

# A Method of Evaluating Profitability and Risk of Multiple Investments Applying Internal Rate of Return

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**Abstract.** In today's uncertain economic environment, economic risk is inherent in making large investments on manufacturing facilities. It is, therefore, practically meaningful to divide investment over multiple periods, reducing the risk of investment. Then, the cash-flow over the entire planning horizon would comprise positive inflow and negative outflow. In this case, in general, evaluation by internal rate of return (IRR) is not feasible, because multiple IRRs are involved. This paper deals with a problem of evaluating profitability, as well as risk, of investment alternatives made in multiple times of investment over the entire horizon. Typically, an additional investment is required after the initial one, for expanding manufacturing capacity or other reasons. The paper pays attention to a unit cash-flow over two periods, decomposing the total cash-flow into a series of unit cash-flow patterns. It is easy to evaluate profitability of a unit cash-flow by using IRR. The total cash-flow can be decomposed into the series of two types of unit cash-flows: an investment type one (negative-positive) and the borrowing type one (positive-negative). This paper, therefore, proposes a method in which only the borrowing type unit cash-flow is eliminated in the series by converting total cash-flow using capital interest rate. Then, a unique IRR can be obtained and the profitability is evaluated. Thus, the paper extends the method of IRR so that it may help decision making in complicated cash-flow pattern observed in practice.

**Keywords:** Profitability, Risk, Unit Cash-flow, Internal Rate of Return, Multiple Investment

## 1. INTRODUCTION

Under recent uncertain economic environment, it is sometimes reasonable in practice to divide investment into multiple times, in such a case as establishing an overseas manufacturing site. It means that initial investment is followed by another one several years later for the purpose of installing additional machinery or starting full capacity operation. In such a case, total cash-flow involves multiple times of investment, and therefore, so-called the internal rate of return (IRR) method is not applicable to judging profitability over the total life, since multiple IRRs may be obtained.

Many studies have been made on the issue of multiple IRRs. Such studies have focused on mathematical analysis of net present value (as well as net future value) function and IRR as roots of an equation of high degree. Teichroew *et al.* (1965) classify investment projects into pure investment and mixed investment. The phenomenon of multiple IRRs occurs only in the situation of a mixed investment. They propose the project investment rate (PIR) which is dependent on the capital interest rate. The mixed investment can be judged by comparison of

PIR and the capital interest rate. As a trial and error approach, a set of computer codes for calculating PIR are proposed (Ward, 1994). Mizumachi *et al.* (2002) developed another algorithm for computing PIR as IRR of certain pure investment obtained by conversion of the original mixed investment based on the capital interest rate. Nevertheless, the proposed algorithm is unsatisfactory in the sense that the calculation of IRR in the mixed investment which might involve multiple IRRs still remains undeleted in the algorithm.

In this paper, we focus on one year investment project which consists of an investment (negative cash-flow) and a positive return. The IRR of one year investment project is calculated as the rate of increase of capital in the project. The project is clearly evaluated by the comparison of its IRR and the capital interest rate. We attempt to apply the IRR criterion of the one year investment project to the evaluation of the investment project over the entire horizon. We decompose total cash-flow of the investment project into a series of unit cash-flows over two periods. The unit cash-flow is classified into two types; the "investment type" in which negative cash-flow (investment) occurs at the end of one period

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and positive cash-flow (return) comes at the end of the next period, and the “borrowing type” in which positive cash-flow occurs at the end of one period and negative cash-flow follows at the end of the next one. We also analyze the cash-flow as a repetition of fundamental cash-flow pattern, namely, one time investment followed by positive returns. It is known that the fundamental cash-flow pattern has a unique IRR.

The practical concerns are: whether the IRR method is applicable to given cash-flow; and how it can be evaluated if the IRR method is not applicable. Major purposes of this paper are as follows:

- (1) To clarify the condition where IRR method is applicable to specific cash-flow pattern which involves two times of investment.
- (2) In case where the IRR method is not applicable, to present a new evaluation method applying IRR. In the method, calculation of multiple IRRs should be avoided.

Application of the procedure to more complicated cash-flow is an area for future study.

The rest of the paper is organized as follows. Chapter 2 presents the model, with assumptions and notations. Chapter 3 provides the fundamental analysis of the profitability of investments by decomposing the total cash-flow into the series of unit cash-flows. Chapter 4 presents the expanded IRR method. The numerical examples are discussed in Chapter 5. Finally the conclusion is presented in Chapter 6.

## 2. MODEL PRESENTATION

### 2.1 Assumptions and Notations

This paper investigates the problem using following assumptions and notations:

1.  $\mathbf{A} = [a_0, a_1, \dots, a_n]_n$  denotes an investment project which generates net cash flows of  $a_t$  at the end of period  $t$ , where  $t = 0, 1, \dots, n$ .
2. The project balances of  $\mathbf{A}$  for interest rate  $i$  are defined as follows:

$$\begin{cases} S_0^{\mathbf{A}}(i) = a_0 \\ S_t^{\mathbf{A}}(i) = S_{t-1}^{\mathbf{A}}(i)(1+i) + a_t, \quad 1 \leq t \leq n \end{cases} \quad (1)$$

The project balance at the end of period  $n$  is equivalent to the Net Future Value (NFV) function.

$$S_n^{\mathbf{A}}(i) = a_0(1+i)^n + a_1(1+i)^{n-1} + \dots + a_{n-1}(1+i) + a_n \quad (2)$$

3. The IRR of  $\mathbf{A}$  is denoted by  $r^{\mathbf{A}}$ , which satisfies

$$S_n^{\mathbf{A}}(r^{\mathbf{A}}) = 0 \quad (3)$$

4. The capital interest rate is given and represented by  $i_0$ .

5. We define a unit cash-flow over two periods. The unit cash-flow consists of  $c_t$  at the end of period  $t-1$  and  $-c_t(1+r)$  at the end of period  $t$ . We denote the unit cash-flow in period  $t$  whose IRR is  $r$  by,

$$\begin{aligned} c_t \mathbf{u}_{t(r)} &= [0, \dots, 0, c_t, -c_t(1+r), 0, \dots, 0]_n \\ &= [c_t, -c_t(1+r)]_t \end{aligned} \quad (4)$$

The subscript  $t$  outside of  $[ ]$  represents the final period of the cash-flows listed in  $[ ]$ . As shown in Fig.1, the case where  $c_t < 0$  is called that of the investment unit cash-flow, and the case where  $c_t > 0$ , that of the borrowing unit cash-flow.

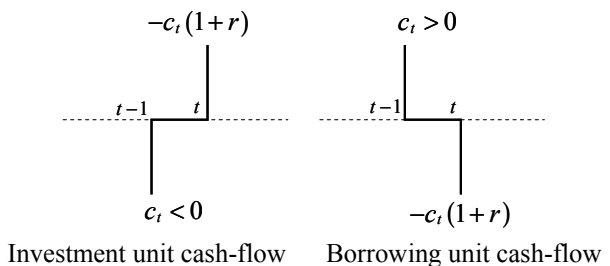


Figure 1. Unit cash-flow.

