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# EXACT SOLUTIONS OF THE MDI AND SAWADA–KOTERA EQUATIONS WITH VARIABLE COEFFICIENTS VIA EXP-FUNCTION METHOD

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ABSTRACT. Based on the Exp-function method and a suitable transformation, new generalized solitonary solutions including free parameters of the MDI and Sawada–Kotera equations with variable coefficients are obtained, form which solitary wave solutions and periodic solutions including some known solutions reported in open literature are derived as special cases. The free parameters in the obtained generalized solitonary solutions might imply some meaningful results in the physical models. It is shown that the Exp-function method provides a very effective and important new method for nonlinear evolution equations with variable coefficients.

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## 1. Introduction

Since the soliton phenomena were first observed by John Scott Russell in 1834 [1] and the KdV equation was solved by the inverse scattering method by Gardner et al. in 1967 [2], finding exact solutions of nonlinear evolution equations (NLEEs) has become one of the most exciting and extremely active areas of research investigation. Many powerful methods for finding exact solutions of NLEEs have been proposed, such as the Hirota's bilinear method [3], Bäcklund transformation [4], homogenous balance method [5], homotopy perturbation method [6], variational iteration method [7], tanh-function method [8], Jacobi elliptic function expansion method [9], and the auxiliary equation method [10].

However, to our knowledge, most of aforementioned methods are related to the constant-coefficient models. Recently, the study of variable-coefficient NLEEs

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has attracted much attention [11–13] because most of real nonlinear physical equations possess variable coefficients. Recently, He and Wu [14] proposed a new method called Exp-function method to obtain exact solutions of NLEEs. The Exp-function method is a straightforward and concise method for obtaining generalized solitonary solutions, solitary wave solutions and periodic solutions of NLEEs [15–19]. Taking full advantage of the generalized solitonary solutions, we can recover some known solutions obtained by the most existing methods. The present paper is motivated by the desire to extend the Exp-function method to the MDI and Sawada–Kotera equations with variable coefficients.

## 2. Exact solutions of the MDI equation

Let us consider the MDI equation with variable coefficients [12]

$$u_t = K_0(t)(u_{xxx} - 6u^2u_x) + 4K_1(t)u_x - h(t)(u + xu_x), \tag{1}$$

which is of importance in mathematical physics field, here $K_0(t)$, $K_1(t)$ and $h(t)$ are arbitrary functions of $t$. It was indicated in [20] that the double-soliton solution of Eq. (1) with some suitable $K_0(t)$, $K_1(t)$ and $h(t)$ can describe, in a qualitative way, the interactions between non-propagation solitons in liquid.

We suppose that

$$u(x,t) = v(x,t)\exp[\alpha(t)]. \tag{2}$$

Substituting Eq. (2) into Eq. (1) and canceling $\exp[\alpha(t)]$ yields the following equation

$$v_t + [\alpha'(t) + h(t)]v + [h(t)x - 4K_1(t)]v_x + 6K_0(t)\exp[2\alpha(t)]v^2v_x \\ -K_0(t)v_{xxx} = 0, \tag{3}$$

where $\alpha'(t) = \mathrm{d}\alpha(t)/\mathrm{d}t$. Further setting

$$\alpha(t) = -\int^t h(\tau)\mathrm{d}\tau, \tag{4}$$

then Eq. (3) becomes

$$v_t + [h(t)x - 4K_1(t)]v_x + 6K_0(t)\exp[2\alpha(t)]v^2v_x - K_0(t)v_{xxx} = 0. \tag{5}$$

We use the transformation

$$v = V(\eta), \quad \eta = p(t)x + q(t), \tag{6}$$

where $p(t)$ and $q(t)$ are functions of $t$ to be determined later, then Eq. (5) becomes

$$[p'(t)x + q'(t) + p(t)h(t)x - 4p(t)K_1(t)]V' \\ + 6p(t)K_0(t)\exp[2\alpha(t)]V^2V' - p^3(t)K_0(t)V''' = 0, \tag{7}$$

where $p'(t) = \mathrm{d}p(t)/\mathrm{d}t$, $q'(t) = \mathrm{d}q(t)/\mathrm{d}t$, $V' = \mathrm{d}V(\eta)/\mathrm{d}\eta$, $V''' = \mathrm{d}^3V(\eta)/\mathrm{d}\eta^3$.

