J. Appl. Math. & Informatics Vol. 28(2010), No. 1 - 2, pp. 163 - 173
Website: http://www.kcam.biz

EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR A CLASS OF SEMIPOSITONE QUASILINEAR ELLIPTIC SYSTEMS WITH DIRICHLET BOUNDARY VALUE PROBLEMS†

ZHOUJIN CUI, ZUODONG YANG* AND RUI ZHANG

ABSTRACT. We consider the system

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega, \\ -\Delta_q v = \mu g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\Delta_q v = \operatorname{div}(|\nabla v|^{q-2}\nabla v)$, $p, q \geq 2$, Ω is a ball in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$, $N \geq 1$, λ, μ are positive parameters, and f, g are smooth functions that are negative at the origin and $f(x) \sim x^m$ and $g(x) \sim x^n$ for x large for some $m, n \geq 0$ with $mn < (p-1)(q-1)$. We establish the existence and uniqueness of positive radial solutions when the parameters λ and μ are large.

AMS Mathematics Subject Classification : 35J65, 35B30.
Key word and phrases : Quasilinear elliptic systems, uniqueness, semipositone.

1. Introduction

In this paper, we consider the following boundary value problem

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega, \\ -\Delta_q v = \mu g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is the open unit ball in $\mathbf{R}^N$; $N \geq 1$, with a smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\Delta_q v = \operatorname{div}(|\nabla v|^{q-2}\nabla v)$, $p, q \geq 2$, λ, μ are positive parameters and f, g satisfy the following assumptions:

Received June 9, 2009. Revised June 13, 2009. Accepted July 12, 2009. *Corresponding author.
†Project Supported by the National Natural Science Foundation of China(Grant No.10871060). Project Supported by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No.08KJB110005).

(H1) $f, g : [0, \infty) \to R$ are continuous, nondecreasing C^1 functions such that $f(0) < 0$ and $g(0) < 0$ (semipositone system), and

$$\lim_{x \to \infty} f(x) > 0, \quad \lim_{x \to \infty} g(x) > 0.$$

(H2) There exist positive numbers A, B and nonnegative numbers m, n with $mn < (p-1)(q-1)$ such that

$$\lim_{x \to \infty} \frac{f(x)}{x^m} = A, \quad \lim_{x \to \infty} \frac{g(x)}{x^n} = B.$$

(H3) For $m_1 > m, n_1 > n$, $\dfrac{f(x)}{x^{m_1}}$ and $\dfrac{g(x)}{x^{n_1}}$ are nonincreasing for x large.

Equations of the above form are mathematical models occurring in studies of the p-Laplace equation, generalized reaction- diffusion theory [2], non-Newtonian fluid theory [3,4], non-Newtonian filtration theory [5,24] and the turbulent flow of a gas in porous medium [6]. In the non-Newtonian fluid theory, the pair (p, q) is a characteristic quantity of the medium. Media with $(p, q) > (2, 2)$ are called dilatant fluids and those with $(p, q) < (2, 2)$ are called pseudoplastics. If $(p, q) = (2, 2)$, they are Newtonian fluids.

In recent years, the existence and uniqueness of the positive solutions for the quasilinear eigenvalue problems:

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(u), \quad x \in \Omega \tag{1.2}$$

$$u(x) = 0, \quad x \in \partial\Omega, \tag{1.3}$$

with $\lambda > 0, p > 1, \Omega \in \mathbf{R}^N, N \geq 2$ have been considered by a number of authors, see [7,8,9,10,11] and the references therein. In [9], Guo and Webb proved existence and uniqueness results of (1.2) for λ large when $f \geq 0$, $(f(x)/x^{p-1})' < 0$ for $x > 0$ and f satisfies some p-sublinearity conditions at 0 and ∞, generalizing a result in [11] where Ω is a ball. Uniqueness results for semilinear equations ($p = 2$) were obtained in [12,13] where the assumption $(f(x)/x)' < 0$ is required only for large x. Similar results for systems were discussed in [15]. Related results for the superlinear case when $f \geq 0$ can be found in [7,16]. The case when $f(0) < 0$ and $p = 2$ was treated in [23], in which uniqueness of positive solution to single equation of (1.1) for λ large was established for sublinear f. See also [17] where this result was extended to the case when Ω is any bounded domain with convex outer boundary.

In [14], D.D. Hai and R. Shivaji considered the following elliptic systems

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega, \\ -\Delta_q v = \lambda g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

in which λ is a positive parameter, and Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. They prove the existence of a large positive solution for λ large. In particular, they do not assume any sign conditions on $f(0)$ or $g(0)$.

In [1], D.D. Hai and R. Shivaji considered the semipositone elliptic systems of (1.1) when $p, q = 2$. They established the uniqueness of radial positive solutions when the parameters λ and μ are large. In this paper, we extend the result in [1] mentioned above for system (1.1).

