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OPTIMALITY AND DUALITY FOR GENERALIZED NONDIFFERENTIABLE FRACTIONAL PROGRAMMING WITH GENERALIZED INVEXITY†

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Abstract. Sufficient optimality conditions for a class of generalized nondifferentiable fractional optimization programming problems are established. Moreover, we prove the weak and strong duality theorems under (V, ρ)-invexity assumption.

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1. Introduction

In this paper, we consider the following generalized nondifferentiable fractional optimization problem (GFP):

$$\text{(GFP)} \qquad \begin{array}{ll} \text{Minimize} & \max\left\{\dfrac{f_i(x)+s(x|C_i)}{g_i(x)-s(x|D_i)} \mid i=1,\cdots,p\right\} \\ \text{subject to} & h_j(x)\le 0, \quad j=1,\cdots,m, \end{array}$$

where $f := (f_1,\cdots,f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $g := (g_1,\cdots,g_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h_1,\cdots,h_m) : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable. We assume that $g_i(x) - s(x|D_i) > 0$, $i = 1,\cdots,p$. For each $i = 1,\cdots,p$, C_i and D_i are compact convex set of $\mathbb{R}^n$ and we define a support function with respect to C_i as follows:

$$s(x|C_i) := \max\{\langle x, y_i\rangle \mid y_i \in C_i\}.$$

Further let, $J(x) = \{j : h_j(x) = 0\}$, for any $x \in \mathbb{R}^n$ and let

$$k_i(x) = s(x|C_i), \quad i = 1,\cdots,p.$$

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Then, k_i is a convex function and we can prove that

$$\partial k_i(x) = \{w_i \in C_i \mid \langle w_i, x\rangle = s(x|C_i)\},$$

where ∂k_i is the subdifferential of k_i.

Many authors have introduced various concepts of generalized convexity and have obtained optimality and duality results for a fractional programming problem ([2]–[9], [11]).

Recently, Kim and Kim [4] consider the following generalized nondifferentiable fractional optimization problem.

$$\text{(FP)} \qquad \begin{array}{ll} \text{Minimize} & \max\left\{\dfrac{f_i(x)+s(x|C_i)}{g_i(x)} \mid i=1,\cdots,p\right\} \\ \text{subject to} & h_j(x)\le 0, \quad j=1,\cdots,m, \end{array}$$

where $f := (f_1,\cdots,f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $g := (g_1,\cdots,g_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h_1,\cdots,h_m) : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable. We assume that $g_i(x) > 0$, $i = 1,\cdots,p$. For each $i = 1,\cdots,p$, C_i are compact convex set of $\mathbb{R}^n$.

In this paper, we apply the approach of Kim and Kim [4] to the generalized nondifferentiable fractional optimization problem (GFP), we establish the necessary and sufficient optimality conditions for a class of generalized nondifferentiable fractional optimization problem (GFP). Moreover, we prove the weak and strong duality theorems under (V,ρ)-invexity assumptions.

We introduce the following definition due to Kuk et al. [5].

Definition 1. A vector function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be (V,ρ)-invex at $u \in \mathbb{R}^n$ with respect to the functions η and $\theta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if there exists $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\rho_i \in \mathbb{R}, i = 1,\cdots,p$ such that for any $x \in \mathbb{R}^n$ and for all $i = 1,\cdots,p$,

$$\alpha_i(x,u)\big[f_i(x) - f_i(u)\big] \ge \nabla f_i(u)\eta(x,u) + \rho_i\|\theta_i(x,u)\|^2.$$

Definition 2. A vector function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be η-invex at $u \in \mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$ and for all $i = 1,\cdots,p$,

$$f_i(x) - f_i(u) \ge \nabla f_i(u)\eta(x,u).$$

We give the following theorem due to Kim et al. [2].