6. We call the cash-flow which consists of one time investment and positive returns, the fundamental cash-flow pattern.

$$\begin{cases} a_0 < 0 \\ a_t > 0, \quad 1 \leq t \leq n \end{cases} \quad (5)$$

### 2.2 Problem to be Solved

This paper investigates the cash-flow involving investment in two times over the entire horizon. We denote the cash-flow pattern by  $\mathbf{D}$ . We define the judgment criteria for evaluating the profitability of  $\mathbf{D}$  based on the comparison of IRR and the capital interest rate.

$$\mathbf{D} = [d_0, d_1, \dots, d_k, \dots, d_n], \quad \begin{cases} d_0 < 0 \\ d_k < 0, \quad (1 < k < n) \\ d_t > 0, \quad (t \neq 0, k) \end{cases} \quad (6)$$

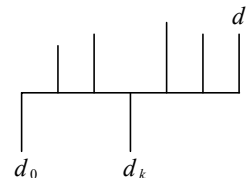


Figure 2. Cash-flow pattern of multiple investments.

Where  $\mathbf{D}$  and the capital interest rate  $i_0$  are given, the problems are:

1. To clarify the condition that  $\mathbf{D}$  is classified into the pure investment to which IRR method is applicable.
2. When  $\mathbf{D}$  is classified into the mixed investment, to present

an evaluation method for **D** by calculating only IRR of the pure investment.

### 3. FUNDAMENTAL ANALYSIS

#### 3.1 Decomposition of Total Cash-flow into Series of Unit Cash-flows

**Theorem 1:** The cash-flow is decomposed into sum of the series of unit cash-flows whose IRR is  $i$  and the NFV function of  $i$  at the end of period  $n$ .

$$\mathbf{A} = \sum_{t=1}^n S_{t-1}^{\mathbf{A}}(i) \mathbf{u}_t(i) + [S_n^{\mathbf{A}}(i)]_n \quad (7)$$

**Proof:** The following equations can be obtained from the Eqs. (1).

$$\begin{cases} a_0 = S_0^{\mathbf{A}}(i) \\ a_t = -S_{t-1}^{\mathbf{A}}(i)(1+i) + S_t^{\mathbf{A}}(i), \quad 1 \leq t \leq n \end{cases} \quad (8)$$

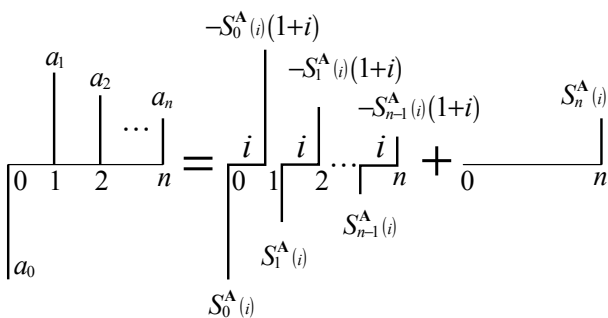
Using Eqs. (8),  $\mathbf{A}$  can be rewritten as follows.

$$\begin{aligned} \mathbf{A} &= [a_0, a_1, \dots, a_n]_n \\ &= [S_0^{\mathbf{A}}(i), -S_0^{\mathbf{A}}(i)(1+i) + S_1^{\mathbf{A}}(i), -S_1^{\mathbf{A}}(i)(1+i) + S_2^{\mathbf{A}}(i), \\ &\quad \dots, -S_{n-2}^{\mathbf{A}}(i)(1+i) + S_{n-1}^{\mathbf{A}}(i), -S_{n-1}^{\mathbf{A}}(i)(1+i) + S_n^{\mathbf{A}}(i)]_n \quad (9) \\ &= S_0^{\mathbf{A}}(i) \mathbf{u}_1(i) + S_1^{\mathbf{A}}(i) \mathbf{u}_2(i) + \dots + S_{n-1}^{\mathbf{A}}(i) \mathbf{u}_n(i) + [S_n^{\mathbf{A}}(i)]_n \\ &= \sum_{t=1}^n S_{t-1}^{\mathbf{A}}(i) \mathbf{u}_t(i) + [S_n^{\mathbf{A}}(i)]_n \end{aligned}$$

Q.E.D.

We denote the series of unit cash-flows by  $\bar{\mathbf{A}}(i)$ .

$$\bar{\mathbf{A}}(i) = \sum_{t=1}^n S_{t-1}^{\mathbf{A}}(i) \mathbf{u}_t(i) \quad (10)$$



**Figure 3.** Decomposition into series of unit cash-flows.

**Corollary 1:** If cash-flow  $\mathbf{A}$  has an IRR  $r^{\mathbf{A}}$ , then  $\mathbf{A}$  is just decomposed into the series of unit cash-flows  $\bar{\mathbf{A}}(r^{\mathbf{A}})$ .

$$\begin{aligned} \mathbf{A} &= \sum_{t=1}^n S_{t-1}^{\mathbf{A}}(r^{\mathbf{A}}) \mathbf{u}_t(r^{\mathbf{A}}) + [S_n^{\mathbf{A}}(r^{\mathbf{A}})]_n \\ &= \sum_{t=1}^n S_{t-1}^{\mathbf{A}}(r^{\mathbf{A}}) \mathbf{u}_t(r^{\mathbf{A}}) \\ &= \bar{\mathbf{A}}(r^{\mathbf{A}}) \end{aligned} \quad (11)$$

#### 3.2 Profitability of Series of Unit Cash-flow Patterns

We classify the series of unit cash-flow patterns into two categories as follows.

$\bar{\mathbf{A}}(i)$  is pure investment series (that is,  $\bar{\mathbf{A}}(i)$  is pure)

$$\Leftrightarrow S_0^{\mathbf{A}}(i) < 0, \quad S_t^{\mathbf{A}}(i) \leq 0 (1 \leq t \leq n-1)$$

$\bar{\mathbf{A}}(i)$  is mixed investment series (that is,  $\bar{\mathbf{A}}(i)$  is mixed)

$$\Leftrightarrow S_0^{\mathbf{A}}(i) < 0, \exists S_t^{\mathbf{A}}(i) > 0 (1 \leq t \leq n-1)$$

The pure investment series comprises only of the unit cash-flow of the investment type. The mixed investment series comprises both of the investment type and the borrowing one.

The NFV of  $c_t \mathbf{u}_t(r)$  is given by

$$\begin{aligned} S_n^{c_t \mathbf{u}_t(r)}(i) &= S_t^{c_t \mathbf{u}_t(r)}(i) \times (1+i)^{n-t} \\ &= -c_t(r-i)(1+i)^{n-t} \end{aligned} \quad (12)$$

Eq. (12) reveals Property 1.

**Property 1:** The unit cash-flow  $c_t \mathbf{u}_t(r)$  has a unique IRR. Profitability of the unit cash-flow is judged as follows:

Investment unit cash-flow ( $c_t < 0$ ) is profitable, if  $r > i$ .  
Borrowing unit cash-flow ( $c_t > 0$ ) is profitable, if  $r < i$ .