Since we look for radial solutions of (1.1), we shall consider the corresponding ODE system

$$\begin{cases} -(r^{N-1}\phi_p(u'))' = \lambda r^{N-1} f(v), \\ -(r^{N-1}\phi_q(v'))' = \mu r^{N-1} g(u), \\ u'(0) = v'(0) = 0, \quad u(1) = v(1) = 0, \end{cases} \tag{1.4}$$

where $\phi_p(s) = |s|^{p-2}s$.

By a modification of the method given in [1,15], we obtain the following results.

Theorem 1.1. *Assume $(H_1) - (H_3)$ hold. Then there exists a unique positive radial solution of problem (1.1) if λ and μ are sufficiently large.*

Theorem 1.2. *Assume (H_1) and (H_2) and let (u, v) be a positive radial solution of (1.4). Then there exist positive constants $M_i, 1 \le i \le 4$, and M independent of u, v such that*

$$M_1(\lambda^{q-1}\mu^m)^{\frac{1}{(p-1)(q-1)-mn}}(1-r) \le u(r) \le M_2(\lambda^{q-1}\mu^m)^{\frac{1}{(p-1)(q-1)-mn}}(1-r),$$
$$M_3(\lambda^n\mu^{p-1})^{\frac{1}{(p-1)(q-1)-mn}}(1-r) \le v(r) \le M_4(\lambda^n\mu^{p-1})^{\frac{1}{(p-1)(q-1)-mn}}(1-r),$$

for $0 < r < 1$, λ, $\mu \ge M$.

This paper is organized as follows. Section 2 deals with the proof of Theorems 1.2. We will prove Theorem 1.1 in Section 3.

2. Proof of Theorem 1.2

Here we will consider the case when $m > 0$ and $n > 0$. When $m = 0$ or $n = 0$ the proof follows by similar arguments.

Let (u, v) be a positive solution of (1.4). Let β_1, θ_1 be the positive zeros of g and G respectively, and β_2, θ_2 be the positive zeros of f and F respectively where $F(x) = \int_0^x f(s)ds$ and $G(x) = \int_0^x g(s)ds$. Let $\rho_1 \in (\beta_1, \theta_1)$, $\rho_2 \in (\beta_2, \theta_2)$, $r_1 \in (0, \beta_1)$ and $r_2 \in (0, \beta_2)$ be fixed.

We will first establish the following claim.

Lemma 2.1. *There exists a constant $m > 0$ such that $u(\frac{1}{2}) \ge \rho_1$ or $v(\frac{1}{2}) \ge \rho_2$ if λ, $\mu \ge m$.*

Proof. Assume not. Then $u(\frac{1}{2}) < \rho_1$ and $v(\frac{1}{2}) < \rho_2$ for λ large. Let $c_1 := \sup_{[0,r_1]} g(x)$. Clearly $c_1 < 0$. Suppose that $u(\frac{3}{4}) < r_1$. Then

$$-(r^{N-1}\phi_q(v'))' = \mu r^{N-1} g(u(r)) \le \mu c_1 r^{N-1} \quad \text{on } [3/4, 1],$$

which implies

$$-\phi_q(v') \geq -\frac{\mu c_1}{r^{N-1}} \int_r^1 s^{N-1} ds \geq \frac{\mu c_1}{N}(1-r), \quad \text{for } r \in [3/4, 1],$$

and so

$$-v'(r) \geq \phi_q^{-1}\{\frac{\mu c_1}{N}(1-r)\} \quad \text{on} \quad [3/4, 1],$$

follows. Integrating this inequality gives

$$\rho_2 > v(3/4) \geq (-\frac{\mu c_1}{N})^{\frac{1}{q-1}} \frac{q-1}{q} (\frac{1}{4})^{\frac{q}{q-1}},$$

a contradiction for $\mu > \frac{N\rho_2^{q-1}4^q}{-c_1}(\frac{q-1}{q})^{1-q}$. Similarly if $\lambda > \frac{N\rho_1^{p-1}4^p}{-c_2}(\frac{p-1}{p})^{1-p}$ where $c_2 := \sup_{[0,r_2]} f(x)$ then $v(\frac{3}{4}) \geq r_2$. Thus there exists a constant $\widetilde{m} > 0$ such that if $\lambda, \mu \geq \widetilde{m}$ then

$$r_1 < u(\frac{3}{4}) \leq u(\frac{1}{2}) < \rho_1 \tag{2.1}$$

and

$$r_2 < v(\frac{3}{4}) \leq v(\frac{1}{2}) < \rho_2. \tag{2.2}$$

Let $H(r) = \phi_p(u')\phi_q(v') + \lambda|v'|^{q-2}F(v(r)) + \mu|u'|^{p-2}G(u(r))$. Then $H(1) = \phi_p(u'(1))\phi_q(v'(1)) \geq 0$ and