Theorem 1. Assume that f and g are vector-valued differentiable functions defined on X_0 and $f(x) + \langle w, x\rangle \ge 0$, $g(x) - \langle \widetilde{w}, x\rangle > 0$ for all $x \in X_0$. If

$f(\cdot)+\langle w,\cdot\rangle$ and $-g(\cdot)+\langle \widetilde{w},\cdot\rangle$ are (V,ρ)-invex at $x_0\in X_0$, then $\frac{f(\cdot)+\langle w,\cdot\rangle}{g(\cdot)-\langle \widetilde{w},\cdot\rangle}$ is (V,ρ)-invex at x_0, where

$$\bar{\alpha}_i(x,x_0)=\frac{g_i(x)-\langle \widetilde{w}_i,x\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\alpha_i(x,x_0),\quad \bar{\theta}_i(x,x_0)=\Big(\frac{1}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)^{\frac{1}{2}}\theta_i(x,x_0),$$

that is, for all i,

$$\begin{aligned}&\alpha_i(x,x_0)\Big[\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big]\\ &\geq \frac{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}\Big[\nabla\Big(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)\eta_i(x,x_0)\\ &\qquad+\rho_i\Big\|\Big(\frac{1}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)^{\frac{1}{2}}\theta_i(x,x_0)\Big\|^2\Big].\end{aligned}$$

Proof. Let $k_i(x)=s(x|C_i)$ and $\widetilde{k}_i(x)=s(x|D_i)$, $i=1,\dots,p$. Choose $w_i\in\partial k_i(x_0)$ and $\widetilde{w}_i\in\partial\widetilde{k}_i(x_0)$. Let $x,x_0\in X_0$. By the (V,ρ)-invexity of $f(\cdot)+\langle w,\cdot\rangle$ and $-g+\langle \widetilde{w},\cdot\rangle$,

$$\begin{aligned}&\alpha_i(x,x_0)\ \Big[\ \frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\ \Big]\\ &=\alpha_i(x,x_0)\Big[\frac{f_i(x)+\langle w_i,x\rangle-f_i(x_0)-\langle w_i,x_0\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}\\ &\qquad-(f_i(x_0)+\langle w_i,x_0\rangle)\frac{g_i(x)-\langle \widetilde{w}_i,x\rangle-g_i(x_0)+\langle \widetilde{w}_i,x_0\rangle}{(g_i(x)-\langle \widetilde{w}_i,x\rangle)(g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle)}\Big]\\ &\geq\frac{1}{g_i(x)-\langle \widetilde{w}_i,x\rangle}\Big[(\nabla f_i(x_0)+w_i)\eta_i(x,x_0)+\rho_i\|\theta_i(x,x_0)\|^2\Big]\\ &\qquad+\frac{f_i(x_0)+\langle w_i,x_0\rangle}{(g_i(x)-\langle \widetilde{w}_i,x\rangle)(g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle)}\Big[(-\nabla g_i(x_0)+\widetilde{w}_i)\eta_i(x,x_0)+\rho_i\|\theta_i(x,x_0)\|^2\Big].\end{aligned}$$

Since $g(x)-\langle \widetilde{w},x\rangle>0$ for all $x\in X_0$, we see that

$$\begin{aligned}&\alpha_i(x,x_0)\Big[\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big]\\ &\geq\frac{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}\Big[\frac{\nabla f_i(x_0)+w_i}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\eta_i(x,x_0)+\rho_i\Big\|\Big(\frac{1}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)^{\frac{1}{2}}\theta_i(x,x_0)\Big\|^2\\ &\qquad-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{(g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle)^2}(\nabla g_i(x_0)-\widetilde{w}_i)\eta_i(x,x_0)+\rho_i\Big\|\Big(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{(g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle)^2}\Big)^{\frac{1}{2}}\theta_i(x,x_0)\Big\|^2\Big].\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\alpha_i(x,x_0)\Big[\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big]\\
&\geq \frac{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}\\
&\Big[\frac{(\nabla f_i(x_0)+w_i)(g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle)-(f_i(x_0)+\langle \widetilde{w}_i,x_0\rangle)(\nabla g_i(x_0)-\widetilde{w}_i)}{(g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle)^2}\eta_i(x,x_0)\\
&\qquad +\rho_i\|\Big(\frac{1}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)^{\frac{1}{2}}\theta_i(x,x_0)\|^2+\rho_i\|\Big(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{(g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle)^2}\Big)^{\frac{1}{2}}\theta_i(x,x_0)\|^2\Big]\\
&=\frac{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}\Big[\nabla\Big(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)\eta_i(x,x_0)\\
&\qquad +\rho_i\|\Big(\frac{1}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)^{\frac{1}{2}}\Big(1+\Big(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)^{\frac{1}{2}}\Big)\theta_i(x,x_0)\|^2\Big].
\end{aligned}$$