According to the Exp-function method [35], we assume that the solution of Eq. (7) can be expressed in the following form

$$V(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{8}$$

where $c$, $d$, $p$ and $q$ are positive integers which are unknown to be further determined, $a_n$ and $b_m$ are unknown constants. Eq. (8) can be re-written in an alternative form [35] as follows

$$V(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{9}$$

In order to determine values of $c$ and $p$, we balance the linear term of highest order in Eq. (7) with the highest order nonlinear term [35]. By simple calculation, we have

$$V''' = \frac{c_1 \exp[(7p+c)\eta] + \cdots}{c_2 \exp(8p\eta) + \cdots} \tag{10}$$

and

$$V^2 V' = \frac{c_3 \exp[(p+3c)\eta] + \cdots}{c_4 \exp(4p\eta) + \cdots} = \frac{c_3 \exp[(5p+3c)\eta] + \cdots}{c_4 \exp(8p\eta) + \cdots}, \tag{11}$$

where $c_i$ are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eqs. (10) and (11), we have

$$7p + c = 5p + 3c, \tag{12}$$

which leads to the result

$$p = c. \tag{13}$$

Similarly to determine values of $d$ and $q$, we balance the linear term of lowest order in Eq. (8)

$$V''' = \frac{\cdots + d_1 \exp[-(7q+d)\eta]}{\cdots + d_2 \exp[(-8q)\eta]} \tag{14}$$

and

$$V^2 V' = \frac{\cdots + d_3 \exp[-(q+3d)\eta]}{\cdots + d_4 \exp[(-4q)\eta]} = \frac{\cdots + d_3 \exp[-(5q+3d)\eta]}{\cdots + d_4 \exp[(-8q)\eta]}, \tag{15}$$

where $d_i$ are determined coefficients only for simplicity. Balancing lowest order of Exp-function in Eqs. (15) and (16), we have

$$-(7q + d) = -(5q + 3d), \tag{16}$$

which leads to the result

$$q = d. \tag{17}$$

We can freely choose the values of $c$ and $d$, but the final solution does not strongly depend upon the choice of values of $c$ and $d$ [35]. For simplicity, we set $p = c = 1$ and $d = q = 1$, then Eq. (9) becomes

$$V(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{18}$$

Substituting Eq. (18) into Eq. (7), and using Mathematica, equating to zero the coefficients of all powers of $x\exp(n\eta)$ and $\exp(n\eta)$ yields a set of algebraic equations for $a_1$, $a_0$, $a_{-1}$, $b_1$, $b_0$, $b_{-1}$, $K_0(t)$, $K_1(t)$, $h(t)$, $p(t)$ and $q(t)$ as follows:

$$(a_1b_0-a_0b_1)\{6a_1^2p(t)K_0(t)\exp[2\alpha(t)]+b_1^2[-4p(t)K_1(t)-p^3(t)K_0(t)+q'(t)]\}=0,$$

$$\begin{aligned}-2\{2p(t)K_0(t)[3a_1(-a_0a_1b_0+a_0^2b_1+a_1a_{-1}b_1-a_1^2b_{-1})]\exp[2\alpha(t)]\\+b_1(-a_1b_0^2+a_0b_0b_1-2a_{-1}b_1^2+2a_1b_1b_{-1})p^2(t)]\\+b_1[-b_1(a_0b_0+a_{-1}b_1)+a_1(b_0^2+b_1b_{-1})][4p(t)K_1(t)-q'(t)]\}=0,\end{aligned}$$

$$\begin{aligned}p(t)K_0(t)\{-6[-a_0^2a_1b_0-a_1^2a_{-1}b_0+a_0^3b_1+a_0a_1(6a_{-1}b_1-5a_1b_{-1})]\exp[2\alpha(t)]\\+[b_1(a_0b_0^2+5a_{-1}b_0b_1-23a_0b_1b_{-1})-a_1b_0(b_0^2-18b_1b_{-1})]p^2(t)\}\\-[-b_1(a_0b_0^2+5a_{-1}b_0b_1+a_0b_1b_{-1})\\+a_1b_0(b_0^2+6b_1b_{-1})][4p(t)K_1(t)-q'(t)]=0,\end{aligned}$$