$$\begin{aligned}\frac{dH}{dr} &= (\phi_p(u'))'\phi_q(v') + \phi_p(u')(\phi_q(v'))' + \lambda|v'|^{q-2}f(v)v' + \mu|u'|^{p-2}g(u)u' \\ &\quad + \lambda(|v'|^{q-2})'F(v(r)) + \mu(|u'|^{p-2})'G(u(r)) \\ &= -\frac{2(N-1)}{r}|u'|^{p-2}|v'|^{q-2}u'v' - \lambda(q-2)|v'|^{q-3}F(v)v'' \\ &\quad - \mu(p-2)|u'|^{p-3}G(u)u'' \leq -\frac{2(N-1)}{r}|u'|^{p-2}|v'|^{q-2}u'v' \leq 0,\end{aligned}$$

Thus $H(r) \geq 0$. Setting

$$K_1 = \sup_{[r_1,\rho_1]} G(x), \quad K_2 = \inf_{[r_1,\rho_1]} G(x), \quad K_3 = \sup_{[r_2,\rho_2]} F(x), \quad K_4 = \inf_{[r_2,\rho_2]} F(x).$$

Note that $K_i < 0$, $(1 \leq i \leq 4)$. Hence

$$\phi_p(u')\phi_q(v') + \lambda|v'|^{q-2}F(v(r)) \geq 0, \tag{2.3}$$

$$\phi_p(u')\phi_q(v') + \mu|u'|^{p-2}G(u(r)) \geq 0. \tag{2.4}$$

By (2.3) and (2.4), we have

$$|v'| \geq [-\mu K_2(-\lambda K_3)^{\frac{1}{1-p}}]^{\frac{p-1}{(p-1)(q-1)-1}}. \tag{2.5}$$

Integrating (2.5) from 1/2 to 3/4, we obtain

$$-v(3/4) + v(1/2) \geq 1/4[-\mu K_2(-\lambda K_3)^{\frac{1}{1-p}}]^{\frac{p-1}{(p-1)(q-1)-1}}. \tag{2.6}$$

By (2.2), the L.H.S. of (2.6) is bounded if $\lambda, \mu \geq \widetilde{m}$ while the R.H.S. of (2.6)$\to \infty$ as λ or $\mu \to \infty$. Hence Lemma 2.1 is proven.

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let λ, $\mu \geq m$ and suppose $u(\frac{1}{2}) \geq \rho_1$. Then

$$v'(r) = -\phi_q^{-1}(\frac{\mu}{r^{N-1}} \int_0^r s^{N-1} g(u) ds) \leq -(\frac{\mu\delta r}{N})^{\frac{1}{q-1}} \quad \text{for } r \leq 1/2, \tag{2.7}$$

in which $\delta := g(\rho_1) > 0$ and

$$\phi_q^{-1}(u) = \begin{cases} u^{\frac{1}{q-1}}, & \text{if } u \geq 0, \\ -(-u)^{\frac{1}{q-1}}, & \text{if } u < 0. \end{cases}$$

Integrating (2.7) from $\frac{1}{4}$ to $\frac{1}{2}$ we obtain

$$v(\frac{1}{2}) - v(\frac{1}{4}) \leq -(\frac{\mu\delta}{N})^{\frac{1}{q-1}} \frac{q-1}{q}[(\frac{1}{2})^{\frac{q}{q-1}} - (\frac{1}{4})^{\frac{q}{q-1}}].$$

Hence

$$v(\frac{1}{4}) \geq c_3 \mu^{\frac{1}{q-1}}, \tag{2.8}$$

where $c_3 := (\frac{\delta}{N})^{\frac{1}{q-1}} \frac{q-1}{q}[(\frac{1}{2})^{\frac{q}{q-1}} - (\frac{1}{4})^{\frac{q}{q-1}}] > 0$. Now

$$u(r) = \int_r^1 \phi_p^{-1}[\lambda s^{1-N} \int_0^s t^{N-1} f(v) dt] ds.$$

Thus

$$\begin{aligned} u(\frac{1}{4}) &= \lambda^{\frac{1}{p-1}} \int_{\frac{1}{4}}^1 \phi_p^{-1}[s^{1-N}(\int_0^{\frac{1}{4}} t^{N-1} f(v) dt + \int_{\frac{1}{4}}^s t^{N-1} f(v) dt)] ds \\ &\geq \lambda^{\frac{1}{p-1}} \int_{\frac{1}{4}}^1 [s^{1-N}(\int_0^{\frac{1}{4}} t^{N-1} dt f(v(\frac{1}{4})) + f(0)]^{\frac{1}{p-1}} ds \\ &= \lambda^{\frac{1}{p-1}} [\alpha_1 f(v(\frac{1}{4})) + f(0)]^{\frac{1}{p-1}} \alpha_2^{\frac{1}{p-1}}, \end{aligned} \tag{2.9}$$

