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Teaching the Derivation of Area Formulas for Polygonal Regions through Dissection-Motion-Operations (DMO): A Visual Reasoning Approach[1]

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Utilizing a structure of operations known as Dissection-Motion-Operations (DMO), a set of mathematics propositions or area-formulas in school mathematics will be introduced through shape-to-shape transforms. The underlying theme for DMO is problem-solving through visual reasoning and proving manipulatively or electronically vs. rote learning and memorization. Visual reasoning is the focus here where two operations that constitute DMO are utilized. One operation is known as Dissection (or Decomposition) operation that operates on a given region in 2D or 3D and dissects it into a number of sub-regions. The second operation is known as Motion (or Composition) operation applied on the resultant sub-regions to form a distinct area (or volume)-equivalent region. In 2D for example, DMO can transform a given polygon into a variety of new and distinct polygons each of which is area-equivalent to the original polygon (*cf.* [Rahim, M. H. & Sawada, D. (1986). Revitalizing school geometry through Dissection-Motion Operations. *Sch. Sci. Math.* **86(3)**, 235–246] and [Rahim, M. H. & Sawada, D. (1990). The duality of qualitative and quantitative knowing in school geometry, *International Journal of Mathematical Education in Science and Technology* **21(2)**, 303–308])

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BACKGROUND

With the pre-service and in-service teachers in mind, the main purpose of this article is to introduce derivations of a set of area formulas through shape-to-shape transforms utilizing a structure of operations known as Dissection-Motion-Operations (DMO) through hands-on manipulation and the use of technology.

A further elaborative note on the structure of DMO outlined in the abstract is perhaps in order. There are three components that formulate DMO operation:

(1) Dissection Theory as it is described by Eves (1972, pp. 194–239). In 2D, through this component a polygonal region can be dissected into a finite number of certain sub-regions as one may choose (dissection is an 'inside of' or 'intra' operation), and in which there can only be vertical, horizontal, or oblique types of dissection. This categorization of the three types of dissections is made with respect to a side of the given shape – the horizontal base of the right triangle (as the given shape here) as shown in Figure 1 below.

(2) Motion (translation, rotation and reflection), through which one or more of the sub-regions are moved to another location without overlapping (motion is an 'among of' or 'inter' operation) and

(3) Recursion through which the above two operations or one of them may be repeated in the process of creating another shape. Together, these three components constitute the operation called Dissection-Motion-Operation (DMO) by which a polygonal region is transformable into another polygonal region of equal area.

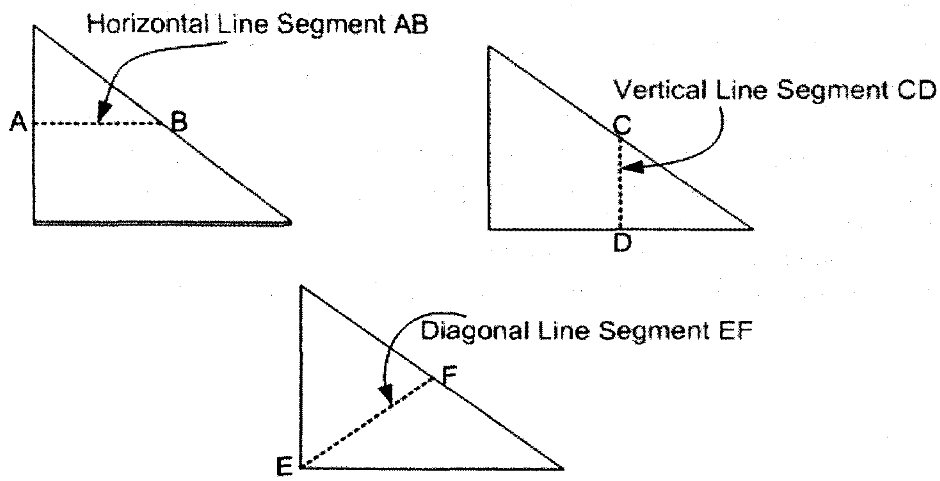

Figure 1. Eves' three suggested dissections of the Δ with respect to the horizontal base

One of the most striking ways to explain a mathematical concept effectively is through the use of manipulatives and model representations, concretely or electronically through a computer. Instructors, through concrete manipulation or graphic model representations, can convey and explain mathematical ideas and solving problems operatively through the combination of hands-and-mind actions (Maturana & Varela, 1998). Instantly, the ideas imbedded in DMO correspond directly to the "connection between action and experience" notion described by Humberto Maturana and Francisco Varela within their conception of cognition; they pointed out that "All doing is knowing and all knowing is doing"; there is no separation between cognition and other activities and no split between body and mind (Maturana & Verala, 1998, p. 26).