**Property 2:** The pure investment series  $\bar{\mathbf{A}}(r)$  has a unique IRR  $r$ .  $\bar{\mathbf{A}}(r)$  satisfies the next conditions.

$$\begin{cases} S_n^{\bar{\mathbf{A}}(r)}(i) > 0, & r > i \\ S_n^{\bar{\mathbf{A}}(r)}(i) = 0, & r = i \\ S_n^{\bar{\mathbf{A}}(r)}(i) < 0, & r < i \end{cases} \quad (13)$$

**Proof.** From Eq. (11), the NFV of  $\bar{\mathbf{A}}(r)$  is calculated as sum of the NFV of each unit cash-flow. Since the pure investment series comprises only of the unit cash-flow of the investment type, the NFV of  $\bar{\mathbf{A}}(r)$  has the same property as that of the investment unit cash-flow shown in Property 1.

Q.E.D.

#### 3.3 Combination of Two Pure Investment Series

**Lemma 1:** We shall consider decomposition of pure investment series  $\bar{\mathbf{A}}(r)$  into the series of unit cash-flow

patterns where the interest rate is  $i$ .

$$\bar{\mathbf{A}}(r) = \sum_{t=1}^n S_{t-1}^{\mathbf{A}}(i) \mathbf{u}_t(i) + [S_n^{\mathbf{A}}(i)]_n = \bar{\mathbf{A}}(i) + [S_n^{\mathbf{A}}(i)]_n \quad (14)$$

1.  $\bar{\mathbf{A}}(i)$  is pure, where  $i \geq r$ .
2. The NFV  $S_n^{\mathbf{A}}(i)$  is a monotone decreasing function of  $i$ , where  $i \geq r$ .

**Proof:** 1. Since  $\bar{\mathbf{A}}(r)$  is pure, the next statement holds.

$$S_0^{\mathbf{A}}(r) < 0, \quad S_t^{\mathbf{A}}(r) \leq 0 \quad (1 \leq t \leq n-1) \quad (15)$$

We shall prove the next statement by mathematical induction.

$$S_t^{\mathbf{A}}(i) \leq S_t^{\mathbf{A}}(r) \leq 0 \quad (0 \leq t \leq n-1), \quad \text{where } i \geq r \quad (16)$$

The following statements hold.

$$S_0^{\mathbf{A}}(i) = S_0^{\mathbf{A}}(r) < 0 \quad (17)$$

$$\begin{aligned} S_1^{\mathbf{A}}(i) &= S_0^{\mathbf{A}}(i)(1+i) + a_1 \\ &\leq S_0^{\mathbf{A}}(r)(1+r) + a_1 = S_1^{\mathbf{A}}(r) \leq 0 \end{aligned} \quad (18)$$

Assuming  $S_t^{\mathbf{A}}(i) \leq S_t^{\mathbf{A}}(r) \leq 0$ , the next statement holds where  $1 \leq t \leq n-2$ .

$$\begin{aligned} S_{t+1}^{\mathbf{A}}(i) &= S_t^{\mathbf{A}}(i)(1+i) + a_{t+1} \\ &\leq S_t^{\mathbf{A}}(r)(1+r) + a_{t+1} = S_{t+1}^{\mathbf{A}}(r) \leq 0 \end{aligned} \quad (19)$$

Then, statement (16) is proved.  $\bar{\mathbf{A}}(i)$  is pure.

2. We shall prove the next statement by mathematical induction.

$$\frac{d}{di} S_n^{\mathbf{A}}(i) < 0, \quad \text{where } i \geq r \quad (20)$$

The next statement holds.

$$\frac{d}{di} S_1^{\mathbf{A}}(i) = \frac{d}{di} [S_0^{\mathbf{A}}(i)(1+i) + a_1] = a_0 < 0 \quad (21)$$

Assuming

$$\frac{d}{di} S_t^{\mathbf{A}}(i) < 0, \quad (22)$$

from Eq. (16), the next statement holds.

$$\begin{aligned} \frac{d}{di} S_{t+1}^{\mathbf{A}}(i) &= \frac{d}{di} [S_t^{\mathbf{A}}(i)(1+i) + a_{t+1}] \\ &= \left[ \frac{d}{di} S_t^{\mathbf{A}}(i) \right] (1+i) + S_t^{\mathbf{A}}(i) < 0 \end{aligned} \quad (23)$$

Then, statement (20) is proved.  $S_n^{\mathbf{A}}(i)$  is a monotone de-

creasing function of  $i$ , where  $i \geq r$  Q.E.D.

**Lemma 2:** We shall consider

$$\mathbf{B} = \bar{\mathbf{A}}(r^{\mathbf{A}}) + [c]_n. \quad (24)$$

If  $\bar{\mathbf{A}}(r^{\mathbf{A}})$  is pure and  $c > 0$ , then  $\mathbf{B}$  is decomposed into pure investment series  $\bar{\mathbf{B}}(r^{\mathbf{B}})$  and  $r^{\mathbf{B}} > r^{\mathbf{A}}$ .

**Proof.** The NFV of  $\mathbf{B}$  is given by

$$S_n^{\mathbf{B}}(i) = S_n^{\mathbf{A}}(i) + c \quad (25)$$

Since  $a_0 < 0$ , the NFV of  $\mathbf{A}$  satisfies the next statement.

$$\lim_{i \rightarrow \infty} S_n^{\mathbf{A}}(i) = -\infty \quad (26)$$

From Lemma 1,

$$S_n^{\mathbf{A}}(i) \text{ is strictly decreasing, where } i \geq r^{\mathbf{A}} \quad (27)$$

From Eqs. (25), (26), statement (27), and  $S_n^{\mathbf{A}}(r^{\mathbf{A}}) = 0$ , there must exist a unique  $i^* > r^{\mathbf{A}}$  that satisfies

$$S_n^{\mathbf{A}}(i^*) = -c. \quad (28)$$

It follows that

$$S_n^{\mathbf{B}}(i^*) = S_n^{\mathbf{A}}(i^*) + c = -c + c = 0 \quad (29)$$

We can consider  $i^*$  as  $r^{\mathbf{B}}$ . Property 2 shows

$$S_n^{\mathbf{A}}(i) > 0, \quad \text{where } i < r^{\mathbf{A}}. \quad (30)$$

From (30) and  $c > 0$ , the next statement holds.

$$S_n^{\mathbf{B}}(i) = S_n^{\mathbf{A}}(i) + c > 0, \quad \text{where } i \leq r^{\mathbf{A}} \quad (31)$$

Therefore,  $\mathbf{B}$  has a unique IRR that satisfies  $r^{\mathbf{B}} > r^{\mathbf{A}}$ . Since the cash-flows at the end of period 0 through  $n-1$  of  $\mathbf{B}$  are identical with those of  $\mathbf{A}$ , the following statement can be derived from the statement (16):

$$S_t^{\mathbf{B}}(r^{\mathbf{B}}) \leq S_t^{\mathbf{A}}(r^{\mathbf{A}}) \leq 0 \quad (0 \leq t \leq n-1). \quad (32)$$

Then  $\mathbf{B}$  is decomposed into the pure investment series  $\bar{\mathbf{B}}(r^{\mathbf{B}})$ . Q.E.D.

