\mu_1), g^{-1}(\mu_2), ...g^{-1}(\mu_n)$. then*

$$\begin{aligned}&\langle g^{-1}([t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n^*}), g^{-1}(1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n^*})\rangle \\ &= \langle [g^{-1}(t_{A_1}) \cdot g^{-1}(t_{A_2}) \cdot ... \cdot g^{-1}(t_{A_n})]_{T_n^*}, 1 - [g^{-1}(f_{A_1}) \\ &\quad \cdot g^{-1}(f_{A_1}) \cdot ... \cdot g^{-1}(f_{A_1})]_{T_n^*}\rangle.\end{aligned}$$

Proof. Let $x \in G$,we have

$$\begin{aligned}g^{-1}([t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n^*})(x) &= ([t_{A_1} \cdot t_{A_2} \cdot ... \cdot t_{A_n}]_{T_n^*})(g(x)) \\ &= T_n^*(t_{A_1}(g(x)), t_{A_2}(g(x)), ... \cdot t_{A_n}(g(x))) \\ &= T_n^*(g^{-1}(t_{A_1})(x), g^{-1}(t_{A_2})(x), ..., g^{-1}(t_{A_n})(x)) \\ &= [g^{-1}(t_{A_1}) \cdot g^{-1}(t_{A_2}) \cdot ... \cdot g^{-1}(t_{A_n})]_{T_n^*}.\end{aligned}$$

and

$$\begin{aligned}1 - g^{-1}([f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n^*})(x) &= (1 - [f_{A_1} \cdot f_{A_2} \cdot ... \cdot f_{A_n}]_{T_n^*})(g(x)) \\ &= T_n^*(1 - f_{A_1}(g(x)), 1 - f_{A_2}(g(x)), ..., \\ &\qquad 1 - f_{A_n}(g(x))) \\ &= T_n^*(1 - g^{-1}(f_{A_1})(x), 1 - g^{-1}(f_{A_2})(x), \\ &\qquad ..., 1 - g^{-1}(f_{A_n})(x)) \\ &= 1 - [g^{-1}(f_{A_1}) \cdot g^{-1}(f_{A_2}) \cdot ... \cdot g^{-1}(f_{A_n})]_{T_n^*}.\end{aligned}$$

This completes the proof.

5. Conclusions

The n-ary group theory has many application in an automata theory ,quantum theory and computer sciences problems.In this paper,we have defined T-vague n-ary subgroups and have studied some of their properties.If the unknown or undecided part $[1 - t_A(x) - f_A(x)]$ is zero for all x(of the group G),then the Biswas's vague group[27] is reduces to a Rosenfeld's fuzzy group [31].It is also justified that interval-valued fuzzy sets [33] are not vague sets.

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