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GRADIENT ESTIMATE OF HEAT EQUATION FOR HARMONIC MAP ON NONCOMPACT MANIFOLDS†

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Abstract. aSuppose that (M, g) is a complete Riemannian manifold with Ricci curvature bounded below by $-K < 0$ and $(N, \overline{g})$ is a complete Riemannian manifold with sectional curvature bounded above by a constant $\mu > 0$. Let $u : M \times [0, \infty) \to B_\tau(p)$ is a heat equation for harmonic map. We estimate the energy density of u.

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1. Introduction

Let M and N be a complete Riemannian manifolds of dimension m and n respectively, and let $\{x^\alpha\}$ and $\{y^i\}$ be the local coordinates of M and N, respectively. Let $u : M \times [0, \infty) \to N$ be a map which is represented by $u = (u^1, \cdots, u^n)$ in terms of the above local coordinates. We say u satisfies the heat equation for harmonic map if it is a solution of the following nonlinear parabolic system:

$$(\Delta - \frac{\partial}{\partial t})u^i(x,t) = g^{\alpha\beta}(x)\Gamma^i_{jk}(u(x,t))\frac{\partial u^j}{\partial x^\alpha}(x,t)\frac{\partial u^k}{\partial x^\beta}(x,t) \tag{1}$$

for $i = 1, \cdots, m$.

In our paper, we give an energy density's estimate, that is, a gradient estimate for the solution of (1). When the target manifold N is the real space $\mathbf{R}$, we have several types of gradient estimate. In [9], Li and Yau proved the gradient estimate for the positive solution of the heat equation. Let the Ricci curvature of M be bounded below by a $-K < 0$. Let $u : M \times [t_0 - T, t_0) \to \mathbf{R}$ is a

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positive solution of the heat equation. Let $a > 0$ and $T > 0$. Then for any $(x,t) \in B_{\frac{a}{2}}(x_0) \times [t_0 - \frac{T}{2}, t_0]$ and $\alpha > 1$,

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{c_n}{R^2} + \frac{c_n}{T} + c_n K.$$

In [7]. Hamilton proved when M is a compact manifold with the Ricci curvature bounded below by a constant $-K < 0$. Let u be a positive solution of the heat equation with $u \leq M$ on $M \times (0, \infty)$.

Then

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K \right) \ln \frac{M}{u}.$$

We prove the local and global gradient estimate for the solution of (1). Let $\rho(p,q)$ be a distance in N between two points $p, q \in N$.

Theorem 1. *Suppose that (M, g) is a complete Riemannian manifold with Ricci curvature bounded below by $-K < 0$ and $(N, \overline{g})$ is a complete Riemannian manifold with sectional curvature bounded above by a constant $\mu > 0$. Assume that $\tau < \min\{\frac{\pi}{2\sqrt{\mu}}$, injectivity radius of N at $p\}$. Let $u : M \times [0, \infty) \to B_\tau(p)$ is a solution of (1) .*
i) Let $a > 0$ and $T > 0$. Then for any $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0, T]$

$$e(u)(x,t) \leq C \left(\frac{T}{a^2 t} + \frac{1}{t} + \frac{KT}{t} \right),$$

for a positive constant $C > 0$ depending only on the dimension n of M and $\sup_{M \times [0,T]} \rho(u(x,t), p) < \infty$.
ii) for any $(x,t) \in M \times (0, \infty]$

$$e(u)(x,t) \leq \frac{C}{t},$$

where $C > 0$ is a constant depending only on the dimension n of M and $\sup_{M \times [0,T]} \rho(u(x,t), p) < \infty$.

Now we give some notations that we use for the proof of Theorem. Let $u : M \times [0, \infty) \longrightarrow N$ be a smooth map. Choose an orthonormal frame $\{e_\alpha, \frac{\partial}{\partial t}\}$ in a neighborhood of $(x,t) \in M \times [0, \infty)$ and an local orthonormal frame $\{f_i\}$ in a neighborhood of $u(x,t) \in N$. Let $\{\theta_\alpha, dt\}$ and $\{\omega_i\}$ be the dual coframes of $\{e_\alpha, \frac{\partial}{\partial t}\}$ and $\{f_i\}$ respectively. Let $\{\theta_{\alpha\beta}\}$ and $\{\omega_{ij}\}$ be the connection forms of M and N respectively.