**Lemma 3:** We shall consider

$$\mathbf{B} = \sum_{t=1}^{n-1} c_t \mathbf{u}_t(r^{\mathbf{X}}) + c_n \mathbf{u}_n(r^{\mathbf{Y}}) \quad (33)$$

If  $\sum_{t=1}^{n-1} c_t \mathbf{u}_t(r^{\mathbf{X}})$  is pure,  $c_n < 0$ , and  $r^{\mathbf{Y}} > r^{\mathbf{X}}$ , then  $\mathbf{B}$  is decomposed into pure investment series  $\bar{\mathbf{B}}(r^{\mathbf{B}})$  and  $r^{\mathbf{X}} < r^{\mathbf{B}} < r^{\mathbf{Y}}$ .

**Proof:** The second term of the right hand side of Eq. (33) can be divided into

$$c_n \mathbf{u}_n (r^Y) = c_n \mathbf{u}_n (r^X) + [-c_n (r^Y - r^X)]_n. \quad (34)$$

It follows that:

$$\begin{aligned} \mathbf{B} &= \sum_{t=1}^{n-1} c_t \mathbf{u}_t (r^X) + c_n \mathbf{u}_n (r^X) + [-c_n (r^Y - r^X)]_n \\ &= \sum_{t=1}^n c_t \mathbf{u}_t (r^X) + [-c_n (r^Y - r^X)]_n. \end{aligned} \quad (35)$$

Since  $-c_n (r^Y - r^X) > 0$ , applying Lemma 2 to Eq. (35),  $\mathbf{B}$  has a unique IRR  $r^B > r^X$  and can be decomposed into pure investment series  $\bar{\mathbf{B}}(r^B)$ .

$$S_n^{\sum_{t=1}^{n-1} c_t \mathbf{u}_t (r^X)} (i) < 0, \text{ where } i \geq r^Y (> r^X) \quad (36)$$

$$S_n^{c_n \mathbf{u}_n (r^Y)} (i) \leq 0, \text{ where } i \geq r^Y. \quad (37)$$

It follows that:

$$S_n^{\mathbf{B}} (i) < 0, \text{ where } i \geq r^Y \quad (38)$$

Therefore,  $r^B < r^Y$  holds. Then,  $r^X < r^B < r^Y$  holds.

Q.E.D.

**Theorem 2:** We shall consider

$$\mathbf{B} = \sum_{t=1}^k c_t \mathbf{u}_t (r^X) + \sum_{t=k+1}^n c_t \mathbf{u}_t (r^Y). \quad (39)$$

If  $\sum_{t=1}^k c_t \mathbf{u}_t (r^X)$  and  $\sum_{t=k+1}^n c_t \mathbf{u}_t (r^Y)$  are pure and  $r^Y > r^X$ ,

then  $\mathbf{B}$  can be decomposed into pure investment series  $\bar{\mathbf{B}}(r^B)$  and  $r^X < r^B < r^Y$ .

**Proof:** We denote the series of unit cash-flows for period 1 through  $k+1$  of Eq. (39) by  $\bar{\mathbf{B}}_{k+1}$ . Applying Lemma 3 to  $\bar{\mathbf{B}}_{k+1}$ , the next equation holds.

$$\begin{aligned} \bar{\mathbf{B}}_{k+1} &= \sum_{t=1}^k c_t \mathbf{u}_t (r^X) + c_{k+1} \mathbf{u}_{k+1} (r^Y) \\ &= \sum_{t=1}^{k+1} c_t \mathbf{u}_t (r') \end{aligned} \quad (40)$$

$\bar{\mathbf{B}}_{k+1}$  has a unique IRR  $r'$  which satisfies  $r^X < r' < r^Y$ , and can be decomposed into the pure investment series  $\bar{\mathbf{B}}_{k+1}(r')$ . Next, applying Lemma 3 to  $\bar{\mathbf{B}}_{k+2}$ , the next equation holds.

$$\begin{aligned} \bar{\mathbf{B}}_{k+2} &= \sum_{t=1}^{k+1} c_t \mathbf{u}_t (r') + c_{k+2} \mathbf{u}_{k+2} (r^Y) \\ &= \sum_{t=1}^{k+2} c_t \mathbf{u}_t (r') \end{aligned} \quad (41)$$

$\bar{\mathbf{B}}_{k+2}$  has a unique IRR  $r''$  which satisfies  $r^X < r' < r'' < r^Y$ , and can be decomposed into the pure investment series  $\bar{\mathbf{B}}_{k+2}(r')$ . Applying Lemma 3 to  $\bar{\mathbf{B}}_{k+3}$ ,  $\bar{\mathbf{B}}_{k+4}$ , ...,  $\bar{\mathbf{B}}_n$ , we obtain the pure investment series  $\bar{\mathbf{B}}_n(r^B)$  whose IRR  $r^B$  satisfies  $r^X < r^B < r^Y$  Q.E.D.

### 3.4 Profitability of Multiple Investment

**Lemma 4:** Fundamental cash-flow pattern is decomposed into pure investment series.

**Proof:** Suppose that  $\mathbf{A}_n = [a_0, a_1, \dots, a_n]_n$  is a fundamental cash-flow. IRR of  $\mathbf{A}_1 = [a_0, a_1]_1$  is  $r_1$  which satisfies

$$S_1^{\mathbf{A}} (r_1) = a_0 (1 + r_1) + a_1 = 0 \quad (42)$$

$\mathbf{A}_1$  has a unique IRR

$$r_1 = -1 + \frac{a_1}{-a_0} \quad (43)$$

$\mathbf{A}_1$  is an investment unit cash-flow.

$$\mathbf{A}_1 = [S_0^{\mathbf{A}} (r_1), -S_0^{\mathbf{A}} (r_1)(1 + r_1)]_1 = S_0^{\mathbf{A}} (r_1) \mathbf{u}_1 (r_1) \quad (44)$$

Assuming that  $\mathbf{A}_t = [a_0, a_1, \dots, a_t]_t$  is decomposed into pure investment series

$$\mathbf{A}_t = \sum_{j=1}^t S_{j-1}^{\mathbf{A}} (r_t) \mathbf{u}_j (r_t), \quad (45)$$

$\mathbf{A}_{t+1}$  is expressed by

$$\begin{aligned} \mathbf{A}_{t+1} &= \mathbf{A}_t + [a_{t+1}]_{t+1} \\ &= \sum_{j=1}^t S_{j-1}^{\mathbf{A}} (r_t) \mathbf{u}_j (r_t) + 0 \times \mathbf{u}_{t+1} (r_t) + [a_{t+1}]_{t+1} \\ &= \sum_{j=1}^{t+1} S_{j-1}^{\mathbf{A}} (r_t) \mathbf{u}_j (r_t) + [a_{t+1}]_{t+1} \end{aligned} \quad (46)$$

Applying Lemma 2,  $\mathbf{A}_{t+1}$  can be decomposed into the pure investment series

$$\mathbf{A}_{t+1} = \sum_{j=1}^{t+1} S_{j-1}^{\mathbf{A}} (r_{t+1}) \mathbf{u}_j (r_{t+1}) \quad (47)$$

By mathematical induction, Lemma 4 is proved.

Q.E.D.

Two fundamental cash-flow patterns can be derived from the total cash-flow of  $\mathbf{D}$  (see Figure 2). The first fundamental pattern is denoted by:

$$\mathbf{X} = [d_0, d_1, \dots, d_{k-1}]_{k-1} \quad (48)$$

and the second one by:

$$\mathbf{Y} = [d_k, d_{k+1}, \dots, d_n]_n \quad (49)$$

**Theorem 3:** Where  $r^X < r^Y$  holds,  $\mathbf{D}$  has a unique IRR  $r^D$  which satisfies  $r^X < r^D < r^Y$ , and can be decomposed into pure investment series.

