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WAVEFRONT SOLUTIONS IN THE DIFFUSIVE NICHOLSON'S BLOWFLIES EQUATION WITH NONLOCAL DELAY

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ABSTRACT. In the present article we consider the diffusive Nicholson's blowflies equation with nonlocal delay incorporated into an integral convolution over all the past time and the whole infinite spatial domain $\mathbb{R}$. When the kernel function takes a special function, we construct a pair of lower and upper solutions of the corresponding travelling wave equation and obtain the existence of travelling fronts according to the existence result of travelling wave front solutions for reaction diffusion systems with nonlocal delays developed by Wang, Li and Ruan (J. Differential Equations, **222**(2006), 185-232).

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1. Introduction

It is well known that many biological and physical problems can be described by differential equations with delays, that is, functional differential equations. For example, to explain the oscillatory phenomenon observed by Nicholoson [6] in the Austrian sheep-blowfly (*Lucia cuprina*) population, Gurney, Blythe and Nisbet [2] proposed the following well known Nicholoson's blowflies equation

$$\frac{du}{dt} = -\delta u(t) + pu(t-\tau)e^{-au(t-\tau)}, \tag{1}$$

where p is the maximum per capita daily egg production rate, $1/a$ is the size at which the blowfly population reproduces at its maximum rate, δ is the per capita daily adult death rate and τ is the generation time. Equation (1) has been studied extensively by many researchers and many interesting results have been obtained (see [1, 7, 8] and the references cited therein). After rescaling

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$u^* = au,\ t^* = \tau t,\ \tau^* = \delta\tau,\ \beta = p/\delta$ and dropping the asterisks, the Nicholoson's blowflies equation (1) becomes

$$\frac{du}{dt} = -\tau u(t) + p\tau u(t-1)e^{-u(t-1)}. \tag{2}$$

In the process of proposing model (1), in fact, it is assumed that the blowfly population encounter each other in proportion to their average density. As pointed out by Law et al.[3], however, spatial structure may make it impossible for organisms to encounter each other in proportion to their average density. The random collision of individuals assumed in the above model may not represent interactions among organisms. Taking the spatial structure into account, Yang and So [14] extended (1.2) to the following diffusive form:

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) - \tau u(x,t) + p\tau u(x,t-1)e^{-u(x,t-1)}. \tag{3}$$

In the case when spatial domain is a bounded open domain in $\mathbb{R}^n (n \in \mathbb{N})$ with a smooth boundary, Yang and So [14] considered the global attractivity of positive steady state and the oscillation of solutions of equation (3) under homogeneous Neumann boundary conditions. Under homogeneous Dirichlet boundary conditions, So and Yang [9] investigated the global attractivity of the equilibrium of equation (3). In addition, some numerical and Hopf bifurcation analysis of this model was carried out by So, Wu and Yang [10]. When the spatial variable is confined on the whole real line $\mathbb{R}$, So and Zou [11] obtained the existence of travelling wave front solutions of equation (3). For the fundamental theories of reaction diffusion equations with delay, we refer to the monograph of Wu [13].

Recently, by incorporating nonlocal delays described by an integral convolution over the whole infinite spatial domain $\mathbb{R}$ and the whole time internal up to now, Li, Ruan and Wang [4] proposed the following Nicholson's blowflies equation with nonlocal delays

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) - \tau u(x,t) + p\tau((g*u)(x,t))e^{-(g*u)(x,t)},\ t>0, x\in\mathbb{R}, \tag{4}$$

where $(g*u)(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{+\infty} g(x-y,t-s)u(y,s)dyds$, and the convolution kernel $g(y,s)$ is an integral and nonnegative function satisfying the following normalized condition

$$\int_{0}^{+\infty}\int_{-\infty}^{+\infty} g(y,s)dyds = 1. \tag{5}$$

By applying the existence result of travelling wave front solutions for reaction-diffusion systems with nonlocal delays developed by Wang, Li and Ruan [12], they investigated the existence of travelling fronts of equation (4) when the $g(x,t)$ takes the following functions

