OPTIMALITY AND DUALITY IN NONSMOOTH VECTOR OPTIMIZATION INVOLVING GENERALIZED INVEX FUNCTIONS†

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ABSTRACT. In this paper, we consider nonsmooth optimization problem of which objective and constraint functions are locally Lipschitz. We establish sufficient optimality conditions and duality results for nonsmooth vector optimization problem given under nearly strict invexity and near invexity-infineness assumptions.

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1. Introduction and Preliminaries

The convexity of sets and the convexity or concavity of functions have been the object of many investigations during the past century. This is mainly due to the development of the theory of mathematical programming, both linear and nonlinear, which is closely tied with convex analysis. Optimality conditions, duality and related algorithms were mainly established for classes of problems involving the optimization of convex objective functions over convex feasible regions. Such assumptions were very convenient, due to the basic properties of convex (or concave) functions concerning optimality conditions. In 1981, Hanson [2] introduced a differentiable invex functions, which is an important generalization of differentiable convex function, and established the Kuhn-Tucker sufficient optimality conditions and duality results for a nonlinear optimization problem involving differentiable invex functions.

Very recently, Lee, Kim and Sach [4] defined the near invexity for locally Lipschitz function, which is a generalization of invexity. Kim [3] studied a sufficient optimality conditions and duality results for a weakly efficient solution of nonsmooth vector optimization problem of which the objective functions are nearly invex and the constraint functions are nearly invex-infine. However, until now, we do not have the duality results for efficient solutions of the nonsmooth vector optimization problem under near invexity and near invexity-infineness assumptions. In this paper, we prove a sufficient optimality conditions and duality results for an efficient solution of the nonsmooth vector optimization problem under nearly strict invexity and near invexity-infineness assumptions.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz if for any $z \in \mathbb{R}^\ltimes$ there exist $K > 0$ and a neighborhood N of z such that, for each $x, y \in N$, $|f(x) - f(y)| \leq K\|x - y\|$.

Consider the following vector optimization problem (P):

$$(P) \quad \text{Minimize} \quad f(x) := (f^1(x), \cdots, f^m(x))$$
$$\text{subject to} \quad g^j(x) \leq 0, \ j = 1, 2, \cdots, p,$$
$$h^l(x) = 0, \ l = 1, 2, \cdots, q,$$
$$x \in C,$$

where $f := (f^1, f^2, \cdots, f^m) : \mathbb{R}^n \to \mathbb{R}^m$, $g := (g^1, g^2, \cdots, g^p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h^1, h^2, \cdots, h^q) : \mathbb{R}^n \to \mathbb{R}^q$ are locally Lipschitz functions. Let

$$Q = \{x \in C : g^j(x) \leq 0, \ j = 1, 2, \cdots, p, \ h^l(x) = 0, \ l = 1, 2, \cdots, q\}$$

and C be a closed subset of $\mathbb{R}^n$. Let $x_0 \in Q$ and let $J(x_0) = \{j : g^j(x_0) = 0\}$.

Now we give solution conceptions for (P) as follows:

Definition 1. [5] A feasible point $\bar{x} \in Q$ is said to be an efficient solution (a Pareto solution) of (P) if there exists no $x \in Q$ such that

$$f(x) \leq f(\bar{x}) \ \text{ and } \ f(x) \neq f(\bar{x}).$$

Definition 2. [1] Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $x \in \mathbb{R}^n$. The Clarke generalized directional derivative of f at x in the direction of $v \in \mathbb{R}^n$ is defined by

$$f^0(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

The Clarke generalized subgradient of a locally Lipschitz function f at x is denoted by

$$\partial^c f(x) = \left\{\xi \in \mathbb{R}^n \mid f^0(x; v) \geq \xi^T v \ \text{ for all } v \in \mathbb{R}^n\right\}.$$