Considering that

$$1+\Big(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)^{\frac{1}{2}}\geq 1,\ i=1,2,\dots,p,$$

we have for all i,

$$\begin{aligned}
&\alpha_i(x,x_0)\Big[\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big]\\
&\geq \frac{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}{g_i(x)-\langle \widetilde{w}_i,x\rangle}\Big[\nabla\Big(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)\eta_i(x,x_0)+\rho_i\|\Big(\frac{1}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)^{\frac{1}{2}}\theta_i(x,x_0)\|^2\Big].
\end{aligned}$$

Therefore, the function $\frac{f(x)+\langle w,x\rangle}{g(x)-\langle \widetilde{w},x\rangle}$ is (V,ρ)-invex, where

$$\begin{aligned}
\bar{\alpha}_i(x,x_0)&=\frac{g_i(x)-\langle \widetilde{w}_i,x\rangle}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\alpha_i(x,x_0),\\
\bar{\theta}_i(x,x_0)&=\Big(\frac{1}{g_i(x_0)-\langle \widetilde{w}_i,x_0\rangle}\Big)^{\frac{1}{2}}\theta_i(x,x_0).
\end{aligned}$$

□

2. Optimality Conditions

Now, we establish the Kuhn-Tucker necessary and sufficient conditions for a solution of (GFP).

Theorem 2. (Kuhn-Tucker Necessary Optimality Theorem) If x_0 is a solution of (GFP), and assume that $0\notin co\{\nabla h_j(x_0)\mid j\in J(x_0)\}$, then there exist

$\lambda_i \geq 0,\ i \in I(x_0) := \{i \mid \max\Big\{\frac{f_i(x_0)+s(x_0|C_i)}{g_i(x_0)-s(x_0|D_i)} \mid i = 1, \cdots, p\Big\}\},\ \sum_{i\in I(x_0)} \lambda_i = 1,\ \mu_j \geq 0,\ j = 1, \cdots, m$ and $w_i \in C_i,\ \widetilde{w}_i \in D_i,\ i \in I(x_0)$ such that

$$\begin{aligned}
&\sum_{i\in I(x_0)} \lambda_i \nabla\Big(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0) - \langle \widetilde{w}_i, x_0\rangle}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,\\
&\langle w_i, x_0\rangle = s(x_0|C_i),\ \langle \widetilde{w}_i, x_0\rangle = s(x_0|D_i),\\
&\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.
\end{aligned}$$

Proof. Let $\varphi_i(x) = \frac{f_i(x)+s(x|C_i)}{g_i(x)-s(x|D_i)}, i = 1, \cdots, p$. Let x_0 be a solution of (GFP) and let $I(x_0) = \{i \mid \max\{\varphi_i(x_0) \mid i = 1, \cdots, p\}\}$. Then by Proposition 2.3.12 in [1] and Corollary 5.1.8 in [10], there exists $\mu_j \geq 0,\ j = 1, \cdots, m$,

$$\begin{aligned}
&0 \in \mathrm{co}\{\partial^c \varphi_i(x_0) \mid i \in I(x_0)\} + \sum_{j=1}^{m} \mu_j \partial^c h_j(x_0)\\
&\text{and}\ \ \mu_j h_j(x_0) = 0.
\end{aligned}$$