$$\frac{1}{\tau_0}e^{-\frac{t}{\tau_0}}\delta(x),\ \delta(t)\frac{1}{\sqrt{4\pi\rho_0}}e^{-\frac{x^2}{4\rho_0}},\ \frac{t}{\tau_0^2}e^{-\frac{t}{\tau_0}}\delta(x),\ \frac{1}{\tau_0}e^{-\frac{t}{\tau_0}}\frac{1}{\sqrt{4\pi\rho_0}}e^{-\frac{x^2}{4\rho_0}},$$

and

$$\frac{1}{\tau_0} e^{-\frac{t}{\tau_0}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}},$$

where $\tau_0 > 0, \rho_0 > 0$ and $\delta(\cdot)$ denotes Dirac delta function. In addition, they also studied the dependence of minimal wave speed on the delay and the mobility of population, and found that delay can induce slow travelling wave fronts and the mobility of population can increase fast travelling wave fronts. More recently, Lin [5] generalized the convolution kernel in [4] and obtained the existence of travelling front solutions of equation (4) when $g(x,t)$ takes the following forms:

$$\frac{1}{(n-1)!} \frac{t^{n-1}}{\tau_0^n} e^{-\frac{t}{\tau_0}} \delta(x), \frac{1}{(n-1)!} \frac{t^{n-1}}{\tau_0^n} e^{-\frac{t}{\tau_0}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \delta(t) \frac{1}{2\rho_0} e^{-\frac{|x|}{\rho_0}}, n = 1, 2, \cdots.$$

In this paper, we consider the existence of travelling wave front solutions of equation (4) when the convolution kernel $g(x,t)$ takes the following function

$$g(x,t) = \frac{1}{(n-1)!} \frac{t^{n-1}}{\tau_0^n} e^{-\frac{t}{\tau_0}} \frac{1}{\sqrt{4\pi\rho_0}} e^{-\frac{x^2}{4\rho_0}}, \tau_0 > 0, \rho_0 > 0, n = 1, 2, \cdots. \tag{6}$$

The remaining part of this paper is organized as follows. In Section 2, we state the existence result of travelling wave front solutions of scalar reaction-diffusion equation with nonlocal delays according to the existence theory of travelling fronts for reaction-diffusion systems with nonlocal delays developed by Wang, Li and Ruan [12]. In Section 3, we construct a pair of lower and upper solutions of the corresponding travelling wave equation and obtain the existence of travelling fronts of the diffusive Nicholson's blowflies equation (4) when the convolution kernel $g(x,t)$ has the form (6).

2. Existence result of travelling wave front solutions

In this section, we state an existence theorem of travelling wave front solutions for a scalar reaction-diffusion equation with nonlocal delays according to the existence theory of travelling fronts for general reaction-diffusion systems with nonlocal delays developed by Wang, Li and Ruan [12]. Consider the following scalar reaction-diffusion equation with nonlocal delays

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t), (g_1 * u)(x,t), \cdots, (g_m * u)(x,t)), t > 0, x \in \mathbb{R}, \tag{7}$$

where $u(x,t) \in \mathbb{R}$, $D > 0$, $f \in C(\mathbb{R}^{1+m}, \mathbb{R})$ and

$$(g_j * u)(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} g_j(x-y, t-s) u(y,s) dy ds.$$

The nonnegative integrable functions $g_j(x,t)(j = 1, \cdots, m)$ are called the delay kernel, and satisfy the conditions

$$g_j(-x,t) = g_j(x,t) \text{ and } \int_0^{+\infty} \int_{-\infty}^{+\infty} g_j(y,s) dy ds = 1 \tag{8}$$

and the hypothesis

(H_0): $\int_{-\infty}^{+\infty} g_j(x,t)dx$ is uniformly convergent for $t \in [0,a]$ with $a > 0 (j = 1, \cdots, m)$, that is, for any given $\varepsilon > 0$, there exists an $M > 0$ such that $\int_M^{+\infty} g_j(x,t)dx < \varepsilon$ for any $t \in [0,a]$.

