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A Few Problems for the Intellectual Development of Students in High Schools and Community Colleges[1]

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It is a truism that mathematics is about relations (cf. [Halford, G. S. (1999). The properties of representations used in higher cognitive processes: Developmental implications. In: Sigel, I. E. (Ed.), *The Development of Mental Representation: Theories and Applications* (pp. 147–168). Mahwah, New Jersey: Erlbaum]). In this article we are considering few problems related to the Viviani's and Routh's Theorems. All Problems are connected by the relation which exists between the distances of the point inside the triangle to it sides. We show how reasoning about the relations could lead the student's problem solving process and give easy to understand solutions of the problems. Among the problems being considered are the proof of the Converse to Viviani's Theorem, the formulas for areas of all figures formed by the sides of triangle and its cevians.

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INTRODUCTION TO THE PROBLEMS SOLUTION. REASONING ABOUT RELATIONS

Let us consider triangle ΔABC and arbitrary point P inside it (Figure. 1). We could assign to any point P uniquely three numbers h_1, h_2, h_3 which are equal to the distances from this point to the sides of the triangle: $h_1 = |PD|$, $h_2 = |PE|$, $h_3 = |PF|$. Any two of these distances, say h_1, h_2, exactly defined single distinct point P. It means that among the distances two are independent and third is dependent or the third distance

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h_3 is some function of h_1, h_2 and we could write

$$h_3 = f(h_1, h_2), \tag{1}$$

$$P = P(h_1, h_2, f(h_1, h_2))$$

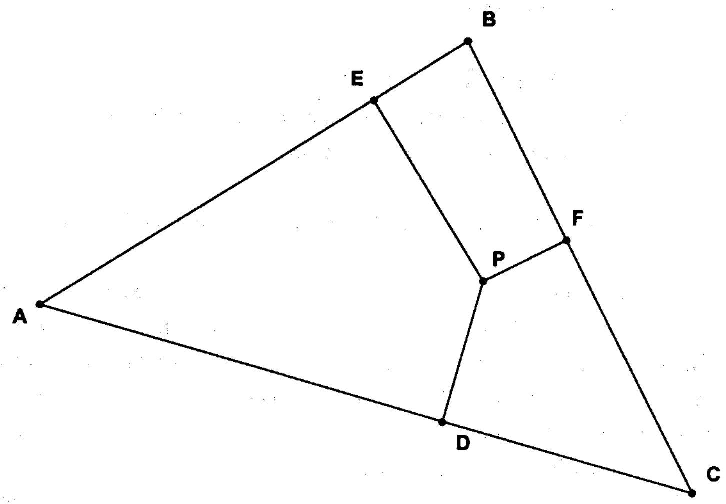

Figure. 1

The equation (1) is the expression of some unique relation

$$S_{\Delta ABC} = S_{\Delta APB} + S_{\Delta BPC} + S_{\Delta APC} \tag{2}$$

due to the fact that the point P is inside the triangle and for the distances h_1, h_2, h_3 it gives us the equation (1). The relation (2) is the very core for the solutions of all problems considered in the article. If, for instance, we find the other relation connecting the distances and which we think is independent of (2), then it gives us some second equation, which must be independent of the equation (1), and then the only one of the distances will be independent. This is impossible because the single distinct point P inside triangle is defined, as we say, by two distances, so the relation (2) must be unique.

PROBLEMS RELATED TO VIVIANI'S THEOREM

Problem 1. Viviani's Theorem

The Viviani's Theorem [http://en.wikipedia.org/wiki/Viviani's_theorem] states that the sum of the distances h_1, h_2, h_3 in the case of equilateral triangle is equal to the triangle's altitude h

$$h_1 = h_1 + h_2 + h_3 \tag{1a}$$

The relation between the distances from any point P inside the equilateral triangle to the sides of triangle is expressed by the equation (1a).

Problem 2. Generalization of Viviani's Theorem for any triangle.

