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τ-CENTRALIZERS AND GENERALIZED DERIVATIONS

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Abstract. In this paper, we show that Jordan τ-centralizers and local τ-centralizers are τ-centralizers under certain conditions. We also discuss a new type of generalized derivations associated with Hochschild 2-cocycles and introduce a special local generalized derivation associated with Hochschild 2-cocycles. We prove that if $\mathcal{L}$ is a CDCSL and $\mathcal{M}$ is a dual normal unital Banach alg$\mathcal{L}$-bimodule, then every local generalized derivation of above type from alg$\mathcal{L}$ into $\mathcal{M}$ is a generalized derivation.

1. Introduction

Let $\mathcal{A}$ be an algebra with identity and let τ be an endomorphism of $\mathcal{A}$.

A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a *left* (*right*) *centralizer* of $\mathcal{A}$ if $f(y) = f(1)y$ ($f(y) = yf(1)$) for any $y \in \mathcal{A}$. If f is a left and right centralizer, then it is to call f a *centralizer*. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a *left* (*right*) *Jordan centralizer* of $\mathcal{A}$ if $f(x^2) = f(x)x$ ($f(x^2) = xf(x)$) for any $x \in \mathcal{A}$. f is called a *Jordan centralizer* of $\mathcal{A}$ if $f(xy+yx) = f(x)y+yf(x) = f(y)x+xf(y)$ for any $x, y \in \mathcal{A}$. In [8], Zalar shows that a left Jordan centralizer of a semiprime ring is a left centralizer and each Jordan centralizer of a semiprime ring is a centralizer.

A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a *left* (*right*) τ*-centralizer* of $\mathcal{A}$ if $f(y) = f(1)\tau(y)$ ($f(y) = \tau(y)f(1)$) for any $x, y \in \mathcal{A}$. If f is a left and right τ-centralizer, then it is to call f a τ*-centralizer*. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a *left* (*right*) *Jordan* τ*-centralizer* of $\mathcal{A}$ if $f(x^2) = f(x)\tau(x)$ ($f(x^2) = \tau(x)f(x)$) for any $x \in \mathcal{A}$. f is called a *Jordan* τ*-centralizer* of $\mathcal{A}$ if

$$f(xy + yx) = f(x)\tau(y) + \tau(y)f(x) = f(y)\tau(x) + \tau(x)f(y)$$

for any $x, y \in \mathcal{A}$. Albaş [1] shows that under some conditions, a left Jordan τ-centralizer of a semiprime ring is a left τ-centralizer and each Jordan τ-centralizer of a semiprime ring is a τ-centralizer.

We call f a *local left centralizer* of $\mathcal{A}$ if for each $x \in \mathcal{A}$, there is a left centralizer f_x of $\mathcal{A}$ such that $f(x) = f_x(x)$. Similarly, we can define *local right*

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centralizer and *local centralizer*. In [2], Hadwin studies local centralizers on von Neumann algebras and nest algebras.

Recently, Nakajima introduced the following definitions. Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Let $\alpha : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ be a bilinear mapping. α is called a *Hochschild 2-cocycle* if

$$x\alpha(y, z) - \alpha(xy, z) + \alpha(x, yz) - \alpha(x, y)z = 0. \tag{1}$$

A linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is called a *generalized derivation* if there is a 2-cocycle α such that

$$\delta(xy) = \delta(x)y + x\delta(y) + \alpha(x, y). \tag{2}$$

We denote it by (δ, α). In [7], Nakajima shows that the usual generalized derivation, left centralizer and (σ, τ)-derivation are also generalized derivations in above sense.

The distribution of this paper is as follows.

In Section 2, we prove that if $\mathcal{A}' \cap \mathcal{A} = \mathbb{C}I$, and τ is an epimorphism of $\mathcal{A}$, then each Jordan τ-centralizer of $\mathcal{A}$ is τ-centralizer. And we also show that if $\mathcal{L}$ is a CDCSL on H and τ is an automorphism of alg$\mathcal{L}$, then each Jordan τ-centralizer of alg$\mathcal{L}$ is τ-centralizer.

We introduce the following definition. We call f a *local left τ-centralizer* of $\mathcal{A}$ if for each $x \in \mathcal{A}$, there is a left τ-centralizer f_x of $\mathcal{A}$ such that $f(x) = f_x(x)$. Similarly, we can define *local right τ-centralizer* and *local τ-centralizer*. In Section 3, we generalize some results of [2] to local τ-centralizer. And we show that if $\mathcal{L}$ is a CDCSL on H and τ is an automorphism of alg$\mathcal{L}$, then each local τ-centralizer of alg$\mathcal{L}$ is a τ-centralizer.

In Section 4, we introduce a new type of local generalized derivations and we show that every local generalized derivation of above type from CDCSL into its dual normal unital Banach $\mathcal{A}$-bimodule is a generalized derivation.

The following notations will be used in our paper.

Let X be a complex Banach space with dual X^* and let $B(X)$ be the set of all bounded linear maps from X into itself. Let H be a complex separable Hilbert space.

A *subspace lattice* on X is a collection $\mathcal{L}$ of subspaces of X with $(0), X$ in $\mathcal{L}$ and such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\wedge M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\wedge M_r$ denotes the intersection of $\{M_r\}$ and $\vee M_r$ denotes the closed linear span of $\{M_r\}$. A totally ordered subspace lattice is called a *nest*. For a subspace lattice $\mathcal{L}$, we define alg$\mathcal{L}$ by

$$\text{alg}\mathcal{L} = \{T \in B(H) : TN \subseteq N, \ \forall N \in \mathcal{L}\}.$$

For any $L \subseteq X$, $L^{\perp} = \{f \in X^*, f(x) = 0 \text{ for all } x \in L\}$. Let $x \in X$, $f \in X^*$ be nonzero. The *rank one operator* $x \otimes f$ is defined by $z \to f(z)x$ for any $z \in X$. For any nonzero $x, y \in H$, the operator $x \otimes y$ is defined by $z \to (z, y)x$ for any $z \in H$. If $\mathcal{L}$ is a subspace lattice of X and $E \in \mathcal{L}$, we define

$$E_- = \vee\{F \in \mathcal{L}, F \not\supseteq E\} \text{ and } E_+ = \wedge\{F \in \mathcal{L}, F \not\subseteq E\}.$$

It is well known that $x \otimes f \in \mathrm{alg}\mathcal{L}$ if and only if there is $E \in \mathcal{L}$ such that $x \in E$ and $f \in (E_-)^\perp$ (equivalently, $x \in E_+$ and $f \in E^\perp$).