Set $u(x,t) = \varphi(z)$ where $z = x + ct$ and $c \geq 0$, then system (7) can be rewritten into the following second order differential equation

$$-D\varphi''(z) + c\varphi'(z) = f(\varphi(z), (g_1 * \varphi)(z), \cdots, (g_m * \varphi)(z)), z \in \mathbb{R}, \tag{9}$$

where

$$(g_j * \varphi)(z) = \int_0^{+\infty} \int_{-\infty}^{+\infty} g_j(y,s)\varphi(z - y - cs)dyds, j = 1, \cdots, m.$$

A travelling wave front solution of (7) with wave speed $c > 0$ is a monotone function $\varphi \in BC^2(\mathbb{R}, \mathbb{R})$ satisfying (9) and the following limit boundary conditions

$$\varphi(-\infty) = 0 \text{ and } \varphi(+\infty) = K \text{ with } K > 0. \tag{10}$$

From [12], to guarantee the existence of travelling wave front solutions of equation (7), we also need the following monotonicity conditions and assumptions:

(H_1): There exists $\gamma > 0$ such that

$$f(\varphi_2(z), (g_1 * \varphi_2)(z), \cdots, (g_m * \varphi_2)(z)) + \gamma\varphi_2(z)$$
$$\geq f(\varphi_1(z), (g_1 * \varphi_1)(z), \cdots, (g_m * \varphi_1)(z)) + \gamma\varphi_1(z),$$

where $\varphi_1, \varphi_2 \in C(\mathbb{R}, \mathbb{R})$ satisfy $0 \leq \varphi_1 \leq \varphi_2 \leq K$ in $z \in \mathbb{R}$;

(H_2): $f(\mu, \cdots, \mu) \neq 0$ for $0 < \mu < K$;

(H_3): $f(0, \cdots, 0) = f(K, \cdots, K) = 0$.

Next, we define the following profile set

$$\Gamma = \left\{ \varphi \in Y : \begin{array}{l} \text{(i) } \varphi \text{ is increasing in } \mathbb{R}; \\ \text{(ii) } 0 \leq \lim\limits_{z \to -\infty} \varphi(z) < K \text{ and } \lim\limits_{z \to +\infty} = K \end{array} \right\},$$

and

$$BC[0,K] = \{\varphi \in BC(\mathbb{R}, \mathbb{R}) : \nvdash \leq \varphi \leq \mathbb{K}\},$$

where $Y = \{\varphi \in BC(\mathbb{R}, \mathbb{R}) : \varphi', \varphi'' \in \mathbb{L}^\infty(\mathbb{R}, \mathbb{R})\}$.

In order to state the existence theorem of travelling wave front solutions for the scalar reaction-diffusion equation (7), we give the definition of upper solutions and lower solutions of equation (9).

Definition 1. *A continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ is called an upper solution of (9) if φ' and φ'' exist almost everywhere and are essentially bounded on $\mathbb{R}$, and φ satisfies*

$$-D\varphi''(z) + c\varphi'(z) \geq f(\varphi(z), (g_1 * \varphi)(z), \cdots, (g_m * \varphi)(z)), a.e. \text{ on } \mathbb{R}. \tag{11}$$

A lower solution of (9) can be defined in a similar way by reversing the inequality (11).

Theorem 1. *Assume that (H_0)-(H_4) hold, $\psi \in BC[0,K] \cap Y$ with $\psi \not\equiv 0$ and $\lim_{z \to -\infty} \psi(z) = 0$, $\phi \in \Gamma$ with $\psi \leq \phi$, are lower solution and upper solution of (9), respectively. Then (7) has a travelling wave front ϕ^* which is increasing and satisfies (10) with $\psi \leq \phi^* \leq \phi$.*