$$\begin{aligned}-4(a_{-1}b_1-a_1b_{-1})\{[6(a_0^2+a_1a_{-1})\exp[2\alpha(t)]-(b_0^2-8b_1b_{-1})p^2(t)]p(t)K_0(t)\\-(b_0^2+b_1b_{-1})[4p(t)K_1(t)-q'(t)]\}=0,\end{aligned}$$

$$\begin{aligned}-p(t)K_0(t)\{-6[-a_0^2a_{-1}b_0-a_1a_{-1}^2b_0+a_0^3b_{-1}\\+a_0a_{-1}(-5a_{-1}b_1+6a_1b_{-1})]\exp[2\alpha(t)]\\+[b_{-1}(a_0b_0^2+5a_1b_0b_{-1}-23a_0b_1b_{-1})-a_{-1}b_0(b_0^2-18b_1b_{-1})]p^2(t)\}\\+[-b_{-1}(a_0b_0^2+5a_1b_0b_{-1}+a_0b_1b_{-1})\\+a_{-1}b_0(b_0^2+6b_1b_{-1})][4p(t)K_1(t)-q'(t)]=0,\end{aligned}$$

$$\begin{aligned}2\{2p(t)K_0(t)[3a_{-1}(-a_0a_{-1}b_0-a_{-1}^2b_1+a_0^2b_{-1}+a_1a_{-1}b_{-1})]\exp[2\alpha(t)]\\+b_{-1}(a_0b_0b_{-1}+2a_{-1}b_1b_{-1}-a_{-1}b_0^2-2a_1b_{-1}^2)p^2(t)\\+b_{-1}[-b_{-1}(a_0b_0+a_1b_{-1})+a_{-1}(b_0^2+b_1b_{-1})][4p(t)K_1(t)-q'(t)]\}=0,\end{aligned}$$

$$\begin{aligned}(-a_{-1}b_0+a_0b_1)\{6a_{-1}^2p(t)K_0(t)\exp[2\alpha(t)]+b_{-1}^2[-4p(t)K_1(t)\\-p^3(t)K_0(t)+q'(t)]\}=0,\end{aligned}$$

$$b_1^2(a_1b_0-a_0b_1)[h(t)p(t)+p'(t)]=0,$$

$$-2b_1[b_1(a_0b_0+a_{-1}b_1)-a_1(b_0^2+b_1b_{-1})][h(t)p(t)+p'(t)]=0,$$

$$\{a_1b_0(b_0^2+6b_1b_{-1})-b_1[5a_{-1}b_0b_1+a_0(b_0^2+b_1b_{-1})]\}[h(t)p(t)+p'(t)]=0,$$

$$-4(a_{-1}b_1-a_1b_{-1})(b_0^2+b_1b_{-1})[h(t)p(t)+p'(t)]=0,$$

$$\{-a_{-1}b_0(b_0^2+6b_1b_{-1})+b_{-1}[5a_1b_0b_{-1}+a_0(b_0^2+b_1b_{-1})]\}[h(t)p(t)+p'(t)]=0,$$

$$2b_{-1}[b_{-1}(a_0b_0+a_1b_{-1})-a_{-1}(b_0^2+b_1b_{-1})][h(t)p(t)+p'(t)]=0,$$

$$b_{-1}^2(-a_{-1}b_0+a_0b_{-1})[h(t)p(t)+p'(t)]=0.$$

Solving the system of algebraic equations by use of Mathematica, we obtain

$$a_1=\pm\frac{b_1P}{2},\quad a_0=a_0,\quad a_{-1}=\pm\frac{4a_0^2-b_0^2P^2}{8b_1P},\quad b_1=b_1, \tag{19}$$

$$b_0=b_0,\quad b_{-1}=-\frac{4a_0^2-b_0^2P^2}{4b_1P^2},\quad p(t)=\pm P\exp[-\int^t h(\tau)\mathrm{d}\tau], \tag{20}$$

$$\begin{aligned} q(t)=\frac{1}{2}\int^t P\exp[-\int^t h(\tau)\mathrm{d}\tau]\{-P^2K_0(\tau)\exp[-2\int^t h(\tau)\mathrm{d}\tau] \\ +8K_1(\tau)\}\mathrm{d}\tau+q_0, \end{aligned} \tag{21}$$

where $P$ is a nonzero constant, $q_0$ is an arbitrary constant.