We could generalize the Viviani's Theorem for any triangle (Figure.1).
Let

$|AC|=a, |AB|=b, |BC|=c, |PD|=h_1, |PE|=h_2, |PF|=h_3, h_a$
– altitude to the side AC, h – AB, h – to BC

Using (2) we could write three different equations

$$ah_a = ah_1 + bh_2 + ch_3, bh_b = ah_1 + bh_2 + ch_3, ch_c = ah_1 + bh_2 + ch_3.$$

Only one of these equations is independent. Remaining two equations could be derived from first one by the relation.

$$ah_a = ah_b = bh_c.$$

It is the proof of the generalized Viviani's theorem: For any ΔABC and for any point P inside the triangle the sum of the products ah_1, bh_2, ch_3 is constant

$$ah_a = ah_1 + bh_2 + ch_3 \tag{1b}$$

We could rewrite (1b) in more beautiful form

$$h_a = h_1 + \frac{b}{a} h_2 + \frac{c}{a} h_3$$

If the triangle is equilateral, then , $h_a = h_b = h_c = h, a = b = c$, and (1b) gives (1a).

Problem 3. Converse to Viviani's Theorem.

In the article (Chen and Liang, 2006) is given the proof of the converse to Viviani's Theorem using vectors. Converse means that if the sum of distances h_1, h_2, h_3 in the triangle is constant for any point P, then the triangle must be equilateral. We show how using the idea that exists only one independent relation between three distances this statement could be easily proved.

Let use the proof by contradiction. Suppose that the triangle is not equilateral. Then by the given we have the equation (1a), and by assumption the equation (1b). These two linear equations, which we rewrite in the form

$$h_2 + h_3 = h - h_1,$$

$$bh_2 = ch_3 + ah_a - ah_1,$$

must be dependent. Here h is some constant. If not, then among h_1, h_2, h_3 only one distance h_1 will be independent and this is impossible because any point P determinates by two distances. It gives us by methods of linear algebra that $b = c = a$. Thus the triangle is equilateral. We get the contradiction, which proves the converse to Viviani's Theorem.

Problem 4. The radius of an inscribe circle of a triangle.

We could use the equation (1b) to find the radius R of an inscribe circle of a triangle. Assume that P is the center of inscribed circle, then $h_1 = h_2 = h_3 = R$ and equation (2) gives

$$ah_a = R(a + b + c),$$

$$2S_{\Delta ABC} = R(a + b + c),$$

$$R = \frac{2S_{\Delta ABC}}{(a + b + c)}.$$

PROBLEMS RELATED TO ROUTH'S THEOREM

Problem 5. Routh's Theorem

The Routh's Theorem [http://en.wikipedia.org/wiki/Routh's_theorem] states there is the relation

$$\frac{2_{\Delta HIG}}{2_{\Delta ABC}} = \frac{(xyz - 1)^2}{(1 + x + xz)(1 + y + yz)(1 + z + zy)} \tag{3}$$

between the area of a given triangle and a triangle formed by the intersections of three cevians (Figure 2). The corresponding Theorem for the equilateral triangle is called one-seventh area triangle or Feynman's Theorem (Cook & Wood, 2004). In (Man, 2009) is given the proof of Routh's Theorem, which can be accessible even at junior secondary level. The solution presenting thereinafter also is very easy for understanding and performing, and could be used in the ordinary HS or CC in teaching the students.

Let us use the same type of reasoning, based on the distances from the point inside the triangle, their dependence and independence. This reasoning will lead us to the formula which is different from the formula (3), but which is the same in point of fact. For the distances from any point inside the triangle the equation (1b) is valid. Let us take one of the cevians intersections – point H. The relation (2) in this case could be written thus

$$S_{\Delta ABC} = S_{\Delta AHB} + S_{\Delta BHC} + S_{\Delta CHA} \quad (4)$$

The point H is uniquely defined by the given cevians AE and BF, that's why there must exist two more independent relations which together with (4) give three independent equations for the distances from point H to the sides of triangle ABC.