3. Travelling fronts of equation (4) with delay kernel (6)

In this section, we study the existence of travelling front solutions of equation (4) when the delay kernel $g(x,t)$ takes the convolution kernel (6) by applying theorem 1. The key purposes in this section are to establish a pair of lower and upper solutions of the travelling wave equation of (4), which satisfy the conditions in theorem 1. It is easy to see that equation (4) has two equilibria $u = 0$ and $u = \ln \beta$ when $\beta > 1$. Thus, it is possible that equation (4) has travelling wave front solutions connecting the equilibria $u = 0$ and $u = \ln \beta := k$ only if $\beta > 1$. To ensure the solutions of travelling wave equation (12) is in the interval $[0,k]$, we always assume $\beta > 1$ in the following discussions. Let $u(x,t) = \varphi(z)$ where $z = x + ct$ and $c \geq 0$. Then the travelling wave equation of (4) is

$$d\varphi''(z) - c\varphi'(z) - \tau\varphi(z) + \beta\tau(g * \varphi)(z)e^{-(g*\varphi)(z)} = 0, z \in \mathbb{R}, \qquad (12)$$

where

$$(g * \varphi)(z) = \int_0^{+\infty} \int_{-\infty}^{+\infty} g(y,s)\varphi(z - y - cs)dyds.$$

From [4], we know that travelling wavefront solutions of equation (4) has the following properties.

Lemma 1. *If $\beta > 1$, then any travelling wave solution $\varphi(z)$ of (4) satisfies $\varphi(z) \leq \beta/e$ everywhere.*

Lemma 2. *If $\beta \leq e$, then any travelling wave front $\varphi(z)$ of (4) satisfies $\varphi(z) \leq \ln \beta \leq 1$ everywhere.*

Lemma 3. *Let $f(\varphi(z), (g * \varphi)(z)) = -\tau\varphi(z) + \beta\tau(g * \varphi)(z)e^{-(g*\varphi)(z)}$. Then $f(\varphi(z), (g * \varphi)(z))$ satisfies the assumption (H_1).*

Linearizing equaiton (12) at zero solution yields the following linear equation

$$d\varphi''(z) - c\varphi'(z) - \tau\varphi(z) + \beta\tau(g * \varphi)(z) = 0, z \in \mathbb{R}. \qquad (13)$$

Seeking solutions of equation (13) proportional to $e^{\lambda z}(\lambda \in \mathbb{R})$, one can find that λ should satisfies the equation

$$d\lambda^2 - c\lambda - \tau + \beta\tau \int_0^{+\infty} \int_{-\infty}^{+\infty} g(y,s)e^{-y-cs}dyds = 0. \qquad (14)$$

If $g(x,t)=\frac{1}{(n-1)!}\frac{t^{n-1}}{\tau_0^n}e^{-\frac{t}{\tau_0}}\frac{1}{\sqrt{4\pi\rho_0}}e^{-\frac{x^2}{4\rho_0}}$ and define the function

$$\Delta_c(\lambda)=\left(\frac{e^{\rho_0\lambda^2}}{(1+\lambda\tau_0 c)^n}-1\right)\beta\tau-[\tau(1-\beta)+c\lambda-d\lambda^2],\lambda\in\mathbb{R},$$

then equation (14) becomes

$$\Delta_c(\lambda)=0. \tag{15}$$

It is easy to see that $\Delta_0(\lambda)=\left(e^{\rho_0\lambda^2}\beta-1\right)\tau+d\lambda^2>0$ for any $\lambda\in\mathbb{R}$ since $\rho_0>0,\tau>0$ and $\beta>1$. Now fix $c>0$ and consider equation (15) when $\lambda>0$. It is also easy to observe that $\Delta_c(0)=(\beta-1)\tau>0$ and

$$\frac{\partial\Delta_c(\lambda)}{\partial\lambda}=\frac{2\left[\rho_0\lambda(1+\lambda\tau_0 c)-n\tau_0 c\right]e^{\rho_0\lambda^2}}{(1+\lambda\tau_0 c)^{2n-1}}\beta\tau-c+2d\lambda. \tag{16}$$

From (16), one can easily see that

$$\left.\frac{\partial\Delta_c(\lambda)}{\partial\lambda}\right|_{\lambda=0}=-\left(2n\tau_0\beta\tau+1\right)c<0 \text{ and } \lim_{\lambda\to+\infty}\frac{\partial\Delta_c(\lambda)}{\partial\lambda}=+\infty.$$

Thus, for fixed $c>0$, there exists at least a $\lambda_c=\lambda(c)>0$ such that $\left.\frac{\partial\Delta_c(\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_c}=0$. In addition, for fixed $\lambda>0$,

$$\lim_{c\to+\infty}\Delta_c(\lambda)=-\infty.$$

Therefore, we have the following result.