where $\alpha_1 = \int_0^{\frac{1}{4}} t^{N-1} dt > 0$, $\alpha_2 = \int_{\frac{1}{4}}^1 s^{1-N} ds > 0$. Now $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} \frac{f(x)}{x^m} = A > 0$. Hence there exists a constant $\overline{m} > 0$ such that if $\mu \geq \overline{m}$ then

$$[\alpha_1 f(v(\frac{1}{4})) + f(0)]^{\frac{1}{p-1}} \geq [\frac{\alpha_1}{2} f(v(\frac{1}{4}))]^{\frac{1}{p-1}} \geq (\frac{\alpha_1 A}{4})^{\frac{1}{p-1}} [v(\frac{1}{4})]^{\frac{m}{p-1}}.$$

Then (2.9) implies

$$u(\frac{1}{4}) \geq (\frac{\lambda \alpha_1 \alpha_2 A}{4})^{\frac{1}{p-1}} [v(\frac{1}{4})]^{\frac{m}{p-1}}. \tag{2.10}$$

In particular, by (2.8) we can assume w.l.o.g that $v(\frac{1}{4}) \geq 1$ and hence

$$u(\frac{1}{4}) \geq c_4 \lambda^{\frac{1}{p-1}}, \tag{2.11}$$

where $c_4 := (\frac{\alpha_1\alpha_2 A}{4})^{\frac{1}{p-1}}$. Now using $v(\frac{1}{4}) = \int_{\frac{1}{4}}^1 \phi_q^{-1}[\mu s^{1-N}\int_0^s t^{N-1}g(u)dt]ds$ and (2.11), by similar arguments there exists $m^* > 0$ such that if $\lambda \geq m^*$, then

$$v(\frac{1}{4}) \geq (\frac{\mu\alpha_1\alpha_2 B}{4})^{\frac{1}{q-1}}[u(\frac{1}{4})]^{\frac{n}{q-1}}. \tag{2.12}$$

Combining (2.10) and (2.12) we have

$$u(\frac{1}{4}) \geq (\frac{\lambda\alpha_1\alpha_2 A}{4})^{\frac{1}{p-1}}\{(\frac{\mu\alpha_1\alpha_2 B}{4})^{\frac{1}{q-1}}[u(\frac{1}{4})]^{\frac{n}{q-1}}\}^{\frac{m}{p-1}},$$

which implies there exists a constant $\widetilde{M_3} > 0$ such that

$$u(\frac{1}{4}) \geq \widetilde{M_3}(\lambda\mu^{\frac{m}{q-1}})^{\frac{q-1}{(p-1)(q-1)-mn}}. \tag{2.13}$$

Similarly, there exists a constant $\widetilde{M_1} > 0$ such that

$$v(\frac{1}{4}) \geq \widetilde{M_1}(\mu\lambda^{\frac{n}{p-1}})^{\frac{p-1}{(p-1)(q-1)-mn}}. \tag{2.14}$$

Now for $r > \frac{1}{4}$,

$$\begin{aligned} -v'(r) &= \phi_q^{-1}[\mu r^{1-N}(\int_0^{\frac{1}{4}} t^{N-1}g(u)dt + \int_{\frac{1}{4}}^r t^{N-1}g(u)dt)] \\ &\geq (\mu r^{1-N})^{\frac{1}{q-1}}\phi_q^{-1}[(\int_0^{\frac{1}{4}} t^{N-1}dt)g(u(\frac{1}{4})) + g(0)]. \end{aligned}$$

Again by (2.11) and using arguments like those used before there exists a constant m^{**} such that if $\lambda \geq m^{**}$ then

$$-v'(r) \geq (\frac{\mu\alpha_1 B}{4})^{\frac{1}{q-1}}[u(\frac{1}{4})]^{\frac{n}{q-1}}. \tag{2.15}$$

Combining (2.13) and (2.15), there exists a constant M_3^* such that

$$-v'(r) \geq M_3^*(\mu^{p-1}\lambda^n)^{\frac{1}{(p-1)(q-1)-mn}} \quad \text{for} \quad r \in (\frac{1}{4}, 1). \tag{2.16}$$

Integrating (2.16) from r to 1 we obtain

$$v(r) \geq M_3^*(\mu^{p-1}\lambda^n)^{\frac{1}{(p-1)(q-1)-mn}}(1-r) \quad \text{for} \quad r \in (\frac{1}{4}, 1). \tag{2.17}$$

