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AN ITERATIVE ALGORITHM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. An iterative algorithm was been studied which can be viewed as an extension of the previously known algorithms for asymptotically nonexpansive mappings. Subsequently, we study the convergence problem of the proposed iterative algorithm for asymptotically nonexpansive mappings under some mild conditions in Banach spaces.

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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E. Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $x, y \in C$.

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout that $F(T) \neq \emptyset$.

Let $T : C \to C$ be a nonexpansive mapping. Let $\{\alpha_n\}$ be a sequence in $(0,1)$, $u \in C$ be a fixed point and $x_0 \in C$ be any initial value. Define a sequence $\{x_n\} \subset C$ in an explicit iterative way by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0. \tag{1}$$

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In 1967, Halpern [1] proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies certain control conditions, two of which are

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0,$

(C2) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$ or, equivalently, $\prod\limits_{n=0}^{\infty}(1-\alpha_n) = 0.$

In 1977, Lions [2] improved Halpern's control conditions by showing the strong convergence of the sequence $\{x_n\}$ if $\{\alpha_n\}$ satisfies (C1), (C2) and the following condition

(C3) $\lim\limits_{n\to\infty} \dfrac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0.$

It was observed that both Halpern's and Lions's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$. This was overcome in 1992 by Wittmann [3] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T by using the control conditions (C1), (C2) and the following condition

(C4) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

In 2002, Xu [4] suggested the following control condition (C5) instead of the conditions (C3) or (C4)

(C5) $\lim\limits_{n\to\infty} \dfrac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}} = 0$ or, equivalently, $\lim\limits_{n\to\infty} \dfrac{\alpha_n}{\alpha_{n+1}} = 1.$

He proved the strong convergence of $\{x_n\}$ by using the control conditions (C1), (C2) and (C5). Xu showed that condition (C3) and condition (C5) are not comparable. He also remarked [4, Remark 3.2] that Halpern [1] observed that conditions (C1) and (C2) are necessary for the strong convergence of the sequence $\{x_n\}$ for all nonexpansive mappings $T : C \to C$. It is unclear if they are sufficient. All of above bring us to the following question:

Open Question. *Are the conditions* (C1) *and* (C2) *sufficient for the strong convergence of the sequence* $\{x_n\}$ *for all nonexpansive mappings* $T : C \to C$?

A particular answer to this question was recently given independently by Chidume-Chidume [5] and Suzuki [6]. They proved that if the nonexpansive mapping T in the algorithm (1) is averaged; that is, $T = (1-\lambda)I + \lambda S$, where $\lambda \in (0,1)$, I is the identity and $S : C \to C$ is another nonexpansive mapping, then conditions (C1) and (C2) are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$ generated by the algorithm (1), i.e.,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(\lambda T x_n + (1-\lambda)x_n), \quad n \geq 0.$$

It is clear that the above algorithm can be rewriten as

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0,$$

where $\beta_n = (1-\lambda)(1-\alpha_n)$, $\gamma_n = \lambda(1-\alpha_n)$ and $\lim\limits_{n\to\infty} \gamma_n = \lambda \in (0,1)$. However, the question in its full statement remains unsolved.

On the other hand, as important generations of nonexpansive mappings, the asymptotically nonexpansive mappings were initially introduced by Goebel and Kirk [11]. They proved that if C is a nonempty bounded closed and convex subset of a uniformly convex Banach space E, and $T : C \to C$ is an asymptotically nonexpansive mapping, then T has a fixed point. Subsequently, many authors have investigated iterative methods for approximating fixed points of asymptotically nonexpansive mappings, see, for example [12-17,19-20,22-24]. In particular, in 2004, Chidume, Li and Udomene [18] proved the following result.