A subspace lattice $\mathcal{L}$ is said to be *completely distributive* if for every family $\{X_{i,j}\}_{i\in I, j\in J}$ of elements in $\mathcal{L}$,

$$\bigwedge_{i\in I}\bigvee_{j\in J} x_{i,j} = \bigvee_{f\in J_I}\bigwedge_{i\in I} x_{i,f(i)} \text{ and } \bigvee_{i\in I}\bigwedge_{j\in J} x_{i,j} = \bigwedge_{f\in J_I}\bigvee_{i\in I} x_{i,f(i)},$$

where J_I denotes the set of all maps from I into J.

A Hilbert space subspace lattice $\mathcal{L}$ is called a *commutative subspace lattice* (CSL) if it consists of mutually commuting projections. If $\mathcal{L}$ is a commutative subspace lattice, then $\mathrm{alg}\mathcal{L}$ is called a *CSL algebra*. If $\mathcal{L}$ is a completely distributive commutative subspace lattice (CDCSL), then $\mathrm{alg}\mathcal{L}$ is called a *CDCSL algebra*.

Given a subspace lattice $\mathcal{L}$ on X, put

$$\mathcal{J}_{\mathcal{L}} = \{K \in \mathcal{L} : K \neq \{0\} \text{ and } K_- \neq X\}.$$

Call $\mathcal{L}$ a *$\mathcal{J}$-subspace lattice* on X if it satisfies the following conditions:

(1) $\vee\{K : K \in \mathcal{J}_{\mathcal{L}}\} = X$;

(2) $\wedge\{K_- : K \in \mathcal{J}_{\mathcal{L}}\} = \{0\}$;

(3) $K \vee K_- = X$ for any $K \in \mathcal{J}_{\mathcal{L}}$;

(4) $K \wedge K_- = 0$ for any $K \in \mathcal{J}_{\mathcal{L}}$.

In this paper, we suppose that $\mathcal{A}$ is a unital algebra and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule.

2. Jordan τ-centralizers

Since the proof of the following lemma is analogous to that of [1, Lemma 3], we omit it.

Lemma 2.1. *Let f be a left Jordan τ-centralizer of an algebra $\mathcal{A}$. Then*

(1) *$f(xy + yx) = f(x)\tau(y) + f(y)\tau(x)$ for all $x, y \in \mathcal{A}$,*

(2) *$f(xyx) = f(x)\tau(y)\tau(x)$ for all $x, y \in \mathcal{A}$,*

(3) *$f(xyz + zyx) = f(x)\tau(y)\tau(z) + f(z)\tau(y)\tau(x)$ for all $x, y \in \mathcal{A}$,*

(4) *$D(x, y) = -D(y, x)$, where $D(x, y) = f(xy) - f(x)\tau(y)$ for all $x, y \in \mathcal{A}$.*

Lemma 2.2. *Each left Jordan τ-centralizer f of a unital algebra $\mathcal{A}$ is a left τ-centralizer.*

Proof. Let I be the identity in $\mathcal{A}$. Since τ is an endomorphism of $\mathcal{A}$, it follows that $\tau(I) = I$. Let $D(x, y) = f(xy) - f(x)\tau(y)$ for any $x, y \in \mathcal{A}$. So $D(x, I) = f(xI) - f(x)\tau(I) = 0$ for all $x \in \mathcal{A}$. By Lemma 2.1(4), we have that $D(I, x) = -D(x, I) = 0$ for all $x \in \mathcal{A}$. Thus

$$f(x) = f(Ix) = f(I)\tau(x) \tag{3}$$

for all $x \in \mathcal{A}$. □

In what follows, we suppose that $\mathcal{A}$ is a unital subalgebra of $B(X)$ such that $\mathcal{A}' \cap \mathcal{A} = \mathbb{C}I$, where I is the identity in $\mathcal{A}$, and τ is an epimorphism of $\mathcal{A}$. And we denote by $Z = \mathcal{A}' \cap \mathcal{A} = \mathbb{C}I$ the center of $\mathcal{A}$.

Lemma 2.3. *Let a be a fixed element in $\mathcal{A}$. If $a\tau(x) - \tau(x)a \in Z$ for all $x \in \mathcal{A}$, then $a \in Z$.*

Proof. Since $a\tau(x) - \tau(x)a \in \mathbb{C}I$, by [4, Question 182], it follows that $a\tau(x) - \tau(x)a = 0$. Since τ is surjective, we have that $a \in Z$. □

Lemma 2.4. *Let a be a fixed element in $\mathcal{A}$, and $f(x) = a\tau(x) + \tau(x)a$ for any $x \in \mathcal{A}$. If f is a Jordan τ-centralizer of $\mathcal{A}$, then $a \in Z$.*