Now for $r \leq \frac{1}{4}$,

$$v(r) \geq v(\frac{1}{4}) \geq \widetilde{M_1}(\mu\lambda^{\frac{n}{p-1}})^{\frac{p-1}{(p-1)(q-1)-mn}} \geq \widetilde{M_1}(\mu\lambda^{\frac{n}{p-1}})^{\frac{p-1}{(p-1)(q-1)-mn}}(1-r).$$

Define $M_3 = \{\widetilde{M_1}, M_3^*\}$, we have

$$v(r) \geq M_3(\mu^{p-1}\lambda^n)^{\frac{1}{(p-1)(q-1)-mn}}(1-r) \quad \text{for} \quad r \in (0, 1). \tag{2.18}$$

Similarly, there exists a constant $M_1 > 0$ such that

$$u(r) \geq M_1(\lambda^{q-1}\mu^m)^{\frac{1}{(p-1)(q-1)-mn}}(1-r) \quad \text{for} \quad r \in (0, 1). \tag{2.19}$$

In particular, the L.H.S. inequalities of Theorem 1.2 hold provided λ, μ are bigger than some constant $\widetilde{M} > 0$. Next since

$$u(r) = \int_r^1 \phi_p^{-1}[\lambda s^{1-N} \int_0^s t^{N-1} f(v) dt] ds, \qquad \text{and}$$
$$v(r) = \int_r^1 \phi_q^{-1}[\lambda s^{1-N} \int_0^s t^{N-1} g(u) dt] ds$$

we obtain

$$\|u\|_\infty \leq \lambda^{\frac{1}{p-1}} f(\|v\|_\infty)^{\frac{1}{p-1}} \quad \text{and} \quad \|v\|_\infty \leq \mu^{\frac{1}{q-1}} g(\|u\|_\infty)^{\frac{1}{q-1}}.$$

Now by (2.14) w.l.o.g. we can assume that $\|v\|_\infty$ is large. Hence

$$\begin{aligned} \|v\|_\infty &\leq \mu^{\frac{1}{q-1}} g(\|u\|_\infty)^{\frac{1}{q-1}} \\ &\leq \mu^{\frac{1}{q-1}} g(\lambda^{\frac{1}{p-1}} f(\|v\|_\infty)^{\frac{1}{p-1}})^{\frac{1}{q-1}} \\ &\leq (\mu 2B)^{\frac{1}{q-1}} (\lambda^{\frac{1}{p-1}} f(\|v\|_\infty)^{\frac{1}{p-1}})^{\frac{n}{q-1}} \\ &\leq (2B)^{\frac{1}{q-1}} (2A)^{\frac{mn}{q-1}} \mu^{\frac{1}{q-1}} \lambda^{\frac{n}{(p-1)(q-1)}} \|v\|_\infty^{\frac{mn}{q-1}}. \end{aligned}$$

In particular, there exists a constant $\widetilde{M_2} > 0$ such that

$$\|v\|_\infty \leq \widetilde{M_2} (\mu^{\frac{1}{q-1}} \lambda^{\frac{n}{(p-1)(q-1)}})^{\frac{q-1}{q-1-mn}}. \tag{2.20}$$

and

$$\begin{aligned} -u'(r) &= \phi_p^{-1}(\frac{\mu}{r^{N-1}} \int_0^r s^{N-1} f(v) ds) \\ &\leq \lambda^{\frac{1}{p-1}} f(\|v\|_\infty)^{\frac{1}{p-1}} \leq (2\lambda A)^{\frac{1}{p-1}} \|v\|_\infty^{\frac{m}{p-1}} \\ &\leq (2\lambda A \widetilde{M}_2^m)^{\frac{1}{p-1}} (\lambda^{\frac{n}{p-1}} \mu)^{\frac{mn}{(p-1)(q-1)-mn}}. \end{aligned}$$

Thus there exists a constant $M_2 > 0$ such that

$$-u'(r) \leq M_2 (\lambda^{q-1} \mu^m)^{\frac{1}{(p-1)(q-1)-mn}} \qquad \text{for } r \in (0,1),$$

and hence integrating from r to 1 we obtain

$$u(r) \leq M_2 (\lambda^{q-1} \mu^m)^{\frac{1}{(p-1)(q-1)-mn}} (1-r) \quad \text{for } r \in (0,1). \tag{2.21}$$

Similarly there exists a constant $M_4 > 0$ such that

$$v(r) \leq M_4 (\lambda^n \mu^{p-1})^{\frac{1}{(p-1)(q-1)-mn}} (1-r) \quad \text{for } r \in (0,1). \tag{2.22}$$

and the R.H.S. inequalities of Theorem 1.2 are established provided λ, μ are bigger than some constant $M^* > 0$. Similarly the result can be established in the case $v(\frac{1}{2}) \geq \rho_2$. Hence Theorem 1.2 is proven.