**Proof:**  $\mathbf{D}$  is represented by

$$\begin{aligned} \mathbf{D} &= \mathbf{X} + \mathbf{Y} \\ &= \sum_{t=1}^{k-1} c_t \mathbf{u}_t(r^X) + \sum_{t=k+1}^n c_t \mathbf{u}_t(r^Y) \\ &= \sum_{t=1}^{k-1} c_t \mathbf{u}_t(r^X) + 0 \times \mathbf{u}_k(r^X) + \sum_{t=k+1}^n c_t \mathbf{u}_t(r^Y) \\ &= \sum_{t=1}^k c_t \mathbf{u}_t(r^X) + \sum_{t=k+1}^n c_t \mathbf{u}_t(r^Y) \end{aligned} \quad (50)$$

Applying Theorem 2, the pure investment series  $\bar{\mathbf{D}}^{(r^D)}$  can be obtained whose IRR  $r^D$  satisfies  $r^X < r^D < r^Y$ .  
Q.E.D.

**Theorem 4:** Where  $r^X > r^Y$  holds,  $\mathbf{D}$  has IRR  $r^D$  which satisfies  $r^Y < r^D < r^X$ , and can be decomposed into mixed investment series.

**Proof:** The NFV of  $\mathbf{X}$  and  $\mathbf{Y}$  satisfy next statements.

$$\begin{cases} S_n^X(i) > 0 \wedge S_n^Y(i) \geq 0, & \text{where } i \leq r^Y \\ S_n^X(i) \leq 0 \wedge S_n^Y(i) < 0, & \text{where } i \geq r^X \end{cases} \quad (51)$$

It follows

$$\begin{cases} S_n^D(i) = S_n^X(i) + S_n^Y(i) > 0, & \text{where } i \leq r^Y \\ S_n^D(i) = S_n^X(i) + S_n^Y(i) < 0, & \text{where } i \geq r^X \end{cases} \quad (52)$$

Therefore,  $\mathbf{D}$  has at least one IRR  $r^D$  which satisfies  $r^Y < r^D < r^X$ . Since the cash-flows at the end of periods 0 through  $k-1$  of  $\mathbf{D}$  are identical with those of  $\mathbf{X}$ , the next statement holds.

$$S_{k-1}^D(i) = S_{k-1}^X(i) > 0, \quad \text{where } i < r^X \quad (53)$$

Therefore, the unit cash-flow at the  $k$ -th period of  $\bar{\mathbf{D}}^{(r^D)}$  is of the borrowing type.  $\bar{\mathbf{D}}^{(r^D)}$  is mixed investment series.

Q.E.D.

Where  $r^X < r^Y$  holds,  $\mathbf{D}$  is decomposed into the pure investment series. Then the profitability of  $\mathbf{D}$  is judged by IRR methods.

$\mathbf{D}$  is profitable if  $r^D > i_0$ .

The pure investment series does not involve multiple IRRs. It implies that the cash-flow which has multiple IRRs is decomposed into the mixed investment series.

We shall consider profitability of  $\mathbf{D}$  which is decomposed into mixed investment series. Where  $i < r^D$ , the investment unit cash-flows among  $\bar{\mathbf{D}}^{(r^D)}$  are profitable and the borrowing unit cash-flows among  $\bar{\mathbf{D}}^{(r^D)}$  are not profitable. On the contrary, where  $i > r^D$ , the investment unit cash-flows are not profitable and the borrowing unit cash-flows are profitable. It follows that, in both cases of  $i < r^D$  and  $i > r^D$ , profitable and non-profitable unit cash-flows coexist. Therefore, in the case of mixed investment series, the borrowing unit cash-flows prevent the profitability judgment based on the IRR criterion for the investment unit cash-flows.

## 4. EXPANDED IRR METHOD

### 4.1 Decomposition into Mixed Patterns Using Neutral Borrowing Unit Cash-flow

In order to resolve the problem inherent in the mixed investment series, we decompose the total cash-flow into the series of unit cash-flows, regarding the capital interest rate  $i_0$  as the value of IRR for the borrowing unit cash-flow.

We define the project balance  $F_t^A(i, i_0)$  under the interest rate  $i$  as follows.

$$\begin{aligned} F_0^A(i, i_0) &= a_0 \\ F_t^A(i, i_0) &= (1+v)F_{t-1}^A(i, i_0) + a_t \\ v &= \begin{cases} i, & \text{where } F_{t-1}^A(i, i_0) \leq 0 \\ i_0, & \text{where } F_{t-1}^A(i, i_0) > 0 \end{cases} \end{aligned} \quad (54)$$

Eq. (55), which follows, can be derived from the same manner by which Eq. (7) was derived from Eq. (1).

$$\mathbf{A} = \sum_{t=1}^n F_{t-1}^A(i, i_0) \mathbf{u}_t(v) + [F_n^A(i, i_0)]_n \quad (55)$$

$\mathbf{A}$  is decomposed into the sum of the series of unit cash-flows and  $F_n^A(i, i_0)$  at the end of period  $n$ . The IRR for the investment unit cash-flow is  $i$ . The IRR for the borrowing unit cash-flow is  $i_0$ . We denote  $\sum_{t=1}^n F_{t-1}^A(i, i_0) \mathbf{u}_t(v)$  by  $\bar{\mathbf{A}}(i, i_0)$ . If

$$F_t^A(i, i_0) \leq 0, \quad 1 \leq t \leq n-1 \quad (56)$$

holds, then there is no borrowing unit cash-flow among  $\bar{\mathbf{A}}(i, i_0)$ . In this case, the next statements hold.

$$F_t^A(i, i_0) = S_t^A(i), \quad 0 \leq t \leq n \quad (57)$$

$$\bar{\mathbf{A}}(i, i_0) = \bar{\mathbf{A}}(i) \quad (58)$$

**Lemma 5:**  $F_n^{\mathbf{A}}(i, i_0)$  is a monotone decreasing function of  $i$ .

**Proof:** We shall prove

$$\frac{d}{di} F_n^{\mathbf{A}}(i, i_0) < 0 \quad (59)$$

by mathematical induction. The next statement holds.

$$\frac{d}{di} F_1^{\mathbf{A}}(i, i_0) = \frac{d}{di} \{F_0^{\mathbf{A}}(i, i_0)(1+i) + a_1\} = a_0 < 0 \quad (60)$$

Assuming

$$\frac{d}{di} F_t^{\mathbf{A}}(i, i_0) < 0 \quad (61)$$

the next statements hold.