From Eqs. (2), (6), (18)–(21) we obtain the following generalized solitonary solution of Eq. (1)

$$u=\frac{\pm\frac{b_1P}{2}\exp(\eta)+a_0\pm\frac{4a_0^2-b_0^2P^2}{8b_1P}\exp(-\eta)}{b_1\exp(\eta)+b_0-\frac{4a_0^2-b_0^2P^2}{4b_1P^2}\exp(-\eta)}\exp[-\int^t h(\tau)\mathrm{d}\tau], \tag{22}$$

where

$$\begin{aligned} \eta=\pm P\exp[-\int^t h(\tau)\mathrm{d}\tau]x+\frac{1}{2}\int^t P\exp[-\int^t h(\tau)\mathrm{d}\tau] \\ \{-P^2K_0(\tau)\exp[-2\times\int^t h(\tau)\mathrm{d}\tau]+8K_1(\tau)\}\mathrm{d}\tau+q_0. \end{aligned}$$

If we set $b_1=2$, $b_0=8r$, $a_0=\mp4p\sqrt{r^2-1}$, $P=-2\sqrt{R}C_1$ and $q_0=C_2$, then Eq. (22) becomes

$$\begin{aligned} u&=\mp\sqrt{R}C_1\frac{8\sqrt{r^2-1}\mp[8\exp(\sqrt{R}\xi)-2\exp(-\sqrt{R}\xi)]}{8\exp(\sqrt{R}\xi)+8r+2\exp(-\sqrt{R}\xi)}\exp[-\int^t h(\tau)\mathrm{d}\tau] \\ &=\mp\sqrt{R}C_1\frac{4\sqrt{r^2-1}\,\mathrm{sech}(\sqrt{R}\xi)\mp[5\tanh(\sqrt{R}\xi)+3]}{5+3\tanh(\sqrt{R}\xi)+4r\mathrm{sech}(\sqrt{R}\xi)}\exp[-\int^t h(\tau)\mathrm{d}\tau], \end{aligned} \tag{23}$$

where

$$\begin{aligned} \xi=\pm2C_1\exp[-\int^t h(\tau)\mathrm{d}\tau]x-\int^t 4C_1\exp[-\int^t h(\tau)\mathrm{d}\tau] \\ \{-RC_1^2K_0(\tau)\exp[-2\times\int^t h(\tau)\mathrm{d}\tau]+2K_1(\tau)\}\mathrm{d}\tau+C_2. \end{aligned}$$

It should be noted that Eq. (23) is exactly Dai et al.'s combined solitary wave solution (34) which was obtained by variable-coefficient generalized projected Ricatti equation expansion method in [33].

Also setting $a_0 = 0$ and $b_0 = \pm 2b_1$ in Eq. (22), then we obtain solitary wave solution

$$u = \frac{\pm P\sinh(\eta)}{\pm 2 + 2\cosh(\eta)}\exp[-\int^t h(\tau)\mathrm{d}\tau], \tag{24}$$

where

$$\eta = \pm P\exp[-\int^t h(\tau)\mathrm{d}\tau]x$$
$$+\frac{1}{2}\int^t P\exp[-\int^t h(\tau)\mathrm{d}\tau]\{P^2K_0(\tau)\exp[-2\int^t h(\tau)\mathrm{d}\tau] + 8K_1(\tau)\}\mathrm{d}\tau + q_0.$$

When $P$ is an imaginary number, namely $P = \mathrm{i}K$, Eq. (24) gives periodic solution

$$u = \frac{\mp K\sin(\zeta)}{\pm 2 + 2\cos(\zeta)}\exp[-\int^t h(\tau)\mathrm{d}\tau], \tag{25}$$

where

$$\zeta = \pm K\exp[-\int^t h(\tau)\mathrm{d}\tau]x + \frac{1}{2}\int^t$$
$$K\exp[-\int^t h(\tau)\mathrm{d}\tau]\{-K^2K_0(\tau)\exp[-2\times\int^t h(\tau)\mathrm{d}\tau] + 8K_1(\tau)\}\mathrm{d}\tau + q_0.$$

Selecting special values of $a_0, b_1, b_0$, from Eq. (22) we can obtain other solitary wave solutions and periodic solutions of Eq. (1), we omit them here for simplicity.