The National Council of Teachers of Mathematics (NCTM, 2000) recommends that through the study of geometry, students should acquire knowledge and skills to:

1. Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
2. Specify locations and describe spatial relationships using coordinate geometry and other representational systems;
3. Apply transformations and using symmetry to analyze mathematical situations, and
4. Use visualization, spatial reasoning, and geometric modeling to solve problems.

These NCTM recommendations are closely associated with the van Hiele's five Levels of geometric thought development where a student can only progress through them by instruction and his/her biological growth has no effect on the progress through the levels. Further, these levels must occur in their order as listed below (van Hiele, 1986). Below is a summary of van Hiele's levels:

Level 0 (Visualization): A geometric figure is viewed as a whole; no attention is given to the components of the figure.

Level 1 (Analysis): A geometric figure is recognized as having components and identified by these components. The relations among shapes are not yet recognized.

Level 2 (Informal Deduction): Simple relations among shapes and among properties of a shape are understood.

Level 3 (Formal Deduction): The role of axioms, definitions, undefined terms, postulates, proofs and theorems are understood. High school geometry is approached at this level.

Level 4 (Rigor): Different axiomatic systems such as Euclidean geometry and non-Euclidean geometry are understood.

Nowadays, as states, provinces and local school districts implement more rigorous assessments and accountability systems, teachers often confronted with long lists of mathematics topics or learning expectations to address at each grade level, with many topics repeated from year to year. Lacking clear, consistent priorities and focus, teachers draw out to find the time to present vital mathematical topics deeply and effectively. Consequently, the NCTM in response to this predicament has recently produced Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence (NCTM, 2006) states that

> To achieve the best results with students when teaching for the depth, understanding, and proficiency sought by the curriculum focal points, teachers themselves will need a deep understanding of the mathematics and facility with the relationships among mathematical ideas. Thus, effective instruction built on the curriculum focal points requires in-depth preparation of preservice and ongoing professional development for in-service teachers [bold added] (p. 7).

Realizing a parallel need for focus and coherence at the high school level, the NCTM has produced a second document for grades 9–12: *Focus in high school mathematics: Reasoning and sense making* (NCTM, 2008) which has put a very strong emphasis on reasoning and constructing convincing arguments in school mathematics.

DERIVATION OF AREA FORMULAS OF POLYGONS: WHERE THEY DID COME FROM?

In mathematics, any treatment has to have a starting point or assumption on which the rest of the treatment will rest. For establishing the formulas for polygonal shapes in 2D, the starting assumption considered is the *area of the rectangle*.

The area of a rectangular region is then taken as a postulate and be accepted as given. This assumption will serve as the starting point here; it will be the base to derive the area formulas of other polygonal regions in school mathematics. In addition to this assumption, DMO will be utilized throughout. As well, below is a recall of the properties of the three basic motions in plane geometry.

1. Translation: moving an object along a *straight* line direction; from a starting point to an end point—a required magnitude for the translation.
2. Rotation: rotates an object about *a centre of turn* and turns in a certain direction: left or right (clockwise or counterclockwise) and for a certain amount of turning angle.
3. Reflection: reflects an object about a line known as *a reflection line*; it is like a mirror with the object and its image *have identical distance* from the mirror (reflection line).

Area Formulas of Commonly Used Shapes in School Mathematics

1) The area of any rectangular region

This is our starting point in establishing the derivation of area formulas for many polygons. First, it is essential to specify that for a given square region with its side equals one unit in length, the area of the square region is one unit squared. The area of any shape then will be expressed by the number of the unit squares it contains (Figure 2).

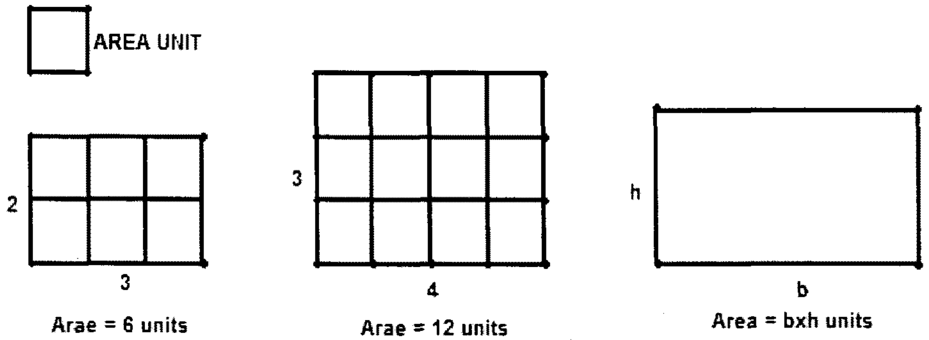

Figure 2. Area of the rectangle exemplified

The above illustration exemplifies the assumption that the area of the rectangle is the product of its base and altitude.