Lemma 4. *There exist $c^*>0$ and $\lambda^*>0$ such that*

(i): $\Delta_{c^*}(\lambda^*)=0$ *and*

$$\left.\frac{\partial\Delta_{c^*}(\lambda)}{\partial\lambda}\right|_{\lambda=\lambda^*}=0;$$

(ii): $\Delta_c(\lambda)>0$ *for* $0<c<c^*$ *and* $\lambda>0$;

(iii): *Equation (15) when $c>c^*$ has two positive roots λ_1 and λ_2 such that $0<\lambda_1<\lambda_2$ and*

$$\Delta_c(\lambda)=\begin{cases}>0, & 0<\lambda<\lambda_1\ ;\\ <0, & \lambda_1<\lambda<\lambda_2,\\ >0, & \lambda>\lambda_2.\end{cases}$$

Next we shall construct a pair of upper and lower solutions of equation (12). We have first have the following result.

Lemma 5. *Let $c>c^*$ and $\lambda_1>0$ with c^* and λ_1 being defined by Lemma 4. Then $\phi(z)=\min\left\{k,ke^{\lambda_1 z}\right\}$ is an upper solution of equation (12) and $\phi\in\Gamma$.*

Proof. $\phi \in \Gamma$ is obvious by the definition of $\phi(z)$ and we need only to verify $\phi(z)$ is an upper solution of equation (12).

From the definition of $\phi(z)$, we know that $\phi(z) = k$ when $z \geq 0$ and thus $\phi'(z) = 0$, $\phi''(z) = 0$ for $z \geq 0$. Therefore, when $z \geq 0$,

$$(g * \phi)(z) \leq k \int_0^{+\infty} \int_{-\infty}^{+\infty} g(y, s) dyds = k$$

and thus

$$\begin{aligned} & d\phi''(z) - c\phi'(z) - \tau\phi(z) + \beta\tau(g * \phi)(z)e^{-(g*\phi)(z)} \\ & \leq -\tau k + \beta\tau k e^{-k} = 0 \end{aligned}$$

since $k = \ln \beta$.

When $z < 0$, $\phi(z) = ke^{\lambda_1 z}$, $\phi'(z) = k\lambda_1 e^{\lambda_1 z}$ and $\phi''(z) = k\lambda_1^2 e^{\lambda_1 z}$. Hence we have from the fact that $0 < \phi(z) \leq ke^{\lambda_1 z}$ for $z \in \mathbb{R}$ that when $z < 0$

$$\begin{aligned} & d\phi''(z) - c\phi'(z) - \tau\phi(z) + \beta\tau(g * \phi)(z)e^{-(g*\phi)(z)} \\ = \; & k \left[(d\lambda_1^2 - c\lambda_1 - \tau)e^{\lambda_1 z} + \frac{\beta\tau}{k}(g * \phi)(z)e^{-(g*\phi)(z)} \right] \\ \leq \; & k \left[(d\lambda_1^2 - c\lambda_1 - \tau)e^{\lambda_1 z} + \frac{\beta\tau}{k}(g * \phi)(z) \right] \\ \leq \; & k \Big[(d\lambda_1^2 - c\lambda_1 - \tau)e^{\lambda_1 z} \\ & + \beta\tau \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{(n-1)!} \frac{s^{n-1}}{\tau_0^n} e^{-\frac{s}{\tau_0}} \frac{1}{\sqrt{4\pi\rho_0}} e^{-\frac{y^2}{4\rho_0}} e^{\lambda_1(z-y-cs)} dyds \Big] \\ = \; & k \left[d\lambda_1^2 - c\lambda_1 + \tau(\beta - 1) + \left(\frac{e^{\rho_0 \lambda_1^2}}{(1 + \lambda_1 \tau_0 c)^n} - 1 \right) \beta\tau \right] = 0. \end{aligned}$$

This proves that $\phi(z) = \min\{k, ke^{\lambda_1 z}\}$ is an upper solution of equation (12).