The Clarke tangent cone and the Clarke normal cone of a subset $C \subset \mathbb{R}^n$ at $x_0 \in C$ are denoted by $T_C(x_0)$ and $N_C(x_0)$, respectively. Recall that

$$T_C(x_0) = \{\eta \in \mathbb{R}^n : \rho_C^0(x_0; \eta) = 0\},$$
$$N_C(x_0) = \{\xi \in \mathbb{R}^n : \xi^T \eta \leq 0 \ \ \forall \eta \in T_C(x_0)\},$$

where $\rho_C(x) = \rho(x, C)$ is the distance from $x \in \mathbb{R}^n$ to C and $\rho_C^0(x_0; \eta)$ is the Clarke generalized directional derivative of ρ_C at x_0 in the direction $\eta \in \mathbb{R}^n$.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz vector-valued function with components φ^i, $i \in I := \{1, 2, \cdots, m\}$:

$$\varphi(x) = (\varphi^1(x), \varphi^2(x), \cdots, \varphi^m(x)), \ x \in \mathbb{R}^n.$$

Let C be a subset of $\mathbb{R}^n$ and let $x_0 \in C$.

The following generalized convexity notion is taken from Sach *et al.* [6].

Definition 3. A vector-valued function φ is said to be invex on C at $x_0 \in C$ if

$$\forall x \in C, \ \forall \xi_i \in \partial \varphi^i(x_0), \ i \in I, \ \exists \eta \in T_C(x_0) \text{ such that}$$
$$\varphi^i(x) - \varphi^i(x_0) \geq \xi_i^T \eta, \ i \in I.$$

When $m = 1$, φ is of class C^1, and $C = \mathbb{R}^n$ this definition reduces to the generalized convexity notion first given by Hanson [2]. Very recently, Lee, Kim and Sach [4] defined the following near invexity for locally Lipschitz functions, which is a generalization of invexity in Definition 3.

Definition 4. A vector-valued function φ is said to be nearly invex on C at $x_0 \in C$ if

$$\forall x \in C, \ \forall \xi_i \in \partial \varphi^i(x_0), \ i \in I, \ \exists \eta_k \in T_C(x_0), \ k = 1, 2, \cdots, \text{ such that}$$
$$\varphi^i(x) - \varphi^i(x_0) \geq \limsup_k \xi_i^T \eta_k, \ i \in I.$$

Definition 5. A vector-valued function φ is said to be nearly strictly invex on C at $x_0 \in C$ if

$$\forall x \in C, \text{with } x \neq x_0, \ \forall \xi_i \in \partial \varphi^i(x_0), \ i \in I, \ \exists \eta_k \in T_C(x_0), \ k = 1, 2, \cdots, \text{ such that}$$
$$\varphi^i(x) - \varphi^i(x_0) > \limsup_k \xi_i^T \eta_k, \ i \in I.$$

Definition 6. A vector-valued function φ is said to be infine on C at $x_0 \in C$ if

$$\forall x \in C, \ \forall \xi_i \in \partial \varphi^i(x_0), \ i \in I, \ \exists \eta \in T_C(x_0) \text{ such that}$$
$$\varphi^i(x) - \varphi^i(x_0) = \xi_i^T \eta, \ i \in I.$$

Definition 7. A vector-valued function φ is said to be nearly infine on C at $x_0 \in C$ if

$$\forall x \in C, \ \forall \xi_i \in \partial \varphi^i(x_0), \ i \in I, \ \exists \eta_k \in T_C(x_0), \ k = 1, 2, \cdots, \ \text{such that}$$
$$\varphi^i(x) - \varphi^i(x_0) = \lim_{k \to \infty} \xi_i^T \eta_k, \ i \in I.$$

2. Sufficient Optimality Theorems and Duality Theorems

We introduce a necessary optimality condition for (P), which will be used for proving strong duality theorems. First we introduce constraint qualification (CQ) for a necessary optimality condition for (P).