Where  $F_t^{\mathbf{A}}(i, i_0) \leq 0$ ,

$$\begin{aligned} \frac{d}{di} F_{t+1}^{\mathbf{A}}(i, i_0) &= \frac{d}{di} \{F_t^{\mathbf{A}}(i, i_0)(1+i) + a_{t+1}\} \\ &= \frac{d}{di} F_t^{\mathbf{A}}(i, i_0)(1+i) + F_t^{\mathbf{A}}(i, i_0) < 0 \end{aligned} \quad (62)$$

Where  $F_t^{\mathbf{A}}(i, i_0) > 0$ ,

$$\begin{aligned} \frac{d}{di} F_{t+1}^{\mathbf{A}}(i, i_0) &= \frac{d}{di} \{F_t^{\mathbf{A}}(i, i_0)(1+i) + a_{t+1}\} \\ &= \frac{d}{di} F_t^{\mathbf{A}}(i, i_0)(1+i) < 0 \end{aligned} \quad (63)$$

Therefore, statement (59) holds. Q.E.D.

**Lemma 6.**  $F_n^{\mathbf{A}}(i, i_0)$  is an increasing function of  $i_0$ .

$$\frac{d}{di_0} F_n^{\mathbf{A}}(i, i_0) \leq 0 \quad (64)$$

**Proof.** Next statement holds.

$$\frac{d}{di_0} F_0^{\mathbf{A}}(i, i_0) = \frac{d}{di_0} a_0 = 0 \quad (65)$$

If  $F_t^{\mathbf{A}}(i, i_0) \leq 0 (1 \leq t \leq n-1)$  holds, then the next statements hold.

$$F_t^{\mathbf{A}}(i, i_0) = S_t^{\mathbf{A}}(i), \quad 1 \leq t \leq n \quad (66)$$

$$\frac{d}{di_0} F_n^{\mathbf{A}}(i, i_0) = 0 \quad (67)$$

Assuming that

$$\begin{aligned} F_1^{\mathbf{A}}(i, i_0) \leq 0, \quad F_2^{\mathbf{A}}(i, i_0) \leq 0, \quad \dots, \quad F_{t-1}^{\mathbf{A}}(i, i_0) \leq 0, \\ \text{and } F_t^{\mathbf{A}}(i, i_0) > 0, \end{aligned} \quad (68)$$

we have

$$\frac{d}{di_0} F_j^{\mathbf{A}}(i, i_0) = 0, \quad 0 \leq j \leq t \quad (69)$$

It yields

$$\begin{aligned} \frac{d}{di_0} F_{t+1}^{\mathbf{A}}(i, i_0) &= \frac{d}{di_0} \{F_t^{\mathbf{A}}(i, i_0)(1+i) + a_{t+1}\} \\ &= \frac{d}{di_0} F_t^{\mathbf{A}}(i, i_0)(1+i) + F_t^{\mathbf{A}}(i, i_0) > 0 \end{aligned} \quad (70)$$

Where  $t+1 < n$ , then the next statements hold.

Where  $F_{t+1}^{\mathbf{A}}(i, i_0) \leq 0$ ,

$$\begin{aligned} \frac{d}{di_0} F_{t+2}^{\mathbf{A}}(i, i_0) &= \frac{d}{di_0} \{F_{t+1}^{\mathbf{A}}(i, i_0)(1+i) + a_{t+2}\} \\ &= \frac{d}{di_0} F_{t+1}^{\mathbf{A}}(i, i_0)(1+i) > 0 \end{aligned} \quad (71)$$

Where  $F_{t+1}^{\mathbf{A}}(i, i_0) > 0$ ,

$$\begin{aligned} \frac{d}{di_0} F_{t+2}^{\mathbf{A}}(i, i_0) &= \frac{d}{di_0} \{F_{t+1}^{\mathbf{A}}(i, i_0)(1+i) + a_{t+2}\} \\ &= \frac{d}{di_0} F_{t+1}^{\mathbf{A}}(i, i_0)(1+i) + F_{t+1}^{\mathbf{A}}(i, i_0) > 0 \end{aligned} \quad (72)$$

By mathematical induction statement (64) holds.

Q.E.D.

We shall consider profitability of **D** where  $r^{\mathbf{X}} > r^{\mathbf{Y}}$ . Where  $i_0 \leq r^{\mathbf{Y}}$ , **D** is judged profitable because both **X** and **Y** are profitable. Where  $i_0 \geq r^{\mathbf{X}}$ , **D** is not profitable because neither **X** nor **Y** are profitable. Therefore, our discussion is focused on the next condition.

$$r^{\mathbf{Y}} < i_0 < r^{\mathbf{X}} \quad (73)$$

**Theorem 5:** Where  $r^{\mathbf{X}} > r^{\mathbf{Y}}$  holds,

1. There exists a unique  $r(i_0)$  that satisfies  $F_n^{\mathbf{D}}(r(i_0), i_0) = 0$ .
2. The unit cash-flow at  $k$ -th period of  $\mathbf{D} = \bar{\mathbf{D}}(r(i_0), i_0)$  is of the borrowing type.

**Proof:** From Lemma 6 and (73), the next conditions hold.

$$F_n^{\mathbf{D}}(r^{\mathbf{Y}}, i_0) \geq F_n^{\mathbf{D}}(r^{\mathbf{Y}}, r^{\mathbf{Y}}) = S_n^{\mathbf{D}}(r^{\mathbf{Y}}) > 0 \quad (74)$$

$$F_n^{\mathbf{D}}(r^{\mathbf{X}}, i_0) \leq F_n^{\mathbf{D}}(r^{\mathbf{X}}, r^{\mathbf{X}}) = S_n^{\mathbf{D}}(r^{\mathbf{X}}) < 0 \quad (75)$$

From (74), (75) and Lemma 5, 1. is proved.

The next equation holds.

$$F_{k-1}^{\mathbf{D}}(r^{\mathbf{X}}, i_0) = S_{k-1}^{\mathbf{D}}(r^{\mathbf{X}}) = 0 \quad (76)$$

From Lemma 5 and (76), we have

$$F_{k-1}^{\mathbf{D}}(i, i_0) > 0, \quad i < r^{\mathbf{X}}. \quad (77)$$

From  $r(i_0) < r^{\mathbf{X}}$  and (77), 2. is proved. Q.E.D.

Where  $r^{\mathbf{X}} > r^{\mathbf{Y}}$  holds, we can decompose  $\mathbf{D}$  into the series of unit cash-flows  $\mathbf{D} = \bar{\mathbf{D}}(r(i_0), i_0)$ . Since the IRR of borrowing unit cash-flow is  $i_0$ , the NFV of the borrowing unit cash-flow under the capital interest rate  $i_0$  equals to 0. The NFV of  $\mathbf{D}$  is calculated as sum of the NFV of pure investment unit cash-flows whose IRR is  $r(i_0)$ . The profitability of  $\mathbf{D}$  can be judged as follows:  $\mathbf{D}$  is profitable if  $r(i_0) > i_0$

#### 4.2 Conversion of Mixed Pattern into Pure Pattern

In order to obtain the value of  $r(i_0)$ , we convert the mixed investment series  $\bar{\mathbf{D}}(r(i_0), i_0)$  into the pure investment series by erasing borrowing unit cash-flows.

**Theorem 6:** Where  $r^{\mathbf{X}} > r^{\mathbf{Y}}$  holds for  $\mathbf{D}$ , we consider  $\mathbf{D}' = [d'_1, d'_2, \dots, d'_n]$  obtained by shifting for one period the product of  $\mathbf{X}$  multiplied by  $(1+i_0)$  and combining with  $\mathbf{Y}$  as follows:

$$\begin{cases} d'_t = (1+i_0)d_{t-1}, & 1 \leq t \leq k-1 \\ d'_k = (1+i_0)d_{k-1} + d_k \\ d'_t = d_t, & k+1 \leq t \leq n \end{cases} \quad (78)$$

$\mathbf{D}'$  is decomposed into  $\bar{\mathbf{D}}'(r(i_0), i_0)$  which has one less borrowing unit cash-flow than  $\bar{\mathbf{D}}(r(i_0), i_0)$ .