3. Existence and uniqueness

Let $C[0,1]\times C[0,1]$ be equipped with the norm $|(u,v)|_0=\max(|u|_0,|v|_0)$, and let $K=\{(u,v)\in C[0,1]\times C[0,1]: u\geq 0, v\geq 0\}$. For each $(u,v)\in K$, define

$$T(u,v)=(\int_r^1 \phi_p^{-1}[\lambda s^{1-N}\int_0^s t^{N-1}f(v)dt]ds, \int_r^1 \phi_q^{-1}[\mu s^{1-N}\int_0^s t^{N-1}g(u)dt]ds)$$

Then $T: K\rightarrow K$ is completely continuous and fixed points of T are nonnegative solutions of (1.4). We shall use the following fixed point theorem in a cone:

Theorem A (Gustafson and Schmitt [25]). *Let K be a cone in a Banach space and $T: K\rightarrow K$ be a completely continuous mapping satisfying*

(*a*) *There exists $k\in K$, $\|k\|=1$, and a number $r>0$ such that all solutions $y\in K$ of $y=Ty+\theta k$, $0<\theta<\infty$ satisfy $\|y\|\neq r$.*

(*b*) *There exists $R>r$ such that all solutions $z\in K$ of $z=\theta Tz$, $0<\theta<1$ satisfy $\|z\|\neq R$.*

Then T has a fixed point $x\in K$, $r\leq\|x\|\leq R$.

Firstly we give a Lemma which will be used later.

Lemma 3.1 ([15]). *Let h be continuous on R^+ and C^1 on $(0,\infty)$ such that*

$$\lim_{x\rightarrow 0^+}\sup xh'(x)<\infty.$$

Let M, ϵ, r be positive numbers with $\epsilon<1$. Then there exists a positive number C such that

$$|h(\gamma x)-\gamma^r h(x)|\leq C(1-\gamma)$$

for $\epsilon\leq\gamma<1$ and $0\leq x\leq M$.

We are now ready to give the

Proof of Theorem 1.1. (Existence): We shall verify the conditions of Theorem A. Let $(u,v)\in K$ satisfy $(u,v)=T(u,v)+\theta(1,1)$ for some $\theta>0$. Then u, v are nonincreasing and since $u,v>0$ on $(0,1)$, it follows as in the proof of Theorem 1.2 that

$$u(\frac{1}{2})\geq C_2(\lambda^{q-1}\mu^m)^{\frac{1}{(p-1)(q-1)-mn}}.$$

Thus $|(u,v)|_0\neq r$ where $0<r<C_2(\lambda^{q-1}\mu^m)^{\frac{1}{(p-1)(q-1)-mn}}$. Next let $(u,v)\in K$ satisfy

$$(u,v)=\theta T(u,v)$$

or some $0<\theta<1$. Then it follows from $\|u\|_0\leq\lambda^{\frac{1}{p-1}}f(\|v\|_0)^{\frac{1}{p-1}}$ and $\|v\|_0\leq\mu^{\frac{1}{q-1}}g(\|u\|_0)^{\frac{1}{q-1}}$ that

$$\|u\|_0\leq\lambda^{\frac{1}{p-1}}f(\mu^{\frac{1}{q-1}}g(\|u\|_0)^{\frac{1}{q-1}})^{\frac{1}{p-1}},\quad \|v\|_0\leq\mu^{\frac{1}{q-1}}g(\lambda^{\frac{1}{p-1}}f(\|v\|_0)^{\frac{1}{p-1}})^{\frac{1}{q-1}}.$$

Since

$$\lim_{x\rightarrow\infty}\frac{\lambda^{\frac{1}{p-1}}f(\mu^{\frac{1}{q-1}}g(x)^{\frac{1}{q-1}})^{\frac{1}{p-1}}}{x}=\lim_{x\rightarrow\infty}\frac{\mu^{\frac{1}{q-1}}g(\lambda^{\frac{1}{p-1}}f(x)^{\frac{1}{p-1}})^{\frac{1}{q-1}}}{x}=0.$$

By (H_2), there exists a number $R > r$ such that $|(u,v)|_0 \neq R$. Hence, Theorem A implies the existence of a nonnegative solution (u,v) of (1.4) with $r \leq |(u,v)|_0 \leq R$. Then it follows from the maximum principle that $u \geq 0$, $v \geq 0$ in $(0,1)$. This completes the existence part of Theorem 1.1.