Proof. Since f is a Jordan τ-centralizer of $\mathcal{A}$, it follows that $f(xy + yx) = f(x)\tau(y) + \tau(y)f(x)$ for all $x, y \in \mathcal{A}$. Hence

$$\begin{aligned} a\tau(xy + yx) + \tau(xy + yx)a &= (a\tau(x) + \tau(x)a)\tau(y) + \tau(y)(a\tau(x) + \tau(x)a), \\ a\tau(y)\tau(x) + \tau(x)\tau(y)a &= \tau(x)a\tau(y) + \tau(y)a\tau(x), \\ \tau(x)(a\tau(y) - \tau(y)a) &= (a\tau(y) - \tau(y)a)\tau(x) \end{aligned}$$

for all $x, y \in \mathcal{A}$. Since τ is surjective, we have that $a\tau(y) - \tau(y)a \in Z$. Hence $a \in Z$ by Lemma 2.3. □

Lemma 2.5. *Every Jordan τ-centralizer f of $\mathcal{A}$ maps Z into Z.*

Proof. For any $c \in Z$, let $a = f(c)$. Since f is a Jordan τ-centralizer of $\mathcal{A}$, we have that

$$2f(cx) = f(cx + xc) = f(c)\tau(x) + \tau(x)f(c) = a\tau(x) + \tau(x)a$$

for all $x \in \mathcal{A}$. Let $g(x) = 2f(cx)$. Then

$$\begin{aligned} g(xy + yx) &= 2f(c(xy + yx)) = 2f(cxy + ycx) \\ &= 2(f(cx)\tau(y) + \tau(y)f(cx)) = g(x)\tau(y) + \tau(y)g(x), \\ g(xy + yx) &= 2f(c(xy + yx)) = 2f(xcy + cyx) \\ &= 2(f(cy)\tau(x) + \tau(x)f(cy)) = g(y)\tau(x) + \tau(x)g(y) \end{aligned}$$

for any $x, y \in \mathcal{A}$. Thus, we have that g is a Jordan τ-centralizer of $\mathcal{A}$. By Lemma 2.4, we have $a = f(c) \in Z$ for all $c \in Z$. □

Theorem 2.6. *Each Jordan τ-centralizer f of $\mathcal{A}$ is τ-centralizer.*

Proof. By Lemma 2.5, we have that

$$2f(x) = f(xI + Ix) = f(I)\tau(x) + \tau(x)f(I) = 2f(I)\tau(x) = 2\tau(x)f(I)$$

for all $x \in \mathcal{A}$. Thus

$$f(x) = f(I)\tau(x) = \tau(x)f(I)$$

for all $x \in \mathcal{A}$. □

Corollary 2.7. *Let $\mathcal{L}$ be a nest on X and let τ be an epimorphism of $alg\mathcal{L}$. Then each Jordan τ-centralizer of $alg\mathcal{L}$ is τ-centralizer.*

Proof. Since $\mathcal{L}$ is a nest on X, we have that $(alg\mathcal{L})' = \mathbb{C}I$. By Theorem 2.6, we conclude the proof. □

Definition 2.8. Let $\mathcal{L}$ be a subspace lattice on X and $L \in \mathcal{L}$. L is said to be a comparable element of $\mathcal{L}$ if for any $M \in \mathcal{L}$, $L \subseteq M$ or $L \supset M$.

Lemma 2.9 ([6, Proposition 2.9])**.** *Suppose that $\mathcal{L}$ is a subspace lattice on X with a nontrivial comparable element M. If there is a subspace N of X such that $X = M \oplus N$, then $(\mathrm{alg}\mathcal{L})' = \mathbb{C}I$.*

By Theorem 2.6 and Lemma 2.9, we can show the following result.

Corollary 2.10. *Let $\mathcal{L}$ be a subspace lattice on X with a nontrivial comparable element M. If there is a subspace N of X such that $X = M \oplus N$ and τ is a surjective endomorphism of* alg$\mathcal{L}$*, then each Jordan τ-centralizer of* alg$\mathcal{L}$ *is a τ-centralizer.*

Remark 2.11. Let $\mathcal{L} = \{(0), K, L, M, X\}$ be a pentagonal lattice on X. Then $(\mathrm{alg}\mathcal{L})'$ is trivial. Hence, by Theorem 2.6, we have that each Jordan τ-centralizer of alg$\mathcal{L}$ is τ-centralizer.

In the following, we give a result of an algebra $\mathcal{A}$ such that the center of $\mathcal{A}$ $\neq \mathbb{C}I$.

Lemma 2.12. *Suppose that $\mathcal{L}$ is a CDCSL on H and τ is an automorphism of* alg$\mathcal{L}$*. Then every Jordan τ-centralizer f of* alg$\mathcal{L}$ *maps I into the center Z.*

Proof. Let $e = e^2 \in \mathrm{alg}\mathcal{L}$. Since τ is an automorphism of alg$\mathcal{L}$, it follows that $P = \tau^{-1}(e)$ such that $P = P^2 \in \mathrm{alg}\mathcal{L}$. Since f is a Jordan τ-centralizer, it follows that

$$2f(P) = f(PI + IP) = f(I)\tau(P) + \tau(P)f(I), \tag{4}$$

$$2f(P) = f(P^2 + P^2) = f(P)\tau(P) + \tau(P)f(P). \tag{5}$$

Thus

$$\tau(P)f(P)\tau(P) = \tau(P)f(I)\tau(P), \tag{6}$$

$$f(P)\tau(P) = \tau(P)f(P) = \tau(P)f(P)\tau(P). \tag{7}$$

By (4), (5), (6), (7), we have that

$$f(I)\tau(P) = 2f(P)\tau(P) - \tau(P)f(I)\tau(P) = \tau(P)f(P)\tau(P),$$
$$\tau(P)f(I) = 2\tau(P)f(P) - \tau(P)f(I)\tau(P) = \tau(P)f(P)\tau(P).$$