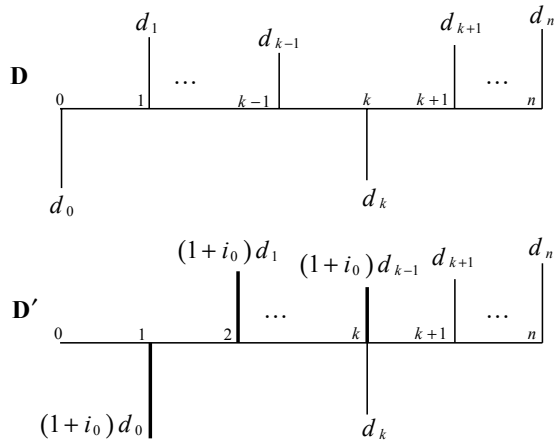


Figure 4. Conversion of  $\mathbf{D}$  to  $\mathbf{D}'$ .

**Proof.**  $\bar{\mathbf{D}}(r(i_0), i_0) = \mathbf{D}$  is expressed by extracting the  $k$ -th borrowing unit cash-flow as follows:  $(F_t^{\mathbf{D}}(r(i_0), i_0))$  is simply denoted as  $F_t^{\mathbf{D}}$

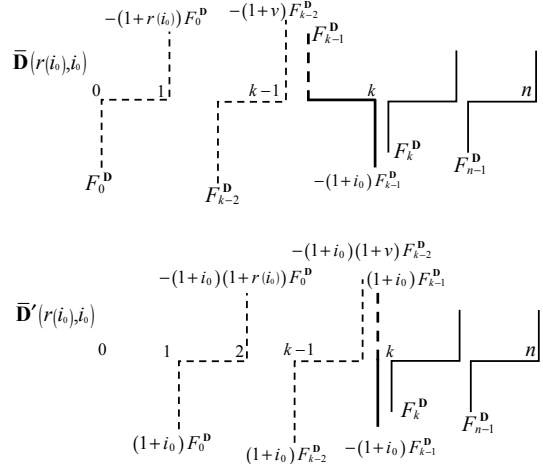


Figure 5. Conversion of  $\bar{\mathbf{D}}(r(i_0), i_0)$  to  $\bar{\mathbf{D}}'(r(i_0), i_0)$ .

$$\begin{aligned} \mathbf{D} &= \sum_{t=1}^{k-1} F_{t-1}^{\mathbf{D}} \mathbf{u}_t(v) + F_{k-1}^{\mathbf{D}} \mathbf{u}_k(i_0) + \sum_{t=k+1}^n F_{t-1}^{\mathbf{D}} \mathbf{u}_t(v) \\ &= \sum_{t=1}^{k-1} F_{t-1}^{\mathbf{D}} \mathbf{u}_t(v) + [F_{k-1}^{\mathbf{D}}, -(1+i_0)F_{k-1}^{\mathbf{D}}]_k + \sum_{t=k+1}^n F_{t-1}^{\mathbf{D}} \mathbf{u}_t(v) \end{aligned} \quad (79)$$

$\mathbf{D}'$  can be obtained by applying the conversion as follows:

$$\begin{aligned} \mathbf{D}' &= \sum_{t=2}^k (1+i_0)F_{t-1}^{\mathbf{D}} \mathbf{u}_t(v) + [(1+i_0)F_{k-1}^{\mathbf{D}} - (1+i_0)F_{k-1}^{\mathbf{D}}]_k \\ &\quad + \sum_{t=k+1}^n F_{t-1}^{\mathbf{D}} \mathbf{u}_t(v) \\ &= \sum_{t=2}^k (1+i_0)F_{t-2}^{\mathbf{D}} \mathbf{u}_t(v) + \sum_{t=k+1}^n F_{t-1}^{\mathbf{D}} \mathbf{u}_t(v) \end{aligned} \quad (80)$$

The borrowing unit cash-flow  $[F_{k-1}^{\mathbf{D}}, -(1+i_0)F_{k-1}^{\mathbf{D}}]_k$ , which used to exist in  $\bar{\mathbf{D}}(r(i_0), i_0)$ , is eliminated. All unit cash-flows converted from  $\bar{\mathbf{D}}(r(i_0), i_0)$  to  $\bar{\mathbf{D}}'(r(i_0), i_0)$  change neither the types of investment or borrowing, nor the value of IRR.

Q.E.D.

Where  $\mathbf{D}'$  is the fundamental cash-flow or  $r^{\mathbf{X}} \leq r^{\mathbf{Y}}$  holds for  $\mathbf{D}' = \mathbf{X}' + \mathbf{Y}'$ , then  $\mathbf{D}'$  is pure. We can obtain the value of  $r(i_0)$  for  $\mathbf{D}$  as the value of IRR for  $\mathbf{D}'$ . In another case, where  $r^{\mathbf{X}} > r^{\mathbf{Y}}$  holds,  $\mathbf{D}'$  is mixed. We should convert  $\mathbf{D}'$  into  $\mathbf{D}''$  again. The value of  $r(i_0)$  for  $\mathbf{D}$  can be calculated as the IRR for pure  $\mathbf{D}^{(j)}$  obtained after  $j$ -times conversion ( $j \leq n-2$ ) eliminating the borrowing unit cash-flow. The procedure for judging the profitability of  $\mathbf{D}$  is described in Figure 6.

In the procedure we calculate only IRR of pure investment. Therefore, we are not faced with the problem of multiple IRRs. The conversion  $\mathbf{D}$  into  $\mathbf{D}'$  does not change the NFV (nor NPV). The proposed procedure leads to an evaluation consistent with the NFV and NPV criteria.



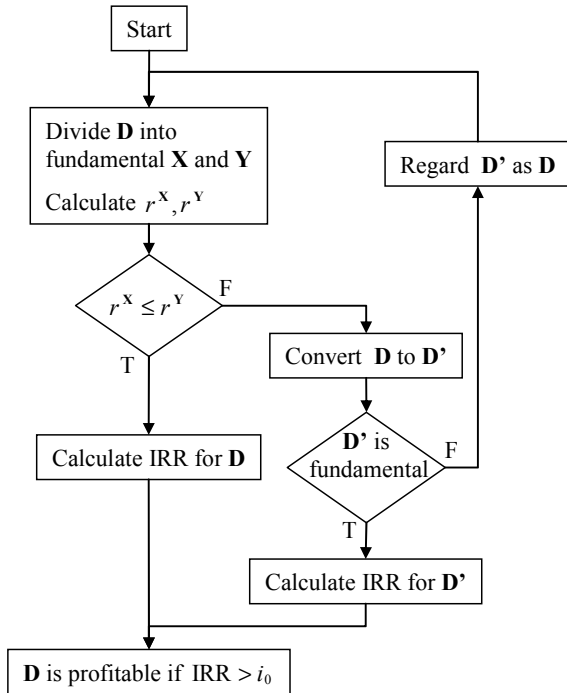


Figure 6. Decision-making procedure.