(Existence): Let (u,v) and (u_1,v_1) be positive solutions of (1.4) and let $\min(\lambda^{q-1}\mu^m, \lambda^n\mu^{p-1})$ be large enough so that Theorem 1.2 applies. By Theorem 1.2,

$$\frac{M_1}{M_2}u_1 \leq u \leq \frac{M_2}{M_1}u_1 \quad \text{on} \quad (0,1).$$

As in [15], let $\alpha = \sup\{c > 0 : u \geq cu_1 \text{ in } (0,1)\}$. Then clearly $\alpha_0 \leq \alpha < \infty$ and $u \geq \alpha_1 u_1$ in $(0,1)$, where $\alpha_0 = \frac{M_1}{M_2}$. We claim that $\alpha \geq 1$. Suppose to the contrary that $\alpha < 1$. Since

$$\begin{aligned} (r^{N-1}\phi_p(u'))' &= -\lambda r^{N-1} f\Big(\int_r^1 \phi_q^{-1}[\lambda s^{1-N}\int_0^s t^{N-1}g(u)dt]ds\Big), \\ (r^{N-1}\phi_p(\alpha u_1'))' &= -\lambda\alpha^{\frac{1}{p-1}} r^{N-1} f\Big(\int_r^1 \phi_q^{-1}[\lambda s^{1-N}\int_0^s t^{N-1}g(u_1)dt]ds\Big), \end{aligned} \tag{3.1}$$

it follows that

$$\begin{aligned} (r^{N-1}(\phi_p(u') - \phi_p(\alpha u_1')))' \leq -\lambda r^{N-1}\{ & f\Big(\int_r^1 \phi_q^{-1}[\lambda s^{1-N}\int_0^s t^{N-1}g(\alpha u_1)dt]ds\Big) \\ & - \alpha^{\frac{1}{p-1}} f\Big(\int_r^1 \phi_q^{-1}[\lambda s^{1-N}\int_0^s t^{N-1}g(u_1)dt]ds\Big)\}. \end{aligned}$$

Let $n_1 > n_2 > n$, $m_1 > m$, and $m_1 n_1 < (p-1)(q-1)$. We claim that

$$\int_0^s t^{N-1}g(\alpha u_1)dt \geq \alpha^{n_1}\int_0^s t^{N-1}g(u_1)dt, \quad s \geq 0. \tag{3.2}$$

Since $\alpha \geq \alpha_0$ and $g(x)/x^{n_2}$ is nonincreasing for $x \gg 1$,

$$g(\alpha x) \geq \alpha^{n_2}g(x), \text{ for } x \gg 1.$$

Let $\frac{1}{2} < T < 1$. By Theorem 1.2,

$$u_1(s) \geq M_1(\lambda^{q-1}\mu^m)^{\frac{1}{(p-1)(q-1)-mn}}(1-T) \gg 1, \ \ s \leq T,$$

and therefore

$$\int_0^s t^{N-1}(g(\alpha u_1) - \alpha^{n_1}g(u_1))dt \geq (\alpha^{n_2} - \alpha^{n_1})\int_0^s t^{N-1}g(u_1)dt \geq 0, \ \ s \leq T.$$

For $s > T$,

$$\begin{aligned} &\int_0^s t^{N-1}(g(\alpha u_1) - \alpha^{n_1}g(u_1))dt \\ &= \int_0^T t^{N-1}(g(\alpha u_1) - \alpha^{n_1}g(u_1))dt + \int_T^s t^{N-1}(g(\alpha u_1) - \alpha^{n_1}g(u_1))dt \\ &\geq (\alpha^{n_2} - \alpha^{n_1})\int_0^T t^{N-1}g(u_1)dt - C(1-T)(1-\alpha), \end{aligned}$$

where Lemma 3.1 with $h = g$. Since

$$\int_0^T t^{N-1} g(u_1) \geq \int_0^{\frac{1}{2}} t^{N-1} g(u_1) \geq \frac{g(u(\frac{1}{2}))}{N2^N} \geq \frac{K_2}{N2^N},$$

and since there exits a positive number $k > 0$ such that

$$\alpha^{n_2} - \alpha^{n_1} \geq k(1-\alpha) \quad \text{for } \alpha_1 \leq \alpha \leq 1,$$

it follows that

$$\int_0^s t^{N-1}(g(\alpha u_1) - \alpha^{n_1} g(u_1))dt > 0 \quad \text{for } s > T$$

if T is sufficiently close to 1. This proves the claim.

4. Acknowledgement

This work was supported by the National Natural Science Foundation of China(Grant No.10871060) and the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No.08KJB110005). The authors express their thanks to the referee for helpful suggestions to improve the paper.