Next we construct a lower solution of equation (12).

Lemma 6. *Let $c > c^*$ and λ_1, λ_2 be two real positive roots of (15) defined by Lemma 4. Choose $\varepsilon > 0$ such that $\varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$ and $\psi(z) = \max\{0, k(1 - Me^{\varepsilon z})e^{\lambda_1 z}\}$ with $M > 1$. Then for sufficiently large M, $\psi(z)$ is a lower solution of equation (12).*

Proof. Let $z_1 = \dfrac{1}{\varepsilon} \ln \dfrac{1}{M}$. Then $z_1 < 0$ and

$$\psi(z) = \begin{cases} 0, & \text{for } z \geq z_1, \\ k(1 - Me^{\varepsilon z})e^{\lambda_1 z}, & \text{for } z < z_1. \end{cases}$$

If $z \geq z_1$, then $\psi(z) = 0$. Notice that $\psi(z - y - cs) \geq 0$ for any $z \in \mathbb{R}$, we have when $z \geq z_1$

$$d\phi''(z) - c\phi'(z) - \tau\phi(z) + \beta\tau(g * \phi)(z)e^{-(g*\phi)(z)} = \beta\tau(g * \phi)(z)e^{-(g*\phi)(z)} \geq 0.$$

If $z < z_1$, then $\psi(z) = k(1 - Me^{\varepsilon z})e^{\lambda_1 z}$ and hence $\psi'(z) = k[\lambda_1 - M(\lambda_1 + \varepsilon)e^{\varepsilon z}]e^{\lambda_1 z}$, $\psi''(z) = k[\lambda_1^2 - M(\lambda_1+\varepsilon)^2 e^{\varepsilon z}]e^{\lambda_1 z}$. Note that $\psi(z) \geq k(1-Me^{\varepsilon z})e^{\lambda_1 z}$ for $z \in \mathbb{R}$, we have when $z < z_1$

$$\begin{aligned}
& (g * \psi)(z) \\
\geq\ & k \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{(n-1)!} \frac{s^{n-1}}{\tau_0^n} e^{-\frac{s}{\tau_0}} \frac{1}{\sqrt{4\pi\rho_0}} e^{-\frac{y^2}{4\rho_0}} (1 - Me^{\varepsilon(z-y-cs)}) \\
& \times e^{\lambda_1(z-y-cs)} dyds \\
=\ & ke^{\lambda_1 z} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{(n-1)!} \frac{s^{n-1}}{\tau_0^n} e^{-\frac{s}{\tau_0}} \frac{1}{\sqrt{4\pi\rho_0}} e^{-\frac{y^2}{4\rho_0}} e^{\lambda_1(-y-cs)} dyds \\
& -kMe^{(\lambda_1+\varepsilon)z} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{(n-1)!} \frac{s^{n-1}}{\tau_0^n} \frac{1}{\sqrt{4\pi\rho_0}} e^{-\frac{y^2}{4\rho_0}} e^{(\lambda_1+\varepsilon)(-y-cs)} dyds \\
=\ & ke^{\lambda_1 z} \frac{e^{\rho_0 \lambda_1^2}}{(1+\lambda_1 \tau_0 c)^2} - kMe^{(\lambda_1+\varepsilon)z} \frac{e^{\rho_0(\lambda_1+\varepsilon)^2}}{(1+\tau_0(\lambda_1+\varepsilon)c)^n},
\end{aligned}$$

and

$$(g * \psi)(z) \leq ke^{\lambda_1 z} \frac{e^{\rho_0 \lambda_1^2}}{(1+\lambda_1 \tau_0 c)^n}.$$