That is,

Area of a rectangular region = Base of Rectangle × Altitude of Rectangle

$$A = b \times a$$

where b is the base and a is the altitude of the rectangle regardless of its orientation (Figure 3).

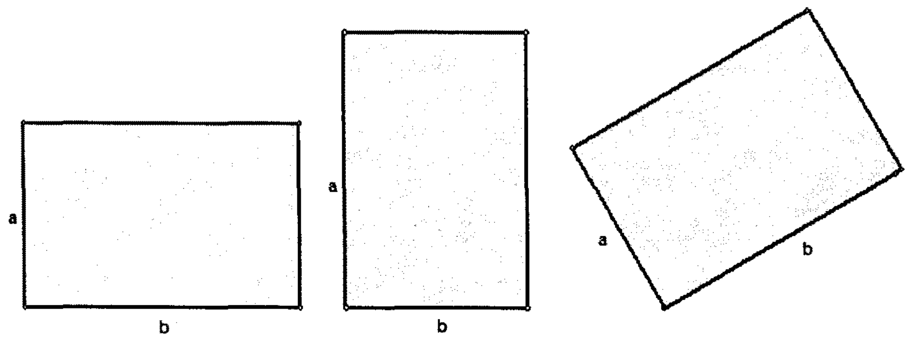

Figure 3. Different orientations of a rectangle

2) The area formula of a parallelogram region

Given a parallelogram region BCDE with base ***b*** and altitude ***a*** units (Figure 4).

By dissecting BCDE vertically at D, as shown in Figure 4 below, and translating (move) region 2 to the *left* a distance equals b units while keeping region 1 as is, the translation produces a rectangular region resembled as FGHI of equal area (*no overlapping*).

Also this result can be reached by translating (move) region 1 to the *right* a distance equals b units while keeping region 2 as is, the translation produces a rectangular region.

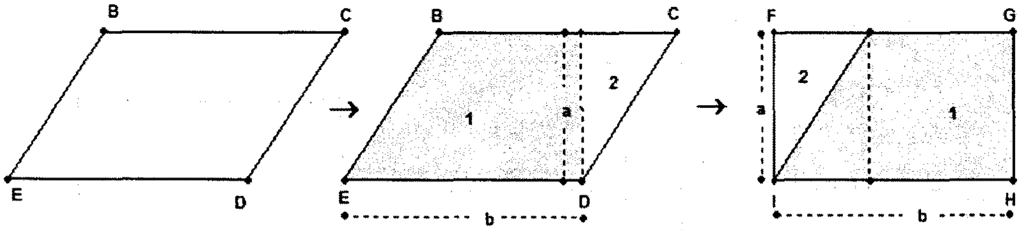

Figure 4. Shape transform using DMO of a parallelogram region into an area equivalent rectangular region

Based on our transformation shown in Figure 4 above, one can deduce that the area of the parallelogram region BCDE equals to the area of the rectangular region FGHI (same pieces with no overlapping - nothing added nor taken away).

Therefore

The area of the given parallelogram region

= the area of the resulting rectangular region

$= b \times a$

(by the assumption made earlier). That is,

Area of a Parallelogram Region = Base × Altitude

$= b \times a.$

Remark 1. Try to justify why FGHI is a rectangle.

3) The area formula of a triangular region

Any triangular region (acute, obtuse or right angle) can be transformed into an *area equivalent* rectangular region with base = **b** units and altitude = ½ **a** units as shown in Figure 5 below. Thus,

Area of the triangular region = Area of the resulting rectangular region

$= ½\, b \times a$

(by the assumption made earlier and Figure 5).