It follows that $f(I)\tau(P) = \tau(P)f(I)$. Thus $f(I)e = ef(I)$ for any $e = e^2 \in \mathrm{alg}\mathcal{L}$. By [3, Lemma 2.3], for any $x \otimes y \in \mathrm{alg}\mathcal{L}$, $x \otimes y \in \mathrm{span}\{e \in \mathrm{alg}\mathcal{L}, e = e^2\}$. We have that

$$f(I)(x \otimes y) = (x \otimes y)f(I).$$

Let $\mathcal{R}_1(\mathrm{alg}\mathcal{L})$ be the algebra generated by all of rank one operators of alg$\mathcal{L}$. By [5, Theorem 3],

$$\overline{\mathcal{R}_1(\mathrm{alg}\mathcal{L})}^{SOT} = \mathrm{alg}\mathcal{L}.$$

It follows that $f(I)T = Tf(I)$ for any T in alg$\mathcal{L}$. So $f(I) \in Z$. □

Theorem 2.13. *If $\mathcal{L}$ is a CDCSL on H and τ is an automorphism of* alg$\mathcal{L}$*, then each Jordan τ-centralizer of* alg$\mathcal{L}$ *is τ-centralizer.*

Proof. Let f be a Jordan τ-centralizer of alg$\mathcal{L}$. We have that

$$2f(x) = f(Ix + xI) = f(I)\tau(x) + \tau(x)f(I).$$

By Lemma 2.12, $f(I) \in Z$, it follows that $f(I)\tau(x) = \tau(x)f(I)$. Thus $f(x) = f(I)\tau(x) = \tau(x)f(I)$. □

3. Local τ-centralizer

In this section, we suppose that $\mathcal{R}$ is a commutative ring with identity, $\mathcal{A}$ is an algebra with identity over $\mathcal{R}$, and τ is an endomorphism of $\mathcal{A}$.

Proposition 3.1. *Suppose $\varphi : \mathcal{A} \to \mathcal{A}$ is a linear mapping and $\tau : \mathcal{A} \to \mathcal{A}$ is an endomorphism such that for any $e = e^2 \in \mathcal{A}$, $\varphi(e) \in \mathcal{A}\tau(e)$ (respectively, $\varphi(e) \in \tau(e)\mathcal{A}$). Then $\varphi(a) = \varphi(I)\tau(a)$ (respectively, $\varphi(a) = \tau(a)\varphi(I)$) for any a in the linear span of all idempotents in $\mathcal{A}$.*

Proof. Suppose that $e = e^2 \in \mathcal{A}$. Since $I - e = (I - e)^2 \in \mathcal{A}$, it follows that there are c, d in $\mathcal{A}$ such that $\varphi(e) = c\tau(e)$ and $\varphi(I - e) = d\tau(I - e)$. Hence $\varphi(I) = \varphi(e) + \varphi(I - e) = c\tau(e) + d\tau(I - e)$. Multiplying by $\tau(e)$, we have that $\varphi(I)\tau(e) = c\tau(e)\tau(e) + d\tau(I - e)\tau(e) = c\tau(e^2) + d\tau((I - e)e) = c\tau(e) = \varphi(e)$. Thus $\varphi(a) = \varphi(I)\tau(a)$ for any a in span$\{e \in \mathcal{A},\ e = e^2\}$.

The proof of the other case is similar. □

Proposition 3.2. *Suppose that $\varphi : \mathcal{A} \to \mathcal{A}$ is a linear mapping and $\tau : \mathcal{A} \to \mathcal{A}$ is an endomorphism such that for any $e = e^2 \in \mathcal{A}$, $\varphi(\mathcal{A}e) \subseteq \mathcal{A}\tau(e)$ (respectively, $\varphi(e\mathcal{A}) \subseteq \tau(e)\mathcal{A}$). Then $\varphi(a) = \varphi(I)\tau(a)$ (respectively, $\varphi(a) = \tau(a)\varphi(I)$) for any a in the algebra generated by all idempotents in $\mathcal{A}$.*

Proof. We first show that for any idempotents $e_1, \ldots, e_n$ in $\mathcal{A}$,

$$\varphi(e_1 \cdots e_n) = \varphi(I)\tau(e_1 \cdots e_n). \tag{8}$$

If $n = 1$, by Proposition 3.1, $\varphi(e_1) = \varphi(I)\tau(e_1)$.

Suppose that if $n = k$, (8) is true. For $n = k + 1$, by assumption, there are $c,\ d$ in $\mathcal{A}$ such that

$$\varphi(e_1 \cdots e_k e_{k+1}) = c\tau(e_{k+1}),\ \varphi(e_1 \cdots e_k(I - e_{k+1})) = d\tau(I - e_{k+1}).$$

Hence

$$\varphi(e_1 \cdots e_k) = c\tau(e_{k+1}) + d\tau(I - e_{k+1}).$$

Multiplying by $\tau(e_{k+1})$, we have that

$$\varphi(e_1 \cdots e_k)\tau(e_{k+1}) = c\tau(e_{k+1}) = \varphi(e_1 \cdots e_{k+1}),$$

and therefore

$$\varphi(e_1 \cdots e_{k+1}) = \varphi(e_1 \cdots e_k)\tau(e_{k+1}) = \varphi(I)\tau(e_1 \cdots e_k)\tau(e_{k+1}) = \varphi(I)\tau(e_1 \cdots e_{k+1}).$$

Thus $\varphi(a) = \varphi(I)\tau(a)$ for any a in the algebra generated by all idempotents in $\mathcal{A}$. $\square$

We call a left (right) ideal $\mathcal{T}$ of $\mathcal{A}$ a separating left (right) set, if for any a in $\mathcal{A}$, $a\mathcal{T} = \{0\}$ ($\mathcal{T}a = \{0\}$) implies $a = 0$. If $\mathcal{T}$ is both a separating left set a separating right set then we call it a separating set.