Theorem CLU. *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, C be a nonempty closed convex and bounded subset of E and $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1,\infty)$. Let $u \in C$ be fixed, $\{t_n\}_n \subset (0,1)$ be such that $\lim_{n\to\infty} t_n = 1$, $t_n k_n < 1$, and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. Define the sequence $\{x_n\}_n$ iteratively by $x_0 \in C$,*

$$x_{n+1} = \left(1 - \frac{t_n}{k_n}\right) u + \frac{t_n}{k_n} T^n x_n, \quad n = 0, 1, 2, \cdots. \tag{2}$$

Then

(i) *for each integer $n \geq 0$, there is a unique $z_n \in C$ such that*

$$z_n = \left(1 - \frac{t_n}{k_n}\right) u + \frac{t_n}{k_n} T^n z_n;$$

and if, in addition,

$$\lim_{n\to\infty} \|z_n - T z_n\| = 0 \tag{IC1}$$

and

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0, \tag{IC2}$$

then

(ii) *$\{x_n\}_n$ defined by (2) converges strongly to a fixed point of T.*

Remark 1.1. Theorem CLU improved and extended the corresponding result of [21]. However, we note that the authors have imposed additional assumptions ($IC1$) and ($IC2$). A natural problem rises: Could we relax them?

It is the purpose of this paper to present a partial answer to the above questions. We construct a new iterative algorithm which can be viewed as an extension of the previously known algorithms for asymptotically nonexpansive mappings. Subsequently, we study the convergence problem of the proposed iterative algorithm for asymptotically nonexpansive mappings under some mild conditions in Banach spaces.

2. Preliminaries

Let E be a real Banach space with dual E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between E and E^*. In the sequel, we denote single-valued normalized duality mapping by j.

Let $S := \{x \in E : \|x\| = 1\}$ denote the unit sphere of a Banach space E. The space E is said to have a Gâteaux differentiable norm (or E is said to be smooth) if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists for each $x, y \in S$, and E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$ the limit (3) is attained uniformly for $x \in S$. Further, E is said to be uniformly smooth if the limit (3) exists uniformly for $(x, y) \in S \times S$.

The following results are well known:

- If E is smooth then the duality mapping J is single valued and strong-weak* continuous.
- If E is a Banach space with a uniformly Gâteaux differentiable norm, then the duality mapping $J : E \to E^*$ is single valued and norm to weak star uniformly continuous on bounded sets of E.

Let C be a nonempty bounded closed convex subset of a Banach space E and let $d(C) = \sup\{\|x - y\| : x, y \in C\}$ be the diameter of C. For each $x \in C$, let $r(x, C) = \sup\{\|x - y\| : y \in C\}$ and let $r(C) = \inf\{r(x, C) : x \in C\}$, the Chebyshev radius of C relative to itself. The normal structure coefficient of E is defined to be

$$N(E) = \inf\{d(C)/r(C) : C \text{ is bounded closed convex subset of } E \text{ with } d(C) > 0\}.$$

A Banach space E for which $N(E) > 1$ is said to have uniform norm structure. It is known that:

- A Banach space with uniform normal structure is reflexive;
- All uniformly convex or uniformly smooth Banach spaces have uniform normal structure.

Let μ be a continuous linear functional on l^∞ and $s = (a_0, a_1, \cdots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(s)$. We call μ a Banach limit if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \cdots) \in l^\infty$. If μ is a Banach limit, then we have the following:

(i) for all $n \geq 1$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$,
(ii) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r,
(iii) $\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n$ for all $(a_0, a_1, \cdots) \in l^\infty$.

Remark 2.1. If $s = (a_0, a_1, \cdots) \in l^\infty$ with $a_n \to a$, then $\mu(s) = \mu_n(a_n) = a$ for any Banach limit μ by (iii). For more details on Banach limits, we refer readers to [8].

In order to prove the main result of this paper, we need the following lemmas:

Lemma 2.1. ([9]) *Suppose that E is a Banach space with uniformly normal structure, C is a nonempty bounded subset of E, and $T : C \to C$ is a uniformly k-Lipschitzian mapping with $k < N(E)^{\frac{1}{2}}$. Suppose that also there exists a nonempty bounded closed convex subset D of C with the following property* (P)*:*

$$x \in D \quad \textit{implies } \omega_w(x) \subset D,$$

where $\omega_w(x)$ is the weak ω-limit set of T at x, i.e., the set

$$\Big\{y \in E : y = weak - \lim_j T^{n_i} x \textit{ for some } n_j \uparrow \infty\Big\}.$$

Then T has a fixed point in D.