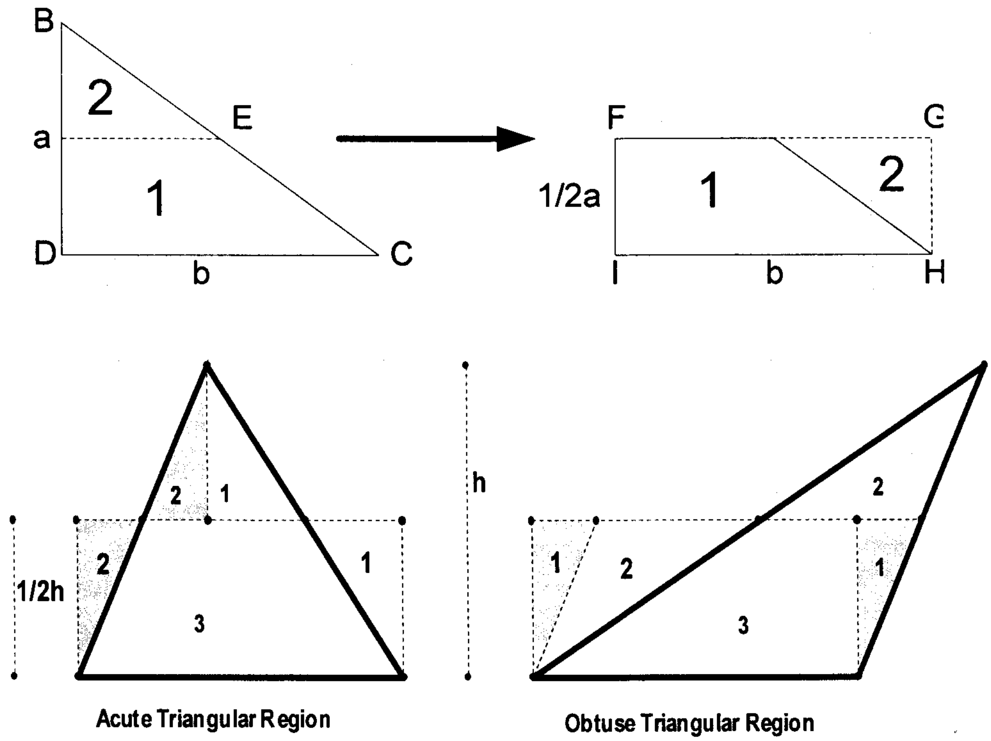

Figure 5. Shape transforms through DMO of all triangular regions into area equivalent rectangular regions

Remark 2. Try to justify why FGHI is a rectangle.

4) The area formula of a Trapezoid region

Figure 6 shows a pair of trapezoid regions, one with a right angle (right trapezoid) and another without a general trapezoid region with no specific condition). Through DMO, each shape is transformed into an area equivalent triangular region.

From Figure 6, it is clear that the area of the trapezoid region ABCD = the area of the resulting triangular region ABE; that is

Area of trapezoid region ABCD = Area of Δ region ABE
= ½ (base) × (height) (established earlier)

$\therefore$ Area of trapezoid region ABCD = ½ (b1 + b2) × (h).

That is,
Area of the trapezoid region ABCD = ½ (Sum of Bases) × (Height).