Thus there exists $\lambda_i \geq 0,\ i \in I(x_0),\ \sum_{i\in I(x_0)} \lambda_i = 1$ such that

$$\begin{aligned}
&0 \in \sum_{i\in I(x_0)} \lambda_i \partial^c \varphi_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) \qquad (2.1)\\
&\text{and}\ \ \mu_j h_j(x_0) = 0.
\end{aligned}$$

By Proposition 2.3.14 in [1],

$$\begin{aligned}
\partial^c \varphi_i(x_0) = \frac{1}{(g_i(x_0) - s(x_0|D_i))^2}\Big((g_i(x_0) - s(x_0|D_i))(\nabla f_i(x_0) + \partial s(x_0|C_i))\\
-(f_i(x_0) + s(x_0|C_i))(\nabla g_i(x_0) - \partial s(x_0|D_i))\Big).
\end{aligned}$$

Since

$$\begin{aligned}
\partial^c \varphi_i(x_0) \ &= \ \Big\{\frac{1}{(g_i(x_0) - \langle \widetilde{w}_i, x_0\rangle)^2}\Big((g_i(x_0) - \langle \widetilde{w}_i, x_0\rangle)(\nabla f_i(x_0) + w_i)\\
&\quad -(f_i(x_0) + \langle w_i, x_0\rangle)(\nabla g_i(x_0) - \widetilde{w}_i)\Big) \mid w_i \in C_i, \langle w_i, x_0\rangle = s(x_0|C_i),\\
&\quad \langle \widetilde{w}_i, x_0\rangle = s(x_0|D_i),\ i \in I(x_0)\}\\
&= \ \Big\{\nabla\Big(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0) - \langle \widetilde{w}_i, x_0\rangle}\Big) | w_i \in C_i,\ \widetilde{w}_i \in D_i,\ \langle w_i, x_0\rangle = s(x_0|C_i),\\
&\quad \langle \widetilde{w}_i, x_0\rangle = s(x_0|D_i), i \in I(x_0)\}
\end{aligned}$$

and hence from (2.1), there exist $\lambda_i \geq 0,\ i \in I(x_0),\ \sum_{i\in I(x_0)} \lambda_i = 1,\ \mu_j \geq 0,\ j = 1, \cdots, m$ and $w_i \in C_i,\ i \in I(x_0)$ such that

$$\begin{aligned}
&\sum_{i\in I(x_0)} \lambda_i \nabla\Big(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0) - \langle \widetilde{w}_i, x_0\rangle}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,\\
&\langle w_i, x_0\rangle = s(x_0|C_i),\ \langle \widetilde{w}_i, x_0\rangle = s(x_0|D_i),\\
&\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.
\end{aligned}$$

□

Theorem 3. (Kuhn-Tucker Sufficient Optimality Theorem) Let x_0 be a feasible solution of (GFP). Suppose that there exist $\lambda_i \geq 0,\ i \in I(x_0)$, $\sum_{i\in I(x_0)} \lambda_i = 1,\ \mu_j \geq 0,\ j = 1, \cdots, m$ and $w_i \in C_i,\ \widetilde{w}_i \in D_i,\ i \in I(x_0)$ such that

$$\begin{aligned}
&\sum_{i\in I(x_0)} \lambda_i \nabla\Big(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0) - \langle \widetilde{w}_i, x_0\rangle}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0, \qquad (2.2)\\
&\langle w_i, x_0\rangle = s(x_0|C_i),\ \langle \widetilde{w}_i, x_0\rangle = s(x_0|D_i),\\
&\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.
\end{aligned}$$

If $f(\cdot)+\langle w, \cdot\rangle$ and $-g(\cdot)+\langle \widetilde{w}, \cdot\rangle$ are (V, ρ)-invex at x_0, and h is η-invex at x_0 with respect to the same η, and $\sum_{i\in I(x_0)} \lambda_i \rho_i \|\bar{\theta}_i(x, x_0)\|^2 \geq 0$, then x_0 is a solution of (GFP).