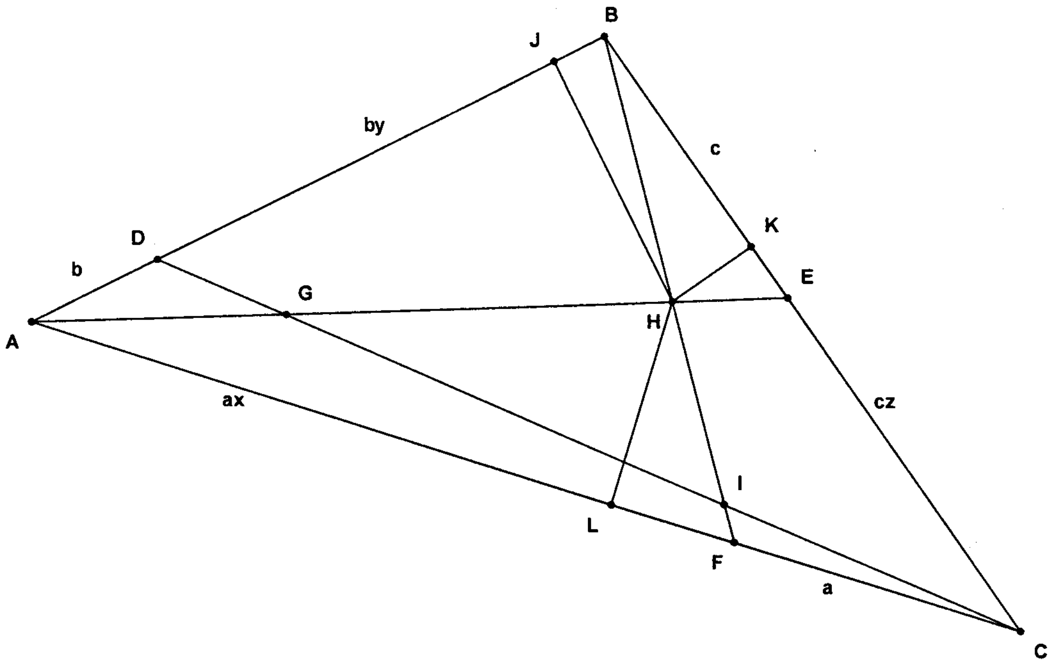

Figure 2.

As these independent relations we could choose

$$S_{\Delta ABF} = S_{\Delta AHF} + S_{\Delta AHB}, \quad (5)$$

$$S_{\Delta ABE} = S_{\Delta AHB} + S_{\Delta BHE}, \quad (6)$$

Relations (4)–(6) give us the following independent linear equations

$$a(1+x)h_a = b(1+y)h_b^H + c(1+z)h_c^H + a(1+x)h_a^H, \quad (4a)$$

$$axh_a = axh_a^H + b(1+y)h_b^H, \quad (5a)$$

$$ch_c = b(1+y)h_b^H + ch_c^H, \tag{6a}$$

where h_a, h_b, h_c - altitudes of the triangle ABC to the sides AC, AB, and CB, and h_a^H, h_b^H, h_c^H - distances from the point H to the sides AC, AB, and CB. We write the solution of the system in the form

$$h_a^H = \frac{1}{a(1+x)}\frac{xz}{(1+x+xz)}2S_{\Delta ABC}, \tag{7}$$

$$h_b^H = \frac{1}{b(1+y)}\frac{x}{(1+x+xz)}2S_{\Delta ABC},$$

$$h_c^H = \frac{1}{c(1+z)}\frac{1}{(1+x+xz)}2S_{\Delta ABC}.$$

Here we used the equality $a(1+x)h_a = b(1+y)h_b = c(1+z)h_c = 2S_{\Delta ABC}$. Similarly, or better using circular permutation, we could find the distances from the points I, G to the sides of the triangle. For instance, we got

$$h_a^I = \frac{1}{a(1+x)}\frac{1}{(1+y+yx)}2S_{\Delta ABC}, \tag{8}$$

$$h_a^G = \frac{1}{a(1+x)}\frac{z}{(1+z+zy)}2S_{\Delta ABC}. \tag{9}$$

Formulas (7)-(9) give us the solution of Routh's Theorem. We have (Figure 2)

$$S_{\Delta HIG} = S_{\Delta AHF} - S_{\Delta AGC} + S_{\Delta FIC}, \qquad \text{or}$$

$$2S_{\Delta HIG} = axh_a^H - a(1+x)h_a^G + ah_a^I =$$
$$= \left(\frac{x}{(1+x)}\frac{xz}{(1+x+xz)} - \frac{z}{(1+z+zy)} + \frac{1}{(1+x)}\frac{1}{(1+y+yx)}\right)2S_{\Delta ABC}$$

And finally for the relation between the area of a given triangle and a triangle formed by the intersections of three cevians we could write

$$\frac{S_{\Delta HIG}}{S_{\Delta ABC}} = \frac{x}{(1+x)}\frac{xz}{(1+x+xz)} - \frac{z}{(1+z+zy)} + \frac{1}{(1+x)}\frac{1}{(1+y+yx)}. \tag{10}$$

Of cause, (3) and (10) are formulas, functions, which expressed the same relation in the different way. If $x=2, y=2, z=2,$ then

$$\frac{S_{\Delta HIG}}{S_{\Delta ABC}} = \frac{1}{7}.$$

Problem 6. Ratios for areas of all figures formed by the sides of triangle and cevians.