Also, notice that $0 < \varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$ and $z < z_1 < 0$, we have

$$2\lambda_1 z = (\lambda_1 + \lambda_1)z < (\lambda_1 + \varepsilon)z.$$

Therefore,

$$[(g * \psi)(z)]^2 \leq k^2 e^{(\lambda_1+\varepsilon)z} \frac{e^{2\rho_0 \lambda_1^2}}{(1+\lambda_1 \tau_0 c)^{2n}}.$$

Thus

$$\begin{aligned}
& d\psi''(z) - c\psi'(z) - \tau\psi(z) + \beta\tau(g * \psi)(z)e^{-(g*\psi)(z)} \\
\geq\ & ke^{\lambda_1 z} \left[d\lambda_1^2 - c\lambda_1 + (\beta-1)\tau + \beta\tau \left(\frac{e^{\rho_0\lambda_1^2}}{(1+\lambda_1\tau_0 c)^n} - 1 \right) \right] \\
& -kMe^{(\lambda_1+\varepsilon)z} \left[d(\lambda_1+\varepsilon)^2 - c(\lambda_1+\varepsilon) + (\beta-1)\tau \right. \\
& \left. + \left(\frac{e^{\rho_0(\lambda_1+\varepsilon)^2}}{(1+\tau_0(\lambda_1+\varepsilon)c)^n} - 1 \right) \right] + \beta\tau(g*\psi)(z) \left[e^{-(g*\psi)(z)} - 1 \right] \\
\geq\ & -kMe^{(\lambda_1+\varepsilon)z} \Delta_c(\lambda_1+\varepsilon) + \beta\tau(g*\psi)(z) \left[e^{-(g*\psi)(z)} - 1 \right] \\
\geq\ & -kMe^{(\lambda_1+\varepsilon)z} \Delta_c(\lambda_1+\varepsilon) - \beta\tau \left[(g*\psi)(z) \right]^2 \\
\geq\ & -kMe^{(\lambda_1+\varepsilon)z} \Delta_c(\lambda_1+\varepsilon) - \beta\tau k^2 e^{(\lambda_1+\varepsilon)z} \frac{e^{2\rho_0\lambda_1^2}}{(1+\lambda_1\tau_0 c)^{2n}} \\
=\ & -ke^{(\lambda_1+\varepsilon)z} \Delta_c(\lambda_1+\varepsilon) \left[M + \frac{\beta\tau k^2 e^{2\rho_0\lambda_1^2}}{\Delta_c(\lambda_1+\varepsilon)(1+\lambda_1\tau_0 c)^{2n}} \right],
\end{aligned}$$

that is,

$$
\begin{aligned}
& d\psi''(z) - c\psi'(z) - \tau\psi(z) + \beta\tau(g * \psi)(z)e^{-(g*\psi)(z)} \\
\geq \quad & -ke^{(\lambda_1+\varepsilon)z}\Delta_c(\lambda_1+\varepsilon)\left[M + \frac{\beta\tau k^2 e^{2\rho_0\lambda_1^2}}{\Delta_c(\lambda_1+\varepsilon)(1+\lambda_1\tau_0 c)^{2n}}\right]. \qquad (17)
\end{aligned}
$$

It is easy to see that the right side of inequality (17) is positive when

$$
M > -\frac{\beta\tau k^2 e^{2\rho_0\lambda_1^2}}{\Delta_c(\lambda_1+\varepsilon)(1+\lambda_1\tau_0 c)^{2n}}
$$

since $\Delta_c(\lambda_1+\varepsilon) < 0$. This completes the proof.

From Lemmas 3, 5, 6 and Theorem 1, we have the following existence theorem of travelling front solutions of equation (4).

Theorem 2. *If $1 < \beta \leq e$, then there exists $c^* > 0$ such that for every $c > c^*$, equation (4) has a travelling wave front solution connecting the trivial equilibrium $u = 0$ with the positive equilibrium $u = \ln\beta$.*

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