Proof. Suppose that x_0 is not a solution of (GFP). Then there exist a feasible solution x of (GFP) such that

$$\max_{1\leq i\leq p} \frac{f_i(x) + s(x|C_i)}{g_i(x) - s(x|D_i)} < \max_{1\leq i\leq p} \frac{f_i(x_0) + s(x_0|C_i)}{g_i(x_0) - s(x_0|D_i)}.$$

Then

$$\frac{f_i(x) + s(x|C_i)}{g_i(x) - s(x|D_i)} < \frac{f_i(x_0) + s(x_0|C_i)}{g_i(x_0) - s(x_0|D_i)}, \quad \text{for all } i \in I(x_0).$$

Since $\langle w_i, x_0\rangle = s(x_0|C_i)$, $w_i \in C_i$, and $\langle \widetilde{w}_i, x_0\rangle = s(x_0|D_i)$, $\widetilde{w}_i \in D_i$, we have for all $i \in I(x_0)$,

$$\begin{aligned}\frac{f_i(x)+\langle w_i, x\rangle}{g_i(x)-\langle \widetilde{w}_i, x\rangle} &\leq \frac{f_i(x)+s(x|C_i)}{g_i(x)-s(x|D_i)}\\ &< \frac{f_i(x_0)+s(x_0|C_i)}{g_i(x_0)-s(x_0|D_i)}\\ &= \frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i, x_0\rangle}\end{aligned}$$

and hence $\bar{\alpha}_i(x, x_0) > 0$,

$$\bar{\alpha}_i(x, x_0)\Big[\frac{f_i(x)+\langle w_i, x\rangle}{g_i(x)-\langle \widetilde{w}_i, x\rangle} - \frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i, x_0\rangle}\Big] < 0.$$

By the (V, ρ)-invexity of $f(\cdot) + \langle w, \cdot\rangle$ and $-g(\cdot) + \langle \widetilde{w}, \cdot\rangle$ at x_0, and by Theorem 1, we have

$$\nabla\Big(\frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i, x_0\rangle}\Big)\eta(x, x_0) + \rho_i\|\bar{\theta}_i(x, x_0)\|^2 < 0.$$

Hence, we have

$$\sum_{i\in I(x_0)} \lambda_i \nabla\Big(\frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i, x_0\rangle}\Big)\eta(x, x_0) + \sum_{i\in I(x_0)} \lambda_i\rho_i\|\bar{\theta}_i(x, x_0)\|^2 < 0.$$

Since $\sum_{i\in I(x_0)} \lambda_i\rho_i\|\bar{\theta}_i(x, x_0)\|^2 \geq 0$,

$$\sum_{i\in I(x_0)} \lambda_i \nabla\Big(\frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)-\langle \widetilde{w}_i, x_0\rangle}\Big)\eta(x, x_0) < 0$$

and so, it follows from (2.2) that

$$\sum_{j=1}^{m} \mu_j \nabla h_j(x_0)\eta(x, x_0) > 0.$$

Then, by the η-invexity of h, we have

$$\sum_{j=1}^{m} \mu_j h_j(x) - \sum_{j=1}^{m} \mu_j h_j(x_0) > 0.$$

Since $\sum_{j=1}^{m} \mu_j h_j(x_0) = 0$, we have $\sum_{j=1}^{m} \mu_j h_j(x) > 0$, which is a contradiction since $\mu_j \geq 0,\ j = 1, \cdots, m$ and x is a feasible solution of (GFP). Consequently, x_0 is a solution of (GFP). □

3. Duality Theorems

Now, we propose the following Mond-Weir type dual problem (DGFP):