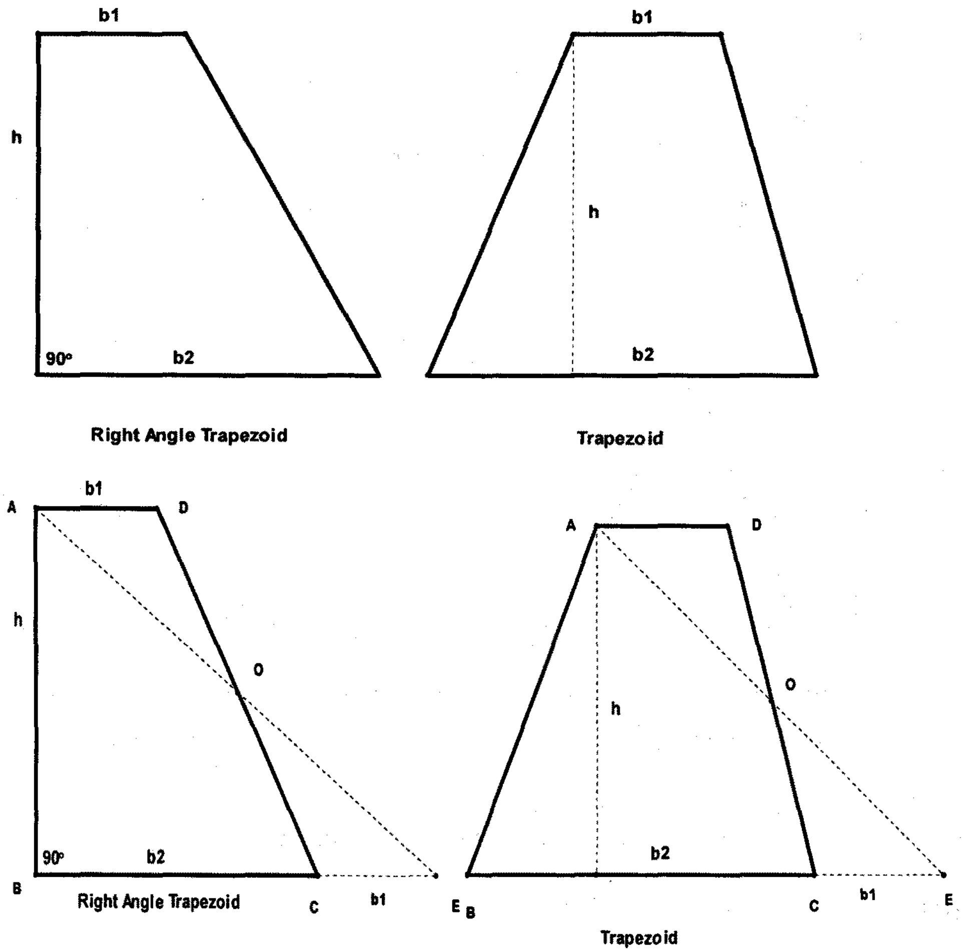

Figure 6. Shape transforms through DMO of trapezoid regions into area equivalent triangular regions

Remark 3. Try to show why AEB is a triangle.

5) The area formula of a Square Region

This is a direct result of the area postulate of a rectangular region since a square is a special case of the rectangle where the base and the height are equal.

Therefore the area of any square = Base × Height (by the assumption)

$$= S \times S = S^2.$$

Remark 4. Try to think of another way of getting the above result utilizing DMO.

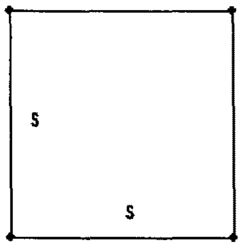

Figure 7. A square region

6) The area formula of a Rhombus Region

As shown in Figure 7a below, a rhombus region is a parallelogram with all sides are equal. To find the area formula for a rhombus region, dissect the region ABCD into three pieces along its diagonals as shown in Figure 7b. Recall that the rhombus diagonals are perpendicular.

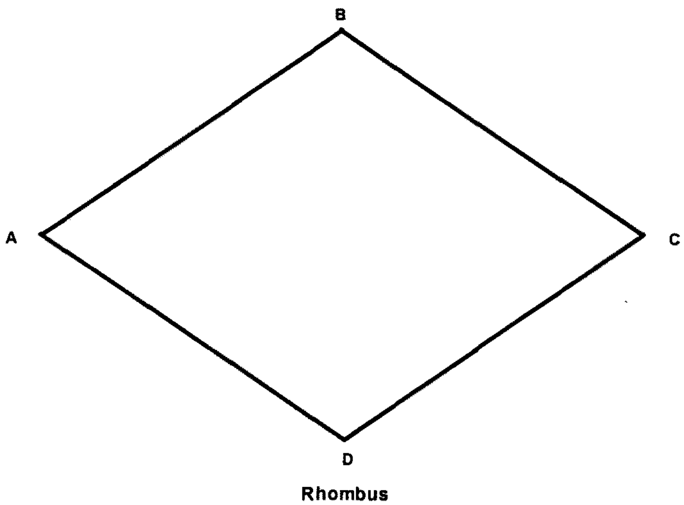

Figure 7a. ABCD is a rhombus region

Then, translate piece 1 along the side segment BC and piece 2 along the line segment BA as shown in Figure 7c below and the area equivalent rectangular region ACEF is constructed. (Try to show why ACEF is a rectangle).

The Figure 7c shows a shape transform of the rhombus region ABCD into an area equivalent rectangle ACEF.

Area of Rhombus region ABCD = Area of Rectangular region ACEF
= Base × Height
= (X)(1/2Y)

∴ Area of the Rhombus region ABCD = ½ (X)(Y).

That is,

Area of the Rhombus = half the product of its diagonals.