We say that condition (CQ) holds at $x_0 \in Q$ if there do not exist $\mu^j \geq 0, \ j \in J(x_0)$ and $\gamma^l \in \mathbb{R}, \ l = 1, 2, \cdots, q$ such that

$$\sum_{j \in J(x_0)} \mu^j + \sum_{l=1}^{q} |\gamma^l| \neq 0$$

and

$$0 \in \sum_{j \in J(x_0)} \mu^j \partial g^j(x_0) + \sum_{l=1}^{q} \gamma^l \partial h^l(x_0) + N_C(x_0).$$

Theorem 1.[1] Let $x_0 \in Q$ and condition (CQ) hold. If $x_0 \in Q$ is an efficient solution of (P), then there exist $\lambda_i \geq 0, \ i = 1, 2, \cdots, m$, not all zero, $\mu^j \geq 0, \ j \in J(x_0), \ \gamma^l \in \mathbb{R}, \ l = 1, 2, \cdots, q$, such that

$$0 \in \sum_{i=1}^{m} \lambda^i \partial f^i(x_0) + \sum_{j \in J(x_0)} \mu^j \partial g^j(x_0) + \sum_{l=1}^{q} \gamma^l \partial h^l(x_0) + N_C(x_0), \quad (1)$$
$$\mu^j g^j(x_0) = 0, \ j \in J(x_0).$$

Theorem 2. Let $x_0 \in Q$. If f is nearly strictly invex on C at x_0, g is nearly invex on C at x_0 and h is nearly infine on C at x_0, then x_0 is an efficient solution of (P).

Proof. From (1) it follows that there exist $\xi_i \in \partial f^i(x_0), \ i = 1, 2, \cdots, m$, $\widetilde{\xi}_j \in \partial g^j(x_0), \ j \in J(x_0), \ \widehat{\xi}_l \in \partial h^l(x_0), \ l = 1, 2, \cdots, q$ and $\overline{\xi} \in N_C(x_0)$ such that

$$\sum_{i=1}^{m} \lambda^i \xi_i + \sum_{j \in J(x_0)} \mu^j \widetilde{\xi}_j + \sum_{l=1}^{q} \gamma^l \widehat{\xi}_l + \overline{\xi} = 0. \quad (2)$$

Multiplying (2) by $\eta_k \in T_C(x_0)$, we have

$$\Big(\sum_{i=1}^{m} \lambda^i \xi_i + \sum_{j \in J(x_0)} \mu^j \widetilde{\xi}_j + \sum_{l=1}^{q} \gamma^l \widehat{\xi}_l\Big)^T \eta_k = -\overline{\xi}^T \eta_k. \quad (3)$$

Suppose that x_0 is not an efficient solution of (P). Then there exists $x^* \in Q$ such that

$$f(x^*) \leq f(x_0), \ f(x^*) \neq f(x_0), \ g(x_0) \leq 0 \ \text{ and } \ h(x_0) = 0.$$

Since $\lambda_i \geq 0, \ i = 1, 2, \cdots, m$, not all zero, and $f(x^*) \leq f(x_0)$, we have

$$\sum_{i=1}^{m} \lambda^i \left\{ f^i(x^*) - f^i(x_0) \right\} \leq 0. \tag{4}$$

Since f is nearly strictly invex on C at x_0, g is nearly invex on C at x_0 and h is nearly infine on C at x_0, there exist $\eta_k \in T_C(x_0), \ k = 1, 2, \cdots$, such that

$$f^i(x^*) - f^i(x_0) > \limsup_k \xi_i^T \eta_k, \ i = 1, 2, \cdots, m,$$

$$g^j(x^*) - g^j(x_0) \geq \limsup_k \widetilde{\xi}_j^T \eta_k, \ j \in J(x_0),$$

$$h^l(x^*) - h^l(x_0) = \limsup_k \widehat{\xi}_l^T \eta_k, \ l = 1, 2, \cdots, q.$$