### 3. Exact solutions of the Sawada–Kotera equation

For the Sawada–Kotera equation with variable coefficients [13]

$$u_t + f(t)u^2u_x + g(t)u_xu_{xx} + h(t)uu_{xxx} + k(t)u_{xxxxx} = 0, \tag{26}$$

where $f(t)$, $g(t)$, $h(t)$ and $k(t)$ are functions of $t$, we use the transformation

$$u = U(\eta), \quad \eta = p(t)x + q(t), \tag{27}$$

where $p(t)$ and $q(t)$ are functions of $t$ to be determined later, then Eq. (26) becomes

$$\begin{aligned}[p'(t)x + q'(t)]U' + f(t)p(t)U^2U' + g(t)p^3(t)U'U'' \\ + h(t)p^3(t)UU''' + k(t)p^5(t)UU^{(5)} = 0,\end{aligned} \tag{28}$$

where $p'(t) = \mathrm{d}p(t)/\mathrm{d}t$, $q'(t) = \mathrm{d}q(t)/\mathrm{d}t$, $U' = \mathrm{d}U(\eta)/\mathrm{d}\eta$, $U''' = \mathrm{d}^3U(\eta)/\mathrm{d}\eta^3$, $U^{(5)} = \mathrm{d}^5U(\eta)/\mathrm{d}\eta^5$.

We assume that the solution of Eq. (26) can be expressed in the following form

$$U(\eta) = \frac{a_c\exp(c\eta) + \cdots + a_{-d}\exp(-d\eta)}{b_p\exp(p\eta) + \cdots + b_{-q}\exp(-q\eta)}. \tag{29}$$

where $c$, $d$, $p$ and $q$ are positive integers which are unknown to be further determined.

By simple calculation, we have

$$U^{(5)} = \frac{c_1\exp[(31p+c)\eta] + \cdots}{c_2\exp(32p\eta) + \cdots} \tag{30}$$

and

$$U^2U' = \frac{c_3\exp[(p+3c)\eta] + \cdots}{c_4\exp(4p\eta) + \cdots} = \frac{c_3\exp[(29p+3c)\eta] + \cdots}{c_4\exp(32p\eta) + \cdots}, \tag{31}$$

where $c_i$ are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eqs. (30) and (31), we have

$$31p + c = 29p + 3c, \tag{32}$$

which leads to the result

$$p = c. \tag{33}$$

As illustrated in the previous section, we can also obtain $q = d$. Here, we also consider the simplest case $p = c = 1$ and $q = d = 1$, then Eq. (29) becomes

$$U(\eta) = \frac{a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{34}$$

Substituting Eq. (34) into Eq. (28), and using Mathematica, equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_1$, $a_0$, $a_{-1}$, $b_1$, $b_0$, $b_{-1}$, $f(t)$, $g(t)$, $h(t)$, $k(t)$, $p(t)$ and $q(t)$. Solving the system of algebraic equations by use of Mathematica, we obtain

$$a_1 = a_1, \quad a_0 = -\frac{5a_1b_0}{b_1}, \quad a_{-1} = \frac{a_1b_0^2}{4b_1^2}, \quad b_1 = b_1, \quad b_0 = b_0, \tag{35}$$

$$b_{-1} = \frac{b_0^2}{4b_1}, \quad k(t) = -\frac{a_1[2a_1f(t) + b_1P^2g(t) + 2b_1P^2h(t)]}{5b_1^2P^4}, \tag{36}$$

$$p(t) = P, \quad q(t) = -\int^t \frac{Pa_1[3a_1f(\tau) - b_1P^2g(\tau) + 3b_1P^2h(\tau)]}{5b_1^2}\mathrm{d}\tau + q_0, \tag{37}$$

where $P$ is a nonzero constant, $q_0$ is an arbitrary constant.