Lemma 2.2. ([7]) *Let E be a real Banach space and let J be the normalized duality mapping. Then for any given $x, y \in E$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.3. ([8]) *Let a be a real number and let $(x_0, x_1, \cdots,) \in l^\infty$ such that $\mu_n x_n \leq a$ for all Banach limits. If $\limsup_{n\to\infty}(x_{n+1} - x_n) \leq 0$, then $\limsup_{n\to\infty} x_n \leq a$.*

Lemma 2.4. ([7]) *Let $\{a_n\}$ and $\{c_n\}$ be two nonnegative real numbers satisfying the following conditions*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n\alpha_n + c_n, \quad n \geq n_0,$$

where n_0 is some nonnegative integer, $\{\alpha_n\} \subset (0, 1)$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\limsup_{n\to\infty} b_n$ < 0 and $\sum_{n=0}^{\infty} c_n < \infty$, then $\{a_n\}$ converges to zero as $n \to \infty$.

3. Main result

We first introduce the following iterative algorithm.

Algorithm 3.1. Let C be a nonempty closed convex subset of a real Banach space E. Let $T : C \to C$ be an asymptotically nonexpansive mapping. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by the following iterative manner:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n x_n, \quad n \geq 0, \tag{4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $(0, 1)$.

Remark 3.1. Suppose that for arbitrary given sequence $\{x_n\}$ is bounded. We define

$$g(x) = \mu_n \|x_n - x\|^2,$$

then $g(x)$ is convex and continuous, also we can easily prove that $g(x) \to \infty$ as $\|x\| \to \infty$. Since E is reflexive, there exists $y \in C$ such that $g(y) = \inf_{x \in C} g(x)$. So the set

$$D = \left\{ y \in C : g(y) = \inf_{x \in C} g(x) \right\} \neq \emptyset.$$

Clearly, D is closed convex subset of C.

Now we state and prove our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a reflexive Banach space E whose norm is uniformly Gâteaux differentiable. Let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$ satisfying $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ or $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$. Suppose that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $(0, 1)$ satisfying the following control conditions*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(ii) $\alpha_n \to 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\gamma_n \to 0$.

Suppose that $D \cap F(T) \neq \emptyset$. For given $x_0 \in C$ arbitrarily, then the sequence $\{x_n\}$ defined by (4) converges strongly to a fixed point of T if and only if $\{x_n\}$ is bounded.

Proof. The necessity is obviously. Now we prove the sufficiency. Suppose that $\{x_n\}$ is bounded.

Now we can take $p \in D \cap F(T)$ and $t \in (0, 1)$. By the convexity of C we have that $(1 - t)p + tf(p) \in C$. It follows that

$$g(p) \leq g((1 - t)p + tf(p)). \tag{5}$$

By Lemma 2.2, we get

$$\|x_n - p - t(f(p) - p)\|^2 \leq \|x_n - p\|^2 - 2t\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle. \tag{6}$$

Taking the Banach limit in (6), we get that

$$\begin{aligned}&\mu_n \|x_n - p - t(f(p) - p)\|^2 \\ &\leq \mu_n \|x_n - p\|^2 - 2t\mu_n \langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle.\end{aligned}$$

This implies

$$2t\mu_n \langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle \leq g(p) - g((1 - t)p + tf(p)). \tag{7}$$

Therefore, from (5) and (7), we have

$$\mu_n\langle f(p)-p, j(x_n-p-t(f(p)-p))\rangle \leq 0. \tag{8}$$

Since the normalized duality mapping j is single-valued and norm-weak* uniformly continuous on bounded subset of E, we have

$$\langle f(p)-p, j(x_n-p)\rangle - \langle f(p)-p, j(x_n-p-t(f(p)-p))\rangle \to 0 \quad \text{as } t \to 0.$$

This implies that for given $\epsilon > 0$ arbitrarily, there exists $\delta > 0$ such that $\forall t \in (0,\delta)$ and for all $n \geq 1$,

$$\langle f(p)-p, j(x_n-p)\rangle - \langle f(p)-p, j(x_n-p-t(f(p)-p))\rangle < \epsilon.$$