### 5. NUMERICAL EXAMPLES AND DISCUSSION

As numerical examples, we shall consider following cash-flows. The capital interest rate  $i_0 = 10\%$  is given.

$$P = [-100, 85, -70, 150]_3$$

$$Q = [-100, 165, -130, 100]_3$$

$$R = [-780, 760, 1620, -2140, 380, 165]_5$$

In accordance with the procedure, we divide  $P$  into  $X^P = [-100, 85]_1$  and  $Y^P = [-70, 150]_3$ . The calculation results in:  $r^{X^P} = -15.0\%$ ,  $r^{Y^P} = 114.3\%$ . Since  $r^{X^P} < r^{Y^P}$  holds, we calculate IRR of  $P$ .  $P$  is judged profitable because  $r^P (= 25\%) > i_0$ .

We divide  $Q$  into  $X^Q = [-100, 165]_1$  and  $Y^Q = [-130, 100]_3$ . The calculation results in:  $r^{X^Q} = 65.0\%$ ,  $r^{Y^Q} = -23.1\%$ . Since  $r^{X^Q} > r^{Y^Q}$  holds, we convert  $Q$  into  $Q'$ .

$$Q' = [-100(1+0.1), 165(1+0.1)-130, 100]_3 = [-110, 51.5, 100]_3 \quad (81)$$

$Q'$  is a fundamental cash-flow, so that we calculate IRR of

$Q'$ .  $Q$  is judged profitable because  $r^{Q'} (= 21.6\%) > i_0$ .

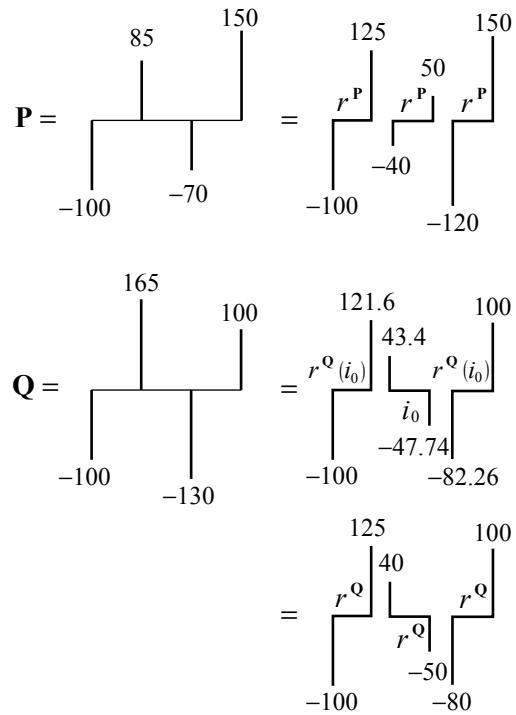


Figure 7. Numerical examples.

The cash-flows and the series of unit cash-flows are shown in Figure 7.  $P$  is decomposed into pure investment series.

$$\bar{P}(r^P) = -100u_1(25\%) - 40u_2(25\%) - 120u_3(25\%) \quad (82)$$

The comparison of IRR of investment unit cash-flow and capital interest rate determines the profitability of the total cash-flow.  $r^P = 25.0\%$  means that the efficiency of investment is 25.0%. At the same time, it shows the maximum rate of fund raising to keep the project  $P$  to be profitable.

$Q$  is decomposed into mixed investment series.  $r^{Q'} = 21.6\%$  is  $r^Q(i_0)$  which satisfies  $F_3^Q(r^Q(i_0), i_0) = 0$ .

$$\bar{Q}(r^Q(i_0), i_0) = -100u_1(21.6\%) + 43.4u_2(10\%) - 82.26u_3(21.6\%) \quad (83)$$

The borrowing unit cash-flow does not influence the profitability of the total cash-flow because its IRR is the capital interest rate. The comparison of  $r^Q(i_0)$  and  $i_0$  determines the profitability of  $Q$ .

We can also decompose  $Q$  into the mixed investment series using IRR of  $Q$ ,  $r^Q = 25.0\%$ .

$$\bar{Q}(r^Q) = -100u_1(25\%) + 40u_2(25\%) - 80u_3(25\%) \quad (84)$$

$r^Q = 25.0\%$  shows not only the efficiency of investment but also the borrowing rate from the project. The statement (84) indicates that, for the firm to earn 25% of interest with the project, it must increase its surplus funds at the rate of 25% through fund management outside of the project. If the firm's ability to earn is at the capital interest rate of  $i_0 = 10\%$ , then the maximum rate of fund raising to keep the project **Q** profitable is  $r^Q(i_0) = 21.6\%$ .

**R** is an example which has multiple IRRs such as  $-36.0\%$ ,  $-4.3\%$ , and  $13.6\%$ . By dividing **R** into **X<sup>R</sup>** and **Y<sup>R</sup>**,  $r^{X^R} = 100.8\%$  and  $r^{Y^R} = -62.0\%$  are obtained. Since  $r^{X^R} > r^{Y^R}$ , **R** is converted to **R'**.

$$\begin{aligned} \mathbf{R}' &= [-780 \times 1.1, 760 \times 1.1, 1620 \times 1.1 - 2140, 380, 165]_5 \\ &= [-858, 836, -358, 380, 165]_5 \end{aligned} \quad (85)$$

**R'** is not a fundamental cash-flow. We divide **R'** into

$$\mathbf{R}' = \mathbf{X}' + \mathbf{Y}' = [-858, 836]_2 + [-358, 380, 165]_3 \quad (86)$$

Since  $r^{X'} (= -2.6\%) < r^{Y'} (= 39.2\%)$  holds, we calculate IRR of **R'**. **R** is judged profitable, because  $r^{R'} (= 10.3\%) > i_0$ .

## 6. CONCLUSION

This paper proposed a new concept of the IRR criterion based on the comparison of the IRR of an investment unit cash-flow and the capital interest rate. The IRR method is applicable to cash-flow which is decomposed into pure investment series consisting only of investment unit cash-flows.

Where the total cash-flow involves investment in two times, we divide the total cash-flow into two fundamental cash-flows **X** and **Y**. The fundamental cash-flow has a unique IRR. Where  $r^X < r^Y$ , the total cash-flow becomes a pure investment pattern with a unique IRR which satisfies  $r^X < r < r^Y$ , and the IRR method is applicable. Where  $r^X > r^Y$ , the total cash-flow has at least one IRR that satisfies  $r^Y < r < r^X$ . In this case, regardless of the number of IRR, the total cash-flow becomes a mixed investment pattern. The existence of the borrowing unit cash-flow in the mixed investment series prevents the criterion, that it is profitable if  $r > i_0$ , from being applicable. We intended to disregard the use of IRR of borrowing unit cash-flow in profitability evaluation. In so doing, we considered decomposition of the total cash-flow into mixed investment series, regarding the capital interest rate as IRR of the borrowing unit cash-flow. The comparison of the IRR of the investment

unit cash-flow  $r(i_0)$  and the capital interest rate  $i_0$  determines the profitability of the total cash-flow. The integrated decision procedure is proposed.

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