Proposition 3.3. *Suppose $\mathcal{A}$ has a left (right) ideal $\mathcal{T}$ that is contained in the algebra generated by all idempotents in $\mathcal{A}$. If $\varphi : \mathcal{A} \to \mathcal{A}$ is a linear mapping and $\tau : \mathcal{A} \to \mathcal{A}$ is an endomorphism of $\mathcal{A}$ such that $\tau(\mathcal{T})$ is a separating left (right) set of $\mathcal{A}$ and $\varphi(\mathcal{A}e) \subseteq \mathcal{A}\tau(e)$ (respectively, $\varphi(e\mathcal{A}) \subseteq \tau(e)\mathcal{A}$) for any $e = e^2 \in \mathcal{A}$. Then $\varphi(a) = \varphi(I)\tau(a)$ (respectively, $\varphi(a) = \tau(a)\varphi(I)$) for any $a \in \mathcal{A}$.*

Proof. We only prove the case that $\mathcal{T}$ is a left ideal and $\tau(\mathcal{T})$ is a separating left set of $\mathcal{A}$, the other case is similar.

We first show that for any idempotents $e_1 \cdots e_n$ in $\mathcal{A}$, a in $\mathcal{A}$,

$$\varphi(a)\tau(e_1 \cdots e_n) = \varphi(ae_1 \cdots e_n). \tag{9}$$

If $n = 1$, since $\varphi(\mathcal{A}e_1) \subseteq \mathcal{A}\tau(e_1)$, $\varphi(\mathcal{A}(I - e_1)) \subseteq \mathcal{A}\tau(I - e_1)$, we know that there are c_1 and d_1 in $\mathcal{A}$ such that $\varphi(ae_1) = c_1\tau(e_1)$, $\varphi(a(I-e_1)) = d_1\tau(I-e_1)$. So

$$\varphi(a) = \varphi(ae_1) + \varphi(a(I - e_1)) = c_1\tau(e_1) + d_1\tau(I - e_1).$$

Thus $\varphi(a)\tau(e_1) = c_1\tau(e_1) = \varphi(ae_1)$.

Suppose that if $n = k$, (9) is true. For $n = k + 1$, by assumption, there are c_{k+1}, d_{k+1} in $\mathcal{A}$ such that

$$\varphi(ae_1 \cdots e_k e_{k+1}) = c_{k+1}\tau(e_{k+1}),\ \varphi(ae_1 \cdots e_k(I - e_{k+1})) = d_{k+1}\tau(e_{k+1}),$$

and therefore

$$\begin{aligned}\varphi(ae_1 \cdots e_k) &= \varphi(ae_1 \cdots e_k e_{k+1}) + \varphi(ae_1 \cdots e_k(I - e_{k+1})) \\ &= c_{k+1}\tau(e_{k+1}) + d_{k+1}\tau(I - e_{k+1}).\end{aligned}$$

It follows that

$$\varphi(ae_1 \cdots e_k)\tau(e_{k+1}) = c_{k+1}\tau(e_{k+1}) = \varphi(ae_1 \cdots e_{k+1}).$$

Thus

$$\varphi(ae_1 \cdots e_{k+1}) = \varphi(ae_1 \cdots e_k)\tau(e_{k+1}) = \varphi(a)\tau(e_1 \cdots e_{k+1}).$$

Hence $\varphi(at) = \varphi(a)\tau(t)$, where t in the algebra generated by idempotents in $\mathcal{A}$. In particular, $\varphi(at) = \varphi(a)\tau(t)$ for any a in $\mathcal{A}$, t in $\mathcal{T}$. Since $\mathcal{T}$ is a left ideal, it follows that

$$\varphi(at) = \varphi(I)\tau(at) = \varphi(I)\tau(a)\tau(t).$$

Thus $(\varphi(a) - \varphi(I)\tau(a))\tau(t) = 0$. Since $\tau(\mathcal{T})$ is a separating left set, it follows that $\varphi(a) = \varphi(I)\tau(a)$ for any $a \in \mathcal{A}$. $\square$

Corollary 3.4. *Suppose that $\mathcal{A}$ has a separating left (right) set $\mathcal{T}$ that is contained in the algebra generated by all idempotents in $\mathcal{A}$. If $\varphi : \mathcal{A} \to \mathcal{A}$ is a linear mapping and $\tau : \mathcal{A} \to \mathcal{A}$ is an automorphism such that for any $e = e^2 \in \mathcal{A}$, $\varphi(\mathcal{A}e) \subseteq \mathcal{A}\tau(e)$ (respectively, $\varphi(e\mathcal{A}) \subseteq \tau(e)\mathcal{A}$), then $\varphi(a) = \varphi(I)\tau(a)$ (respectively, $\varphi(a) = \tau(a)\varphi(I)$) for any $a \in \mathcal{A}$.*

Corollary 3.5. *Suppose that a subspace lattice $\mathcal{L}$ satisfies one of the following conditions:*
(1) *$\mathcal{L}$ is a $\mathcal{J}$-subspace lattice on a Banach space X,*
(2) *$\mathcal{L}$ is CDCSL on a separable Hilbert space H,*
(3) *$\mathcal{L}$ satisfies $0_+ \neq \{0\}$, $X_- \neq X$,*
and τ is an automorphism of alg$\mathcal{L}$.
If $\varphi : \text{alg}\mathcal{L} \to \text{alg}\mathcal{L}$ is a local τ-centralizer, then φ is a τ-centralizer.

Proof. Case 1. $\mathcal{L}$ satisfies Condition (1). Let $\mathcal{I} = \text{span}\{T : T \in \text{alg}\mathcal{L}, \text{rank } T = 1\}$. Then $\mathcal{I}$ is an ideal of alg$\mathcal{L}$. By [3, Lemma 2.10], $\mathcal{I}$ is contained in the linear span of the idempotents in alg$\mathcal{L}$. By [3, Lemma 2.11], $\mathcal{I}$ is a separating set of alg$\mathcal{L}$.