Taking the Banach limit and noting that (8), thus we have

$$\mu_n\langle f(p)-p, j(x_n-p)\rangle \leq \mu_n\langle f(p)-p, j(x_n-p-t(f(p)-p))\rangle + \epsilon \leq \epsilon.$$

By the arbitrariness of ϵ, we obtain

$$\mu_n\langle f(p)-p, j(x_n-p)\rangle \leq 0. \tag{9}$$

At the same time, we note that

$$\|x_{n+1}-x_n\| \leq \alpha_n\|f(x_n)-x_n\| + \gamma_n\|T^n x_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$

Since $\{x_n - p\}$ and $\{f(p)-p\}$ are bounded and the duality mapping j is single-valued and norm topology to weak star topology uniformly continuous on bounded sets in Banach space E with uniformly Gâteaux differentiable norm, then we get that

$$\lim_{n\to\infty}\{\langle f(p)-p, j(x_{n+1}-p)\rangle - \langle f(p)-p, j(x_n-p)\rangle\} = 0. \tag{10}$$

From (9),(10) and Lemma 2.3, we conclude that

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_{n+1}-p)\rangle \leq 0.$$

Finally, we prove that $x_n \to p$. Indeed, applying Lemma 2.2 to (4), we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&= \|\alpha_n(f(x_n)-p) + \beta_n(x_n-p) + \gamma_n(T^n x_n - p)\|^2 \\
&\leq \|\beta_n(x_n-p) + \gamma_n(T^n x_n - p)\|^2 + 2\alpha_n\langle f(x_n)-p, j(x_{n+1}-p)\rangle \\
&\leq [\beta_n\|x_n-p\| + \gamma_n k_n\|x_n-p\|]^2 + 2\alpha_n\langle f(x_n)-f(p), j(x_{n+1}-p)\rangle \\
&\quad + 2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle \\
&\leq [1-\alpha_n+(k_n-1)\gamma_n]^2\|x_n-p\|^2 + 2\alpha\alpha_n\|x_n-p\|\|x_{n+1}-p\| \\
&\quad + 2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle \\
&\leq [(1-\alpha_n)^2 + 2\gamma_n(1-\alpha_n)(k_n-1) + \gamma_n^2(k_n-1)^2]\|x_n-p\|^2 \\
&\quad + \alpha\alpha_n(\|x_n-p\|^2 + \|x_{n+1}-p\|^2) + 2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle,
\end{aligned}$$

which implies that

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \left(1 - \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}\right)\|x_n - p\|^2 + \frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n - p\|^2 \\
&\quad + \frac{[2\gamma_n(1-\alpha_n) + \gamma_n^2(k_n - 1)]\|x_n - p\|^2}{1-\alpha\alpha_n}(k_n - 1) \\
&\quad + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(p) - p, j(x_{n+1} - p)\rangle \\
&\leq \left(1 - \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}\right)\|x_n - p\|^2 + \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} \\
&\quad \times \left\{\frac{\alpha_n}{2(1-\alpha)}\|x_n - p\|^2 + \frac{1}{1-\alpha}\langle f(p) - p, j(x_{n+1} - p)\rangle\right\} + M(k_n - 1) \\
&= (1-\delta_n)\|x_n - p\|^2 + \delta_n\sigma_n + M(k_n - 1),
\end{aligned}
$$

where $\delta_n = \dfrac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$, $\sigma_n = \dfrac{\alpha_n}{2(1-\alpha)}\|x_n - p\|^2 + \dfrac{1}{1-\alpha}\langle f(p) - p, j(x_{n+1} - p)\rangle$ and M is some constant such that $\sup_{n\geq 0}\left\{\dfrac{[2\gamma_n(1-\alpha_n) + \gamma_n^2(k_n - 1)]\|x_n - p\|^2}{1-\alpha\alpha_n}\right\} \leq M$. It is easily seen that $\sum_{n=0}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \leq 0$. Also it is clear that $\sum_{n=0}^{\infty} M(k_n - 1) < \infty$ or $\lim_{n\to\infty}\dfrac{M(k_n - 1)}{\delta_n} = 0$. Hence, by Lemma 2.4, we have that $x_n - p \to 0$ as $n \to \infty$. This completes the proof.