(DGFP)

$$\begin{aligned}
&\text{Maximize} \quad \max\left\{\frac{f_i(u)+s(u|C_i)}{g_i(u)-s(u|D_i)} \mid i=1,\cdots,p\right\}\\
&\text{subject to} \quad \sum_{i\in I(u)} \lambda_i \nabla\Big(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)-\langle \widetilde{w}_i,u\rangle}\Big)+\sum_{j=1}^{m}\mu_j\nabla h_j(u)=0, \qquad (3.1)\\
&\qquad w_i\in C_i,\ \widetilde{w}_i\in D_i,\ \langle w_i,u\rangle=s(u|C_i),\ \langle \widetilde{w}_i,u\rangle=s(u|D_i),\ i\in I(u)\\
&\qquad \sum_{j=1}^{m}\mu_j h_j(u)=0,\\
&\qquad \lambda_i\geq 0,\ i\in I(u),\ \sum_{i\in I(u)}\lambda_i=1,\ \mu_j\geq 0,\ j=1,\cdots,m.
\end{aligned}$$

Now we show that the following weak duality theorem holds between (GFP) and (DGFP).

Theorem 4. (Weak Duality) Let x be a feasible for (GFP) and let (u,λ,μ,w) be feasible for (DGFP). Assume that $f(\cdot)+\langle w,\cdot\rangle$ and $-g(\cdot)+\langle \widetilde{w},\cdot\rangle$ are (V,ρ)-invex at u, and let h is η-invex at u with respect to the same η, and $\sum_{i\in I(u)}\lambda_i\rho_i\|\bar{\theta}_i(x,u)\|^2\geq 0$. Then the following holds:

$$\max\left\{\frac{f_i(x)+s(x|C_i)}{g_i(x)-s(x|D_i)} \mid i=1,\cdots,p\right\}\geq \max\left\{\frac{f_i(u)+s(u|C_i)}{g_i(u)-s(u|D_i)} \mid i=1,\cdots,p\right\}.$$

Proof. Let x be any feasible for (GFP) and let (u,λ,μ,w) be any feasible for (DGFP). Then we have

$$\sum_{j=1}^{m}\mu_j h_j(x)\leq 0\leq \sum_{j=1}^{m}\mu_j h_j(u).$$

By the η-invexity of $h_j(u),\ j=1,\cdots,m$, we have

$$\sum_{j=1}^{m}\mu_j\nabla h_j(u)\eta(x,u)\leq 0.$$

Using (3.1), we obtain

$$\sum_{i\in I(u)}\lambda_i\nabla\Big(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)-\langle \widetilde{w}_i,u\rangle}\Big)\eta(x,u)\geq 0. \qquad (3.2)$$

Now suppose that

$$\max\left\{\frac{f_i(x)+s(x|C_i)}{g_i(x)-s(x|D_i)} \mid i=1,\cdots,p\right\}< \max\left\{\frac{f_i(u)+s(u|C_i)}{g_i(u)-s(u|D_i)} \mid i=1,\cdots,p\right\}.$$

Then

$$\frac{f_i(x)+s(x|C_i)}{g_i(x)-s(x|D_i)} < \frac{f_i(u)+s(u|C_i)}{g_i(u)-s(u|D_i)}, \quad \text{for all } i \in I(u).$$

Since $\langle w_i, u\rangle = s(u|C_i)$ and $\langle \widetilde{w}_i, u\rangle = s(u|D_i)$, we have for all $i \in I(u)$,

$$\frac{f_i(x)+\langle w_i, x\rangle}{g_i(x)-\langle \widetilde{w}_i, x\rangle} < \frac{f_i(u)+\langle w_i, u\rangle}{g_i(u)-\langle \widetilde{w}_i, u\rangle}.$$