Case 2. $\mathcal{L}$ satisfies Condition (2). Let $\mathcal{I} = \text{span}\{T : T \in \text{alg}\mathcal{L}, \text{rank } T = 1\}$. Then $\mathcal{I}$ is an ideal of alg$\mathcal{L}$. By [3, Lemma 2.3], $\mathcal{I}$ is contained in the linear span of the idempotents in alg$\mathcal{L}$. It follows from [5, Theorem 3] that $\mathcal{I}$ is a separating set of alg$\mathcal{L}$.

Case 3. $\mathcal{L}$ satisfies Condition (3). Let $\mathcal{I} = \text{span}\{x \otimes f_0,\ x_0 \otimes f : x \in X, f_0 \in (X_-)^\perp, x_0 \in 0_+, f \in X^*\}$. Then $\mathcal{I}$ is an ideal of alg$\mathcal{L}$ and $\mathcal{I}$ is a separating set of alg$\mathcal{L}$. For any $x \in X$, $0 \neq f_0 \in (X_-)^\perp$, then $x \otimes f_0 \in \text{alg}\mathcal{L}$. If $f_0(x) \neq 0$, then $\frac{1}{f_0(x)} x \otimes f_0$ is an idempotent in $\mathcal{I}$. If $f_0(x) = 0$, choose $x_1 \in X$ such that $f_0(x_1) = 1$, we have that $x \otimes f_0 = \frac{1}{2}(x_1 + x) \otimes f_0 - \frac{1}{2}(x_1 - x) \otimes f_0$, both $(x_1 + x) \otimes f_0$ and $(x_1 - x) \otimes f_0$ are idempotents. The case of $x_0 \otimes f$ is similarly. Thus $\mathcal{I}$ is contained in the algebra generated by the idempotents in alg$\mathcal{L}$.

Thus, by Cases 1, 2 and 3, if $\mathcal{L}$ satisfies one of above conditions, alg$\mathcal{L}$ has an ideal $\mathcal{I}$ which is contained in a subalgebra of alg$\mathcal{L}$ generated by its idempotents and $\mathcal{I}$ separates alg$\mathcal{L}$.

Since φ is a local τ-centralizer, we have that for each x in alg$\mathcal{L}$, there is a τ-centralizer φ_x such that $\varphi(x) = \varphi_x(x)$. It follows that for any $e = e^2 \in \text{alg}\mathcal{L}, a \in \text{alg}\mathcal{L}$,

$$\varphi(ae) = \varphi_{ae}(ae) = \varphi_{ae}(a)\tau(e) \in (\text{alg}\mathcal{L})\tau(e).$$

By Corollary 3.4, $\varphi(a) = \varphi(I)\tau(a)$ for any $a \in \text{alg}\mathcal{L}$. Thus φ is a left τ-centralizer. Similarly, φ is also a right τ-centralizer. Hence φ is a τ-centralizer. □

4. Generalized derivations associate with Hochschild 2-cocycles

In this section, we suppose that $\mathcal{A}$ is a unital algebra and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule.

Motivated by Nakajima [7], we introduce a new type of local generalized derivation. A map (δ, α) is called a *local generalized derivation* if for any $x \in \mathcal{A}$, there is a generalized derivation (δ_x, α) such that $\delta(x) = \delta_x(x)$. If $\alpha = 0$, then δ is a local derivation.

Lemma 4.1. *Let δ be a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ and $\alpha : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ be a Hochschild 2-cocycle bilinear mapping. Then the following relations are equivalent*

(i) $P^\perp \delta(PAQ)Q^\perp = P^\perp \alpha(PA, Q)Q^\perp$,

(ii) $\delta(PAQ) = \delta(PA)Q + P\delta(AQ) - P\delta(A)Q + \alpha(PA, Q) - P\alpha(A, Q)$, *where* $P = P^2, Q = Q^2, A \in \mathcal{A}$.

Proof. It is obvious that (ii) implies (i).

Suppose that (i) is true. Let $h(x, y) = \delta(xy) - \alpha(x, y)$. Then

$$P^\perp h(PA, Q)Q^\perp = 0,$$

$$Ph(A, Q)Q^\perp = Ph(PA, Q)Q^\perp = (I - P^\perp)h(PA, Q)Q^\perp = h(PA, Q)Q^\perp.$$

Therefore, we have that

$$\begin{aligned} h(PA, Q) - Ph(A, Q) &= (h(PA, Q) - Ph(A, Q))Q \\ &= h(PA, I)Q - h(PA, Q^\perp)Q - Ph(A, Q)Q \\ &= h(PA, I)Q - Ph(A, Q^\perp)Q - Ph(A, Q)Q \\ &= h(PA, I)Q - Ph(A, I)Q. \end{aligned}$$

Then

$$\begin{aligned} & \delta(PAQ) - \alpha(PA, Q) - P\delta(AQ) + P\alpha(A, Q) \\ = & \ \delta(PA)Q - \alpha(PA, I)Q - P\delta(A)Q + P\alpha(A, I)Q. \end{aligned}$$

Thus

$$\begin{aligned} \delta(PAQ) &= P\delta(AQ) + \delta(PA)Q - P\delta(A)Q + \alpha(PA, Q) - P\alpha(A, Q) \\ & \quad -\alpha(PA, I)Q + P\alpha(A, I)Q. \end{aligned}$$

Since α is Hochschild 2-cocycle, we have that

$$P\alpha(A, I) - \alpha(PA, I) + \alpha(P, A) - \alpha(P, A) = 0.$$

Hence

$$\delta(PAQ) = P\delta(AQ) + \delta(PA)Q - P\delta(A)Q + \alpha(PA, Q) - P\alpha(A, Q). \qquad \square$$

Let δ be a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ and $\alpha : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ be a Hochschild 2-cocycle bilinear mapping. We say that (δ, α) satisfies the condition $(*)$ if

$$\delta(PAQ) = P\delta(AQ) + \delta(PA)Q - P\delta(A)Q + \alpha(PA, Q) - P\alpha(A, Q)$$

and $\delta(I) = -\alpha(I, I)$ hold for each $A \in \mathcal{A}$ and any idempotents P, Q in $\mathcal{A}$.