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Banach space E which possesses uniform normal structure and whose norm is uniformly Gâteaux differentiable. Let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1,\infty)$ satisfying $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ or $\lim_{n\to\infty}\dfrac{k_n - 1}{\alpha_n} = 0$. Suppose that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying the following control conditions*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(ii) $\alpha_n \to 0, \sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $\gamma_n \to 0$.

Suppose that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. For given $x_0 \in C$ arbitrarily, then the sequence $\{x_n\}$ defined by (4) converges strongly to a fixed point of T if and only if $\{x_n\}$ is bounded.

Proof. First we note that if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then from [9] we can conclude that D has the property (P). Hence, from Lemma 2.1, we have $D \cap F(T) \neq \emptyset$. At the same time, we observe that all Banach spaces with uniform normal structure are reflexive. Consequently, from Theorem 3.1, we can derive our conclusion. This completes the proof.

Remark 3.2. Comparing Corollary 3.1 with Theorem CLU, we have the following observations:

(1) The boundedness of C in Corollary 3.1 is dropped. Only the boundedness of the sequence $\{x_n\}$ is needed.
(2) Our iterative algorithm (4) is different from the iterative algorithm (2). Our iterative algorithm can be viewed as an extension of the previously known iterative algorithms.
(3) We drop the assumption $(IC1)$.
(4) Our proofs are very simple.

Our results improve and extend Theorem CLU.

Corollary 3.2. *Let C be a nonempty closed convex subset of a reflexive Banach space E whose norm is uniformly Gâteaux differentiable. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying the following control conditions*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;
(ii) $\alpha_n \to 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\gamma_n \to 0$.

For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by the following iterative manner:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0. \tag{11}$$

Suppose that $D \cap F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (11) converges strongly to a fixed point of T.

Proof. Take a fixed point $p \in F(T)$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|Tx_n - p\| \\
&\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| \\
&\leq (1 - (1-\alpha)\alpha_n)\|x_n - p\| + \alpha_n \|f(p) - p\| \\
&\leq \max\left\{\frac{1}{1-\alpha}\|f(p) - p\|, \|x_0 - p\|\right\}.
\end{aligned}$$

This implies that $\{x_n\}$ is bounded. Hence the conclusion follows from Theorem 3.1. This completes the proof.

From Theorem 3.1, we also obtain the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a reflexive Banach space E whose norm is uniformly Gâteaux differentiable. Let $T : C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$ satisfying $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ or $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$. Suppose that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying the following control conditions*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(ii) $\alpha_n \to 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\gamma_n \to 0$.

For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T^n x_n, \quad n \geq 1. \tag{12}$$

Suppose that $D \cap F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (12) converges strongly to a fixed point of T.

We note the following Lemma proved by Bruck [10], which is related to a sequence of nonexpansive mappings.

Lemma 3.1. *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in N\}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by*

$$Sx = \sum_{n=1}^{\infty} \lambda_n T_n x$$

for $x \in C$ is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

Using Corollary 3.2 and Lemma 3.1, we obtain the following.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let $\{T_n : n \in N\}$ be a sequence of nonexpansive mappings on C and $f : C \to C$ be a contraction. Suppose that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Define a nonexpansive mapping U on C by*

$$Ux = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda_n T_n x$$

for $x \in C$, *where* $\{\lambda_n\}$ *is a sequence of positive numbers with* $\sum_{n=1}^{\infty} \lambda_n \leq 1$ *and* $\lambda = \sum_{n=1}^{\infty} \lambda_n$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be three real sequences in* $(0,1)$ *satisfying the following control conditions*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(ii) $\alpha_n \to 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\gamma_n \to 0$.

For given $x_0 \in C$ *arbitrarily, let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U x_n, \quad n \geq 1. \tag{13}$$

Suppose that $D \cap F(U) \neq \emptyset$. *Then the sequence* $\{x_n\}$ *defined by* (13) *converges strongly to a common fixed point of* $\{T_n : n \in N\}$.

Proof. Note that all uniformly convex Banach spaces are reflexive. It follows from Corollary 3.2 and Lemma 3.1 that we can conclude the result. This completes the proof.

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