References

1. D.D. Hai, R. Shivaji, *Uniqueness of positive solutions for a class of semipositone elliptic systems*, Nonlinear Anal., **66** (2007) 396–402.
2. H.B. Keller, D.S. Cohen, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech., **16** (1967), 1361–1376.
3. G. Astrita, G. Marrucci, *Principles of Non-Newtonian Fluid Mechanics*, McGraw-Hill, New York, 1974.
4. L.K. Martinson, K.B. Pavlov, *Unsteady shear flows of a conducting fluid with a rheological power law*, Magnit. Gidrodinamika, **2** (1971), 50–58.
5. A.S. Kalashnikov, *On a nonlinear equation appearing in the theory of non-stationary filtration*, Trudy. Sem. Petrovsk., **5** (1978), 60–68.
6. J.R. Esteban, J.L. Vazquez, *On the equation of turbulent filtration in one-dimensional porous media*, Nonlinear Anal., **10** (1982), 1303–1325.
7. L. Erbe, M. Tang, *Uniqueness theorems for positive radial solutions of quasilinear elliptic equations in a ball*, J. Differential Equations, **138** (1997), 351–379.
8. M. Garcia-Huidobro, R. Manasevich, K. Schmitt, *Positive radial solutions of quasilinear elliptic partial differential equations in a ball*, Nonlinear Anal., **35** (1999), 175–190.
9. Z.M. Guo, J.R.L. Webb, *Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large*, Proc. Roy. Soc. Edinburgh, **124A** (1994), 189–198.
10. D.D. Hai, K. Schmitt, *On radial solutions of quasilinear boundary value problems, Topics in Nonlinear Analysis, Progress in Nonlinear Differential Equations and their Applications*, Birkhauser, Basel, 1999, pp.349–361.
11. Z.M. Guo, *Existence and uniqueness of positive radial solutions for a class of quasilinear elliptic equations*, Appl. Anal., **47** (1992), 173–190.
13. E.N. Dancer, *On the number of positive solutions of weakly nonlinear elliptic equations when a parameter is large*, Proc. London Math. Soc., **53** (1986), 429–452.
14. S.S. Lin, *On the number of positive solutions for nonlinear elliptic equations when a parameter is large*, Nonlinear Anal., **16** (1991), 283–297.

15. D.D. Hai, R. Shivaji, *An existence result on positive solutions for a class of p-Laplacian systems*, Nonlinear Anal., **56** (2004), 1007–1010.
16. D.D. Hai, *Uniqueness of positive solutions for a class of semilinear elliptic systems*, Nonlinear Anal., **52** (2003), 595–603.
17. L. Erbe, M. Tang, *Structure of positive radial solutions of semilinear elliptic equations*, J. Differential Equations, **133** (1997), 179–202.
18. A. Castro, M. Hassanpour, R. Shivaji, *Uniqueness of nonnegative solutions for a semipositone problem with concave nonlinearity*, Comm. Partial Differential Equations, **20** (1995), 1927–1936.
19. P.L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev., **24** (1982), 441–467.
20. H. Berestycki, L.A. Caffarelli, L. Nirenberg, *Inequalities for second order elliptic equations with applications to unbounded domains*, A Celebration of John F.Nash Jr., Duke Math. J., **81** (1996), 467–494.
21. A. Castro, C. Maya, R. Shivaji, *Positivity of nonnegative solutions for semipositone cooperative systems*, Proc.Dyn. Syst. Appl., **3** (2001), 113–120.
22. W.C. Troy, *Symmetric properties in systems of semilinear elliptic equations*, J. Differential Equations, **42** (1981) 400–413.
23. D.D. Hai, R. Shivaji, *An existence result on positive solutions for a class of semilinear elliptic systems*, Proc. Roy. Soc. Edinburgh, **134A** (2004), 137–141.
23. I. Ali, A. Castro, R. Shivaji, *Uniqueness and stability of nonnegative solutions for semipositone problems in a ball*, Proc. Amer. Math. Soc., **117** (1993), 775–782.
25. Zuodong Yang, Qishao Lu, *Nonexistence of positive radial solutions to a quasilinear elliptic system and blow-up estimates for Non-Newtonian filtration system*, Applied Math. Letters, **16(4)** (2003), 581–587.
26. G.B. Gustafson, K. Schmitt, *Nonzero solutions of boundary value problems for second order ordinary and delay differential equations*, J. Differential Equations, **12** (1972), 129–147.

Zhoujin Cui received his BS from Yangzhou University and MS at Nanjing Normal University under the direction of Professor Zuodong Yang. Since 2007 he has been at the PLA University of Science and Technology. His research interests focus on the theory of quasi-linear elliptic and parabolic equations.

Institute of Mathematics, School of Mathematics and Computer Science, Nanjing Normal University, Jiangsu Nanjing 210097, China
Institute of Science, PLA University of Science and Technology, Jiangsu Nanjing 211101, China
e-mail: cuizhoujin@126.com

Zuodong yang He research interests focus on the structure of solution for a class of quasilinear elliptic equations(systems) and parabolic equations(systems).

Institute of Mathematics, School of Mathematics and Computer Science, Nanjing Normal University, Jiangsu Nanjing 210097, China
School of Zhongbei, Nanjing Normal University, Jiangsu Nanjing 210046, China
e-mail: zdyang_jin@263.net

Rui Zhang,
Department of Basic Education, Jinling Institute of Technology, Jiangsu Nanjing 210001, China