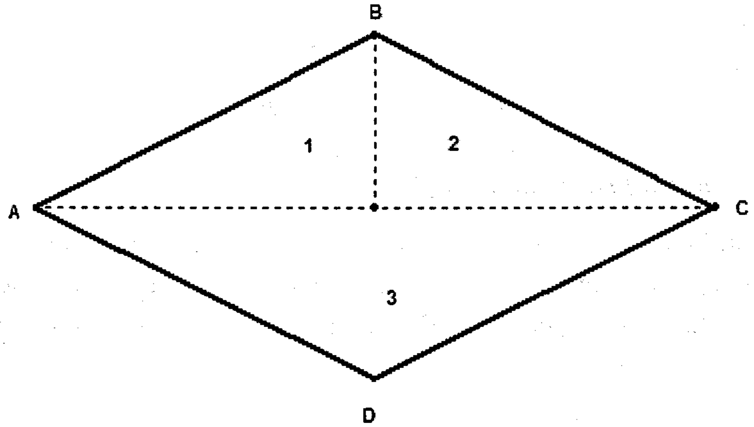

Figure 7b. Rhombus ABCD dissected into here pieces 1, 2 and 3

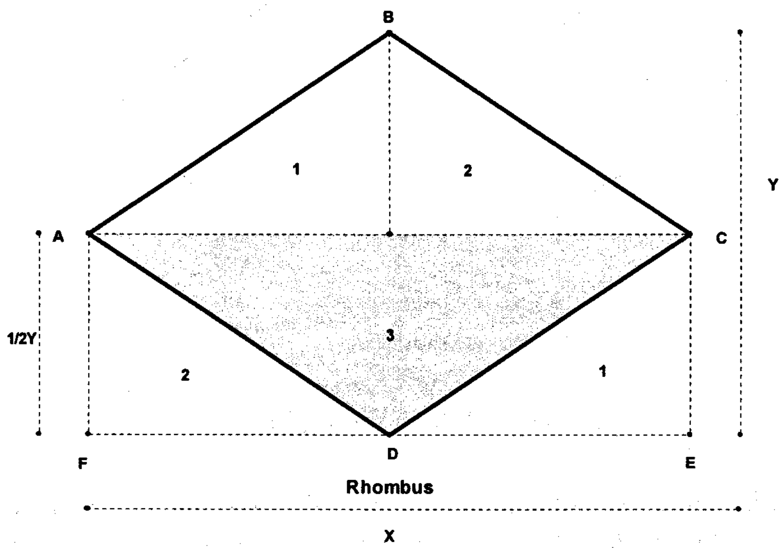

Figure 7c. Rhombus ABCD transformed by DMO into rectangle ACEF

Remark 5. Try to verify why ACEF is a rectangle.

Note that the dissection shown in Figure 7b may take an alternative form as shown in the upper part of Figure 7d below, and one will get the same result as illustrated in the lower part of Figure 7d.

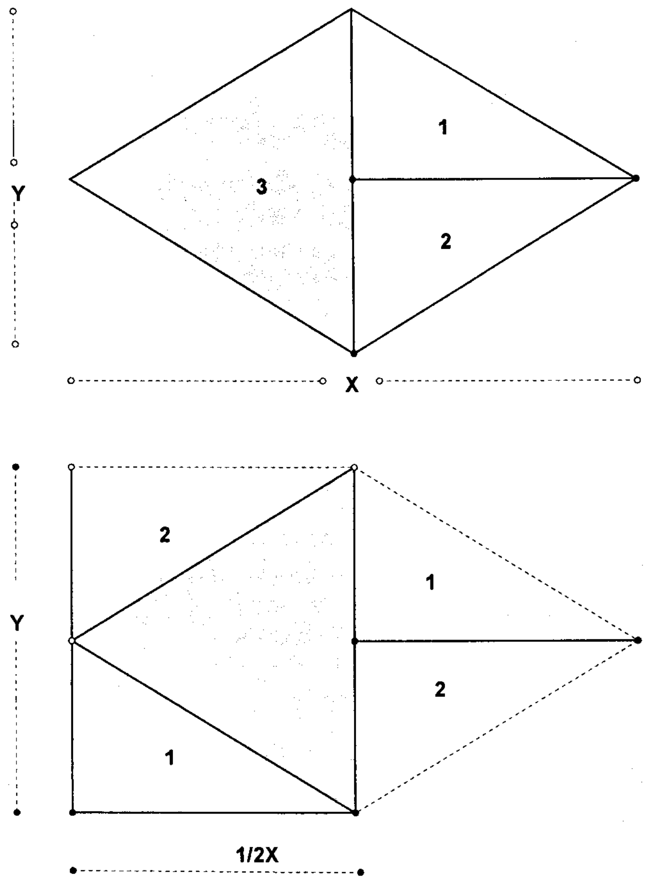

Figure 7d. Alternative shape transform of the rhombus region into an area equivalent rectangular region through DMO

7) Area of a Regular pentagon Region

In Figure 8a below, the left figure is a regular pentagon region ABCDE with O being the center of the circle passing through its vertices (circumscribed circle – not shown); the figure at the right shows ABCDE being dissected into five congruent triangles.