By the (V,ρ)-invexity of $f(\cdot)+\langle w,\cdot\rangle$ and $-g(\cdot)+\langle \widetilde{w},\cdot\rangle$ at x_0, and by Theorem 1, we have,

$$\begin{aligned} 0 &> \bar{\alpha}_i(x,u)\Big[\frac{f_i(x)+\langle w_i, x\rangle}{g_i(x)-\langle \widetilde{w}_i, x\rangle} - \frac{f_i(u)+\langle w_i, u\rangle}{g_i(u)-\langle \widetilde{w}_i, u\rangle}\Big] \\ &\geq \nabla\Big(\frac{f_i(u)+\langle w_i, u\rangle}{g_i(u)-\langle \widetilde{w}_i, u\rangle}\Big)\eta(x,u) + \rho_i\|\bar{\theta}_i(x,u)\|^2. \end{aligned}$$

By using $\lambda_i \geq 0,\ i \in I(u)$, we have,

$$\sum_{i\in I(u)} \lambda_i \nabla\Big(\frac{f_i(u)+\langle w_i, u\rangle}{g_i(u)-\langle \widetilde{w}_i, u\rangle}\Big)\eta(x,u) + \sum_{i\in I(u)} \lambda_i\rho_i\|\bar{\theta}_i(x,u)\|^2 < 0.$$

Since $\sum_{i\in I(u)} \lambda_i\rho_i\|\bar{\theta}_i(x,u)\|^2 \geq 0$, we have

$$\sum_{i\in I(u)} \lambda_i \nabla\Big(\frac{f_i(u)+\langle w_i, u\rangle}{g_i(u)-\langle \widetilde{w}_i, u\rangle}\Big)\eta(x,u) < 0,$$

which contradicts (3.2). Hence the result holds. □

Now we give a strong duality theorem which holds between (GFP) and (DGFP).

Theorem 5. (Strong Duality) If $\bar{x}$ be a solution of (GFP) and suppose that $0 \notin \mathrm{co}\{\nabla h_j(\bar{x}) \mid j \in J(\bar{x})\}$. Then there exist $\bar{\lambda} \in \mathbb{R}^p,\ \bar{\mu} \in \mathbb{R}^m$ and $\bar{w} \in C$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \overline{\widetilde{w}})$ is feasible for (DGFP). Moreover if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \overline{\widetilde{w}})$ is a solution of (DGFP).

Proof. By Theorem 2, there exist $\bar{\lambda} \in \mathbb{R}^p,\ \bar{\mu} \in \mathbb{R}^m$ and $\bar{w}_i \in C_i,\ \overline{\widetilde{w}} \in D_i,\ i \in I(\bar{x})$, such that

$$\begin{aligned} &\sum_{i\in I(\bar{x})} \bar{\lambda}_i \nabla\Big(\frac{f_i(\bar{x})+\langle \bar{w}_i, \bar{x}\rangle}{g_i(\bar{x})-\big\langle \overline{\widetilde{w}}_i, \bar{x}\big\rangle}\Big) + \sum_{j=1}^{m} \bar{\mu}_j \nabla h_j(\bar{x}) = 0, \\ &\langle \bar{w}_i, \bar{x}\rangle = s(\bar{x}|C_i),\ \big\langle \overline{\widetilde{w}}_i, u\big\rangle = s(\bar{x}|D_i), \\ &\sum_{j=1}^{m} \mu_j h_j(\bar{x}) = 0, \\ &\lambda_i \geq 0,\ i \in I(\bar{x}),\ \sum_{i\in I(\bar{x})} \lambda_i = 1. \end{aligned}$$

Thus $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \overline{\widetilde{w}})$ is a feasible for (DGFP). On the other hand, by weak duality (Theorem 4),

$$\max\left\{\frac{f_i(\bar{x}) + s(\bar{x}|C_i)}{g_i(\bar{x}) - s(\bar{x}|D_i)} \mid i = 1, \cdots, p\right\} \geq \max\left\{\frac{f_i(u) + s(u|C_i)}{g_i(u) - s(u|D_i)} \mid i = 1, \cdots, p\right\}$$

for any (DGFP) feasible solution $(u, \lambda, \mu, w, \widetilde{w})$. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \widetilde{w})$ is a solution of (DGFP). □

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