Summing up all these inequalities and equalities, and noting that

$$g^j(x^*) \leq 0, \ g^j(x_0) = 0, \ h^l(x^*) = h^l(x_0) = 0,$$

it follows from (3) that

$$\sum_{i=1}^{m} \lambda^i \left\{ f^i(x^*) - f^i(x_0) \right\} \ > \ \limsup_k \left(\sum_{i=1}^{m} \lambda^i \xi_i + \sum_{j \in J(x_0)} \mu^j \widetilde{\xi}_j + \sum_{l=1}^{q} \gamma^l \widehat{\xi}_l \right)^T \eta_k$$

$$= \ \limsup_k \left(-\overline{\xi}^T \eta_k \right).$$

Since $\overline{\xi} \in N_C(x_0)$ and $\eta_k \in T_C(x_0)$, $\limsup_k \left(-\overline{\xi}^T \eta_k \right) \geq 0$. Thus we have

$$\sum_{i=1}^{m} \lambda^i \left\{ f^i(x^*) - f^i(x_0) \right\} > 0,$$

which contradicts (4). Thus the conclusion holds. $\square$

Now we propose the following Wolfe type dual problem (P):

$(D)_W$

Maximize $f(u) + \mu^T g(u) e + \gamma^T h(u) e$

subject to $0 \in \sum_{i=1}^{m} \lambda^i \partial f^i(u) + \sum_{j \in J(u)} \mu^j \partial g^j(u) + \sum_{l=1}^{q} \gamma^l \partial h^l(u) + N_C(u)$ (5)

$\lambda \in \mathbb{R}^m, \ \mu \in \mathbb{R}^p, \ \gamma \in \mathbb{R}^q, \ \lambda \geq 0, (\lambda_1, \cdots, \lambda_m) \neq 0, \ \mu \geq 0,$ (6)

where $e = (1, \cdots, 1) \in \mathbb{R}^m$.

We establish weak and strong duality theorems between (P) and $(D)_W$.

Theorem 3. (Weak Duality) Let x be any feasible solution for (P) and let $(u, \lambda, \mu, \gamma)$ be any feasible solution for (D)$_W$. If f is nearly strictly invex on C at x_0, g is nearly invex on C at u and h is nearly infine on C at u, then the following do not hold:

$$f(x) \le f(u) + \mu^T g(u)e + \gamma^T h(u)e \quad \text{and} \quad f(x) \ne f(u) + \mu^T g(u)e + \gamma^T h(u)e.$$

Proof. Let x be any feasible solution for (P) and let $(u, \lambda, \mu, \gamma)$ be any feasible solution for (D)$_W$. Then $g(x) \le 0$, $h(x) = 0$ and $(u, \lambda, \mu, \gamma)$ satisfy (5)-(6). According to (5), there exist $\xi_i \in \partial f^i(u)$, $i = 1, 2, \cdots, m$, $\widetilde{\xi}_j \in \partial g^j(u)$, $j \in J(u)$, $\widehat{\xi}_l \in \partial h^l(u)$, $l = 1, 2, \cdots, q$ and $\overline{\xi} \in N_C(u)$ such that

$$\sum_{i=1}^{m} \lambda^i \xi_i + \sum_{j \in J(u)} \mu^j \widetilde{\xi}_j + \sum_{l=1}^{q} \gamma^l \widehat{\xi}_l + \overline{\xi} = 0. \tag{7}$$

Assume to the contrary that

$$f^i(x) \le f^i(u) + \sum_{j \in J(u)} \mu^j g^j(u) + \sum_{l=1}^{q} \gamma^l h^l(u), \quad i = 1, 2, \cdots, m,$$

$$f^k(x) \ne f^k(u) + \sum_{j \in J(u)} \mu^j g^j(u) + \sum_{l=1}^{q} \gamma^l h^l(u), \quad \text{for some } k.$$

Multiplying $\lambda_i \ge 0$, $i = 1, 2, \cdots, m$, not all zero,

$$\sum_{i=1}^{m} \lambda_i f^i(x) \le \sum_{i=1}^{m} \lambda_i f^i(u) + \sum_{j \in J(u)} \mu^j g^j(u) + \sum_{l=1}^{q} \gamma^l h^l(u). \tag{8}$$