Lemma 4.2. *Suppose that δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ and $\alpha : \mathcal{A}\times\mathcal{A} \to \mathcal{M}$ is a Hochschild 2-cocycle bilinear mapping satisfying the condition $(*)$. Then*

$$\begin{aligned}\delta(P_1\cdots P_nAQ_1\cdots Q_m) = {} & \delta(P_1\cdots P_nA)Q_1\cdots Q_m + P_1\cdots P_n\delta(AQ_1\cdots Q_m)\\ & - P_1\cdots P_n\delta(A)Q_1\cdots Q_m + \alpha(P_1\cdots P_nA, Q_1\cdots Q_m)\\ & - P_1\cdots P_n\alpha(A, Q_1\cdots Q_m)\end{aligned} \tag{10}$$

for any idempotents $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ in $\mathcal{A}$ and any A in $\mathcal{A}$.

Proof. We first show that for any positive integer n,

$$\begin{aligned}\delta(P_1\cdots P_nAQ) \quad = \quad & \delta(P_1\cdots P_nA)Q + P_1\cdots P_n\delta(AQ) - P_1\cdots P_n\delta(A)Q\\ & +\alpha(P_1\cdots P_nA, Q) - P_1\cdots P_n\alpha(A, Q).\end{aligned} \tag{11}$$

If $n = 1$, by the condition $(*)$, (11) is obvious.

Suppose that if $n = k$, (11) is true. For $n = k + 1$, by the condition $(*)$, it follows

$$\begin{aligned}& \delta(P_1\cdots P_{k+1}AQ)\\ = \quad & \delta(P_1\cdots P_{k+1}A)Q + P_1\delta(P_2\cdots P_{k+1}AQ) - P_1\delta(P_2\cdots P_{k+1}A)Q\\ & +\alpha(P_1\cdots P_{k+1}A, Q) - P_1\alpha(P_2\cdots P_{k+1}A, Q)\\ = \quad & \delta(P_1\cdots P_{k+1}A)Q + P_1(P_2\cdots P_{k+1}\delta(AQ) - P_2\cdots P_{k+1}\delta(A)Q\\ & -P_2\cdots P_{k+1}\alpha(A, Q)) + \alpha(P_1\cdots P_{k+1}A, Q)\\ = \quad & \delta(P_1\cdots P_{k+1}A)Q + P_1\cdots P_{k+1}\delta(AQ) - P_1\cdots P_{k+1}\delta(A)Q\\ & -P_1\cdots P_{k+1}\alpha(A, Q) + \alpha(P_1\cdots P_{k+1}A, Q).\end{aligned}$$

Now we show that (10) is true.

If $m = 1$, by (11), we have that (10) is true.

Suppose that if $m = k$, (10) is true. For $m = k + 1$, by the condition $(*)$ and (11), we have

$$\begin{aligned}& \delta(P_1\cdots P_nAQ_1\cdots Q_{k+1})\\ = \quad & \delta(P_1\cdots P_nAQ_1\cdots Q_k)Q_{k+1} + P_1\cdots P_n\delta(AQ_1\cdots Q_{k+1})\\ & -P_1\cdots P_n\delta(AQ_1\cdots Q_k)Q_{k+1} + \alpha(P_1\cdots P_nAQ_1\cdots Q_k, Q_{k+1})\\ & -P_1\cdots P_n\alpha(AQ_1\cdots Q_k, Q_k + 1)\\ = \quad & \delta(P_1\cdots P_nA)Q_1\cdots Q_{k+1} + P_1\cdots P_n\delta(AQ_1\cdots Q_{k+1})\\ & -P_1\cdots P_n\delta(A)Q_1\cdots Q_{k+1} + \alpha(P_1\cdots P_nA, Q_1\cdots Q_k)Q_{k+1}\\ & +\alpha(P_1\cdots P_nAQ_1\cdots Q_k, Q_{k+1}) - P_1\cdots P_n(\alpha(AQ_1\cdots Q_k, Q_{k+1})\\ & +\alpha(A, Q_1\cdots Q_k)Q_{k+1})\\ = \quad & \delta(P_1\cdots P_nA)Q_1\cdots Q_{k+1} + P_1\cdots P_n\delta(AQ_1\cdots Q_{k+1})\\ & -P_1\cdots P_n\delta(A)Q_1\cdots Q_{k+1} + P_1\cdots P_nA\alpha(Q_1\cdots Q_k, Q_{k+1})\\ & +\alpha(P_1\cdots P_nA, Q_1\cdots Q_{k+1}) - P_1\cdots P_n(A\alpha(Q_1\cdots Q_k, Q_{k+1})\\ & +\alpha(A, Q_1\cdots Q_{k+1}))\end{aligned}$$

$$\begin{aligned} &= \delta(P_1\cdots P_nA)Q_1\cdots Q_{k+1}+P_1\cdots P_n\delta(AQ_1\cdots Q_{k+1}) \\ &\quad -P_1\cdots P_n\delta(A)Q_1\cdots Q_{k+1}+\alpha(P_1\cdots P_nA,Q_1\cdots Q_{k+1}) \\ &\quad -P_1\cdots P_n\alpha(A,Q_1\cdots Q_{k+1}). \end{aligned}$$ □

Let $\mathcal{I}$ be an ideal of $\mathcal{A}$. We say that $\mathcal{I}$ is a *separating set* of $\mathcal{M}$ if for any $m,n\in\mathcal{M}$, $m\mathcal{I}=\{0\}$ implies $m=0$ and $\mathcal{I}n=\{0\}$ implies $n=0$.