And, Δ ABO has base of b units and height h units.

Because the five triangular regions are congruent,

$\therefore$ Area of the regular pentagon region ABCDE

= 5(area of one of the triangular regions)

= 5(area of Δ OAB)

= 5(½ b × h)

= 5/2 (b × h).

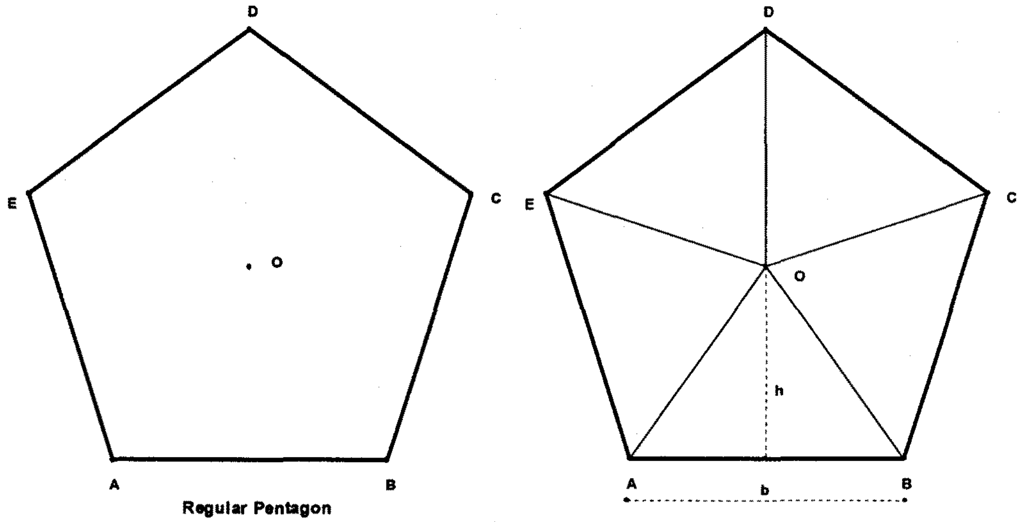

Figure 8a. Regular pentagon region ABCDE dissected
into five congruent triangular sub-regions

Remark 6. Try to justify why the five triangular regions are congruent.

Alternative Way: Another interesting way to find the area formula for a regular pentagon region is based on dissecting one triangular sub-region into two pieces, say Δ ABO, in through a perpendicular dissection with respect to its base AB as shown in Figure 8a above. Then rearrange the two pieces into a rectangular region. Repeat this process on each of the other triangular sub-regions to get 5 corresponding rectangular regions that can be arranged into a single rectangular region with base 5(½ b) and height h units as shown in Figure 8b below:

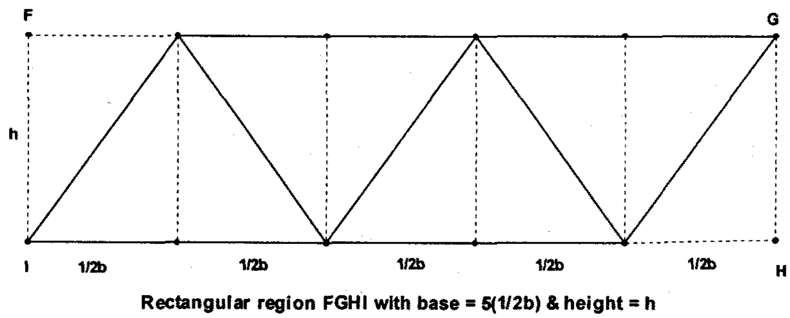

Figure 8b. Alternative transform of pentagon ABCDE
into a rectangle of equal area through DMO

Another alternative method is shown in Figure 9a. Below is a brief description of this method.

1) The dissection of Δ ABO into pieces 1, 2 and 3 is now based on two dissections, a horizontal and perpendicular with respect to the side AB (see Figure 9a).