We therefore obtain the following generalized solitonary solution of Eq. (26)

$$\begin{aligned} u &= \frac{a_1\exp(\eta) - \frac{5a_1b_0}{b_1} + \frac{a_1b_0^2}{4b_1^2}\exp(-\eta)}{b_1\exp(\eta) + b_0 + \frac{b_0^2}{4b_1}\exp(-\eta)} \\ &= \frac{a_1}{b_1} - \frac{\frac{6a_1b_0}{b_1}}{b_1\exp(\eta) + b_0 + \frac{b_0^2}{4b_1}\exp(-\eta)}, \end{aligned} \tag{38}$$

where

$$\eta = Px - \int^t \frac{Pa_1[3a_1f(\tau) - b_1P^2g(\tau) + 3b_1P^2h(\tau)]}{5b_1^2}\mathrm{d}\tau + q_0,$$

$k(t)$ is determined in Eq. (36).

Setting $a_1 = -\frac{a_2}{3}$, $b_1 = 1$, $b_0 = 2$, $P = 2p$ and $q_0 = C$, then Eq. (38) becomes solitary wave solution

$$u = -\frac{a_2}{3} + a_2\text{sech}^2(\theta_3), \quad k(t) = \frac{a_2[-a_2 f(t) + 6P^2 g(t) + 12P^2 h(t)]}{360P^4}, \tag{39}$$

where

$$\theta_3 = px - \int^t \frac{pa_2[a_2 f(\tau) + 4p^2 g(\tau) - 12p^2 h(\tau)]}{15} \mathrm{d}\tau + C.$$

If $a_2 = 1$, solution (39) is equivalent to solution (3.37) obtained by Liu and Zhu [34] by using modified mapping method.

Setting again $b_1 = 8$ and $b_0 = \pm 8$, then Eq. (38) changes into combined solitary wave solution

$$u = -\frac{a_1}{8} - \frac{3a_1}{3\sinh(\eta) + 5\cosh(\eta) \pm 4}, \quad k(t) = -\frac{a_1[a_1 f(t) + 4P^2 g(t) + 8P^2 h(t)]}{160P^4},$$

where

$$\eta = Px - \int^t \frac{Pa_1[3a_1 f(\tau) - 8P^2 g(\tau) + 24P^2 h(\tau)]}{320} \mathrm{d}\tau + q_0.$$

If $P = \mathrm{i}K$, then it becomes periodic solution

$$u = -\frac{a_1}{8} - \frac{3a_1}{3\mathrm{i}\sin(\zeta) + 5\cos(\zeta) \pm 4}, \quad k(t) = -\frac{a_1[a_1 f(t) - 4K^2 g(t) - 8K^2 h(t)]}{160K^4},$$

where

$$\zeta = Kx - \int^t \frac{Ka_1[3a_1 f(\tau) + 8K^2 g(\tau) - 24K^2 h(\tau)]}{320} \mathrm{d}\tau + q_0.$$

When $P = \mathrm{i}K$ and $b_0 = \pm 2b_1$, Eq. (38) gives periodic solution

$$u = \frac{a_1}{b_1} \mp \frac{6a_1}{b_1\cos(\zeta) \pm b_1}, \quad k(t) = -\frac{a_1[2a_1 f(t) - b_1 K^2 g(t) - 2b_1 K^2 h(t)]}{5b_1^2 K^4},$$

where

$$\zeta = Kx - \int^t \frac{Ka_1[3a_1 f(\tau) + b_1 K^2 g(\tau) - 3b_1 K^2 h(\tau)]}{5b_1^2} \mathrm{d}\tau + q_0.$$

**Remark 1.** Unlike that directly did for Eq. (6), we first introduced the transformation (2) for Eq. (1) since it is easy to deal with Eq. (3) derived from Eq. (2). If not, we could obtain the same solutions except for more difficulty in finding these results.

**Remark 2.** With the aid of Mathematica, we have checked all the solutions obtained in this paper by putting them back into the original Eqs. (1) and (26), respectively.

## 4. Conclusion

In this paper, the Exp-function method and a suitable transformation have been successfully used to obtain generalized solitonary solutions including free parameters of MDI and Sawada–Kotera equations with variable coefficients. Taking full advantages of the generalized solitonary solutions with free parameters, some solitary wave solutions and periodic solutions including known solutions reported in open literature for the considered equations are derived as special cases. The free parameters in the obtained generalized solitonary solutions might imply some meaningful results in the physical models. The results show that the Exp-function method is a very effective and important method for solving NLEEs with variable coefficients arising. The solution procedure of the Exp-function method used in this paper can also be applied to other NLEEs with variable coefficients.

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