Theorem 4.3. *Let $\mathcal{I}$ be a separating set of $\mathcal{M}$. Suppose that $\mathcal{I}$ is contained in the algebra generated by the idempotents in $\mathcal{A}$. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ and $\alpha:\mathcal{A}\times\mathcal{A}\to\mathcal{M}$ is a Hochschild 2-cocycle bilinear mapping satisfying the condition $(*)$, then (δ,α) is a generalized derivation.*

Proof. Since $\mathcal{I}$ is contained in the algebra generated by the idempotents in $\mathcal{A}$, by Lemma 4.2, for any S and T in $\mathcal{I}$,

$$\begin{aligned} \delta(ST) &= \delta(S)T+S\delta(T)-S\delta(I)T+\alpha(S,T)-S\alpha(I,T) \\ &= \delta(S)T+S\delta(T)+\alpha(S,T)+S\alpha(I,I)T-S\alpha(I,T) \\ &= \delta(S)T+S\delta(T)+\alpha(S,T). \end{aligned}$$

Let A belongs to $\mathcal{A}$. Since $\mathcal{I}$ is an ideal of $\mathcal{A}$, it follows that

$$\delta(SAT)=\delta(SA)T+SA\delta(T)+\alpha(SA,T).$$

By Lemma 4.2, we have that

$$\delta(SAT)=\delta(SA)T+S\delta(AT)-S\delta(A)T+\alpha(SA,T)-S\alpha(A,T).$$

Thus

$$S\delta(AT)=SA\delta(T)+S\delta(A)T+S\alpha(A,T). \tag{12}$$

Since $\mathcal{I}$ is a separating set of $\mathcal{M}$, by (12), it follows that

$$\delta(AT)=A\delta(T)+\delta(A)T+\alpha(A,T). \tag{13}$$

For any $A,B\in\mathcal{A},T\in\mathcal{I}$, by (13),

$$\begin{aligned} \delta(BAT) &= BA\delta(T)+\delta(BA)T+\alpha(BA,T), \\ \delta(BAT) &= B\delta(AT)+\delta(B)AT+\alpha(B,AT) \\ &= B\delta(A)T+BA\delta(T)+B\alpha(A,T)+\delta(B)AT+\alpha(B,AT). \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \delta(BA)T &= B\delta(A)T+\delta(B)AT+B\alpha(A,T)-\alpha(BA,T)+\alpha(B,AT) \\ &= B\delta(A)T+\delta(B)AT+\alpha(B,A)T. \end{aligned}$$

Since $\mathcal{I}$ is a separating set of $\mathcal{M}$, it follows that $\delta(BA)=B\delta(A)+\delta(B)A+\alpha(B,A)$. □

Corollary 4.4. *Let $\mathcal{I}$ be a separating set of $\mathcal{M}$. Suppose that $\mathcal{I}$ is contained in the algebra generated by idempotents in $\mathcal{A}$. If (δ,α) is a local generalized derivation from $\mathcal{A}$ into $\mathcal{M}$, then (δ,α) is a generalized derivation.*

Proof. Since (δ, α) is a local generalized derivation, we have that

$$\begin{aligned} P^{\perp}\delta(PAQ)Q^{\perp} &= P^{\perp}\delta_{PAQ}(PAQ)Q^{\perp} \\ &= P^{\perp}(\delta_{PAQ}(PA)Q + PA\delta_{PAQ}(Q) + \alpha(PA, Q))Q^{\perp} \\ &= P^{\perp}\alpha(PA, Q)Q^{\perp} \end{aligned}$$

for each $A \in \mathcal{A}$ and any idempotents P, Q in $\mathcal{A}$. And

$$\delta(I) = \delta_I(I)I + I\delta_I(I) + \alpha(I, I) = 2\delta(I) + \alpha(I, I).$$

Thus $\delta(I) = -\alpha(I, I)$. By Lemma 4.1, δ satisfies the condition $(*)$. By Theorem 4.3, (δ, α) is a generalized derivation. □

Let $\mathcal{A}$ be an ultraweakly closed subalgebra of $B(H)$. The Banach space $\mathcal{M}$ is said to be a dual normal Banach $\mathcal{A}$-bimodule if $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule, $\mathcal{M}$ is a dual space, and for any $m \in \mathcal{M}$, the maps $\mathcal{A} \ni a \to am$ and $\mathcal{A} \ni a \to ma$ are ultraweak to weak* continuous.

Corollary 4.5. *Let $\mathcal{L}$ be a CDCSL on a complex separable Hilbert space H. If δ is a linear mapping from* alg$\mathcal{L}$ *into a dual normal unital Banach* alg$\mathcal{L}$-*bimodule $\mathcal{M}$ and $\alpha : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is a Hochschild 2-cocycle bilinear mapping satisfying condition $(*)$, then (δ, α) is a generalized derivation.*

Proof. Let $\mathcal{I} = \text{span}\{T : T \in \text{alg}\mathcal{L}, \text{rank}T = 1\}$. Then $\mathcal{I}$ is an ideal of alg$\mathcal{L}$. By [3, Lemma 2.3], $\mathcal{I}$ is contained in the linear span of the idempotents in alg$\mathcal{L}$. By [5, Theorem 3], $\mathcal{I}$ is a separating set of $\mathcal{M}$. By Theorem 4.3, (δ, α) is a generalized derivation. □

Corollary 4.6. *Let $\mathcal{L}$ be a CDCSL on a complex separable Hilbert space H. If (δ, α) is a local generalized derivation from alg$\mathcal{L}$ into a dual normal unital Banach* alg$\mathcal{L}$-*bimodule $\mathcal{M}$, then (δ, α) is a generalized derivation.*

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