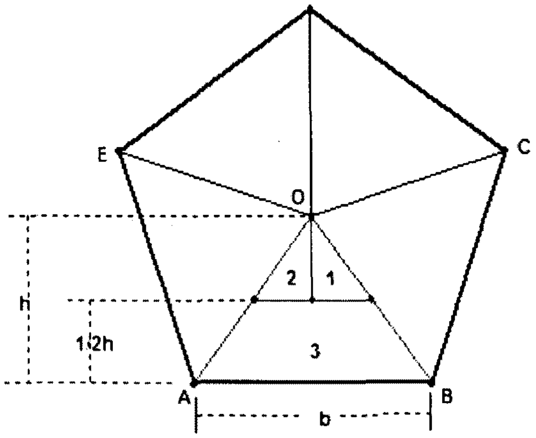

Figure 9a. Alternative dissection of ΔABO into three pieces 1, 2 &3

2) Dissect the pentagon region ABCDE into five triangular sub-regions and unfold the pieces as shown in Figure 9b.

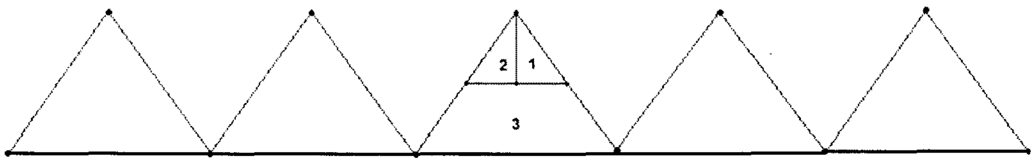

Figure 9b. Unfolding the five triangular sub-regions

3) Repeat the dissection made in step 1 for each of the remaining triangular sub-regions (Figure 9c).

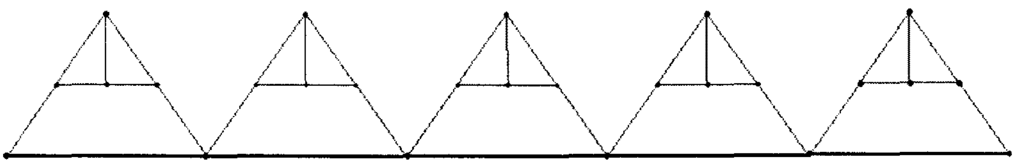

Figure 9c. Dissecting each Δ into three pieces

4) At each triangular sub-region, apply a 180° rotation on each of the pieces 1 and 2

(with center of turn being the midpoint of the sides resembled by OA and OB. The result should be identical to the long rectangular region show at the bottom of Figure 9d.

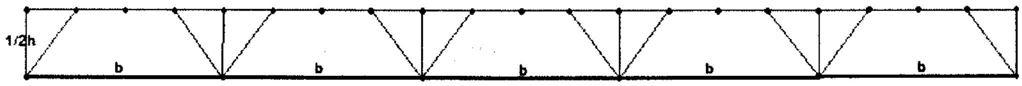

Figure 9d. Alternative way to transform pentagon ABCDE into area equivalent rectangular region

Thus, area of ABCDE = area of the long rectangular region shown in Figure 9d (with base 5b and height 1/2h units) →

$$\begin{aligned} \text{ABCDE area} &= (5b) \times (1/2h) \\ &= (5/2)\, b \times h. \end{aligned}$$

Finally, the above methods for establishing the area formula of a regular pentagon can be applied to find the area formula for all regular *n*-gon regions with $n \geq 3$.

APILOGUE

As it is presented above through many problem solving situations on construction and exploring 2D shapes, dissection-motion-operation (DMO) process seem to created a conceptual and developmental environment for close visual understanding of vital concepts in school mathematics. Construction, exploration and justification of shapes interrelationships by utilizing dissection-motion-operation (DMO) do offer an effective medium for understanding basic properties of a variety of shapes that are commonly used in school mathematics. The notion of 'mind and body' insuperability introduced by Humberto Maturana and Francisco Varela (1998) is certainly there in the midst of decomposition-composition processes within shape-to-shape transforms. Further, a close look into the two NCTM documents (2006 & 2008), one will immediately recognize the vitality of the DMO process especially for pre-service and in-survive teachers ... it is there!

In the technology arena, technology-based decomposition-composition constructions, that exemplify the presented cases here, will be suitable means for visual and spatial reasoning experiences through, for example, the Geometer's Sketchpad or Cabri.

It is my belief that the Maturana and Varela' proposition that "All doing is knowing, and all knowing is doing" is in the heart of DMO process; and the idea: there is no separation between cognition and other activities and no split between body and mind, ca be felt there too within the DMO process.

For more information on DMO, the reader may refer to Rahim (1986; 2003); Rahim & Sawada (1986; 1989; 1990) and Rahim, Sawada & Strasser (1996).

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