Since f is nearly strictly invex on C at u, g is nearly invex on C at u and h is nearly infine on C at u, there exist $\eta_k \in T_C(u)$, $k = 1, 2, \cdots$, such that

$$f^i(x^*) - f^i(u) > \limsup_k \xi_i^T \eta_k, \quad i = 1, 2, \cdots, m,$$

$$g^j(x^*) - g^j(u) \ge \limsup_k \widetilde{\xi}_j^T \eta_k, \quad j \in J(u),$$

$$h^l(x^*) - h^l(u) = \limsup_k \widehat{\xi}_l^T \eta_k, \quad l = 1, 2, \cdots, q.$$

Summing up all these inequalities and equalities, and noting that $g^j(x^*) \leq 0$, $g^j(u) = 0$, $h^l(x^*) = h^l(u) = 0$, it follows from (7) and (8) that

$$
\begin{aligned}
0 &\geq \sum_{i=1}^{m} \lambda^i \left\{ f^i(x) - f^i(u) \right\} - \sum_{j \in J(u)} \mu^j g^j(u) - \sum_{l=1}^{q} \gamma^l h^l(u) \\
&> \limsup_k \Big(\sum_{i=1}^{m} \lambda^i \xi_i + \sum_{j \in J(u)} \mu^j \widetilde{\xi}_j + \sum_{l=1}^{q} \gamma^l \widehat{\xi}_l \Big)^T \eta_k \\
&= \limsup_k \big(-\overline{\xi}^T \eta_k \big).
\end{aligned}
$$

Thus we have $0 > \limsup_k \big(-\overline{\xi}^T \eta_k \big)$. However, since $\overline{\xi} \in N_C(u)$ and $\eta_k \in T_C(u)$, $\limsup_k \big(-\overline{\xi}^T \eta_k \big) \geq 0$, which contradicts. Thus the conclusion holds. $\qquad \square$

Theorem 4. (Strong Duality) Let $\bar{x}$ be an efficient solution of (P) and let condition (CQ) hold. Then there exists $\bar{\lambda} \in \mathbb{R}^m$, $\bar{\mu} \in \mathbb{R}^p$ and $\bar{\gamma} \in \mathbb{R}^q$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ is feasible solution for $(D)_W$. In addition, if f is nearly strictly invex on C at x_0, g is nearly invex on C at x_0 and h is nearly infine on C at x_0, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ is a efficient solution of $(D)_W$ and the optimal values of (P) and $(D)_W$ are equal.

Proof. From Theorem 1, there exist $\bar{\lambda} \in \mathbb{R}^m$, $\bar{\mu} \in \mathbb{R}^p$ and $\bar{\gamma} \in \mathbb{R}^q$ such that

$$
0 \in \sum_{i=1}^{m} \bar{\lambda}^i \partial f^i(\bar{x}) + \sum_{j \in J(\bar{x})} \bar{\mu}^j \partial g^j(\bar{x}) + \sum_{l=1}^{q} \bar{\gamma}^l \partial h^l(\bar{x}) + N_C(\bar{x})
$$

$$
\bar{\mu}^j g^j(\bar{x}) = 0, \ j \in J(\bar{x}),
$$
$$
\bar{\lambda} \geq 0, \ (\lambda_1, \cdots, \lambda_m) \neq 0, \ \bar{\mu} \geq 0.
$$

So, there exist $\bar{\lambda} \in \mathbb{R}^m$, $\bar{\mu} \in \mathbb{R}^p$ and $\bar{\gamma} \in \mathbb{R}^q$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ is feasible solution for $(D)_W$. Since weak duality holds between (P) and $(D)_W$,

$$
f(\bar{x}) = f(\bar{x}) + \bar{\mu}^T g(\bar{x}) e + \bar{\gamma}^T h(\bar{x}) e \not\leq f(u) + \mu^T g(u) e + \gamma^T h(u) e,
$$

for any feasible solution $(u, \lambda, \mu, \gamma)$ for $(D)_W$. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ is a efficient solution of $(D)_W$ and the optimal values of (P) and $(D)_W$ are equal. $\qquad \square$

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