We could find, based on the expressions for the distances from the points of intersections of cevians to the sides of triangle, the ratios of the areas of all figures inside the triangle (Figure 2) formed by the sides of triangle and cevians

$$\frac{S_{\Delta CFI}}{S_{\Delta ABC}} = \frac{1}{(1+x)(1+y+yx)}, \frac{S_{FIGA}}{S_{\Delta ABC}} = \frac{z}{1+z+zy} - \frac{1}{(1+x)(1+y+yx)},$$

$$\frac{S_{\Delta AGD}}{S_{\Delta ABC}} = \frac{1}{(1+y)(1+z+zy)}, \frac{S_{DGHB}}{S_{\Delta ABC}} = \frac{x}{1+x+xz} - \frac{1}{(1+y)(1+z+zy)},$$

$$\frac{S_{\Delta BHE}}{S_{\Delta ABC}} = \frac{1}{(1+z)(1+x+xz)}, \frac{S_{EHIC}}{S_{\Delta ABC}} = \frac{y}{1+y+yx} - \frac{1}{(1+z)(1+x+xz)}.$$

Thus in the case $x = 2, y = 2, z = 2,$ for the equilateral triangle (Figure 3)

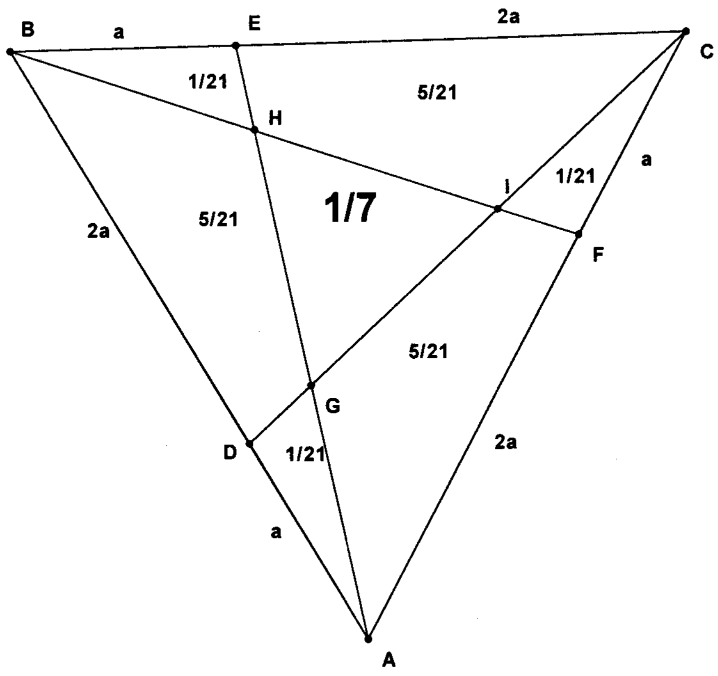

Figure 3.

we could get the following formulas

$$\frac{S_{\Delta ADG}}{S_{\Delta ABC}} = \frac{S_{\Delta BEH}}{S_{\Delta ABC}} = \frac{S_{\Delta CFI}}{S_{\Delta ABC}} = \frac{1}{21},$$

$$\frac{S_{BDHG}}{S_{\Delta ABC}} = \frac{S_{ECIH}}{S_{\Delta ABC}} = \frac{S_{AFIG}}{S_{\Delta ABC}} = \frac{5}{21}$$

CONCLUSION

For all the problems we considered the points inside the triangle. All the problems easily could be solved if the points are on one or two sides of the triangle.

For the teachers in High Schools and instructors in Community Colleges these problems give a possibility to create variety of really demanding problems for the classroom activities, the problems which needs not only solutions by means of some standard procedures, but mostly reasoning about the problems, the relations in them and connections with other topics of mathematics . Here it is also interesting that for the solution of geometrical problems students are using algebraic methods. We didn't formulate the problems exactly how they could appear in front of students; they could be formulated by teachers differently. We believe that the ideas which are in the core of reasoning and solution of problems will be also useful for the intellectual development of students.

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