Denote $d = d_M + \frac{\partial}{\partial t} dt$ to be the exterior differentiation on $M \times [0, \infty)$ where d_M is the exterior differentiation on M. Define $u_{i\alpha}$ by

$$u^* \omega_i = \sum_\alpha u_{i\alpha} \theta_\alpha + u_{it} dt.$$

Define the covariant derivative $u_{i\alpha\beta}$ of $u_{i\alpha}$ by

$$\sum_{\beta} u_{i\alpha\beta}\theta_{\beta} + u_{i\alpha t}dt = du_{i\alpha} - \sum_{j} u_{j\alpha}u^{*}\omega_{ji} - \sum_{\beta} u_{i\beta}\theta_{\beta}\alpha.$$

Since $du_{i\alpha} = d_M u_{i\alpha} + u_{i\alpha t}dt$, we have

$$\sum_{\beta} u_{i\alpha\beta}\theta_{\beta} = d_M u_{i\alpha} - \sum_{j} u_{j\alpha}u^{*}\omega_{ji} - \sum_{\beta} u_{i\beta}\theta_{\beta\alpha}.$$

It is well known that the harmonic map heat equation (1) is equivalent to

$$u_{it} = u_{i\alpha\alpha},$$

for $i = 1, \cdots, n$. We define the energy density function $e(u)$ of u by $e(u) = \sum_{i\alpha} u_{i\alpha}^2$. Then an easy computation gives the following Bochner type formula:

$$\frac{1}{2}(\Delta - \frac{\partial}{\partial t})e(u) = \sum_{i,\alpha,\beta} u_{i\alpha\beta}^2 - \sum_{i,j,k,l,\alpha,\beta} R_{ijkl}u_{i\alpha}u_{j\beta}u_{k\alpha}u_{l\beta} + \sum_{\alpha,\beta,i} K_{\alpha\beta}u_{i\alpha}u_{i\beta},$$

where R_{ijkl} is the curvature tensor of N and $K_{\alpha\beta}$ is the Ricci tensor of M.

2. The Proof of Main Theorem

The proof given here is a modification of the method of [4]. First scaling the metric of N, we may assume without loss of generality that $\mu = 1$. Let $\rho(p, q)$ denote the intrinsic distance function on N between p and q. Define

$$\phi(u)(x, t) = 1 - \cos(\rho(u(x, t), p)).$$

Since the image of u on $M \times [0, T]$ contained in $B_\tau(p)$, we have $\cos(\rho(u(x, t), p)) < \cos\tau < 1$. Let $B_a(x_0)$ be the closed geodesic ball of radius $a > 0$ and center x_0 in M and let γ be the distance function in M from $x_0 \in M$. Let $a > 0$ and $T > 0$ and $b = 2\sup_{M\times(0,T]} \phi(u)(x, t) < b.$. Consider the function $\Phi : B_a(x_0) \times [0, T] \to \mathbf{R}$ defined by

$$\Phi = \frac{t\,(a^2 - \gamma^2)^2 e(u)}{(b - \phi(u))^2}.$$

Let (x_1, t_1) be a point in $B_a(x_0) \times [0, T]$ such that

$$\Phi(x_1, t_1) = \max_{B_a(x_0)\times[0,T]} \frac{t\,(a^2 - \gamma^2)^2 e(u)}{(b - \phi(u))^2}.$$

Since $\Phi(x, 0) = 0$ for $x \in B_a(x_0)$ and $\Phi(x, t) = 0$ for $x \in \partial B_a(x_0)$, we get the maximum point (x_1, t_1) of Φ in $B_a(x_0) \times (0, T]$.

At $(x_1, t_1) \in B_a(x_0) \times (0, T]$, Φ has the following properties :

$$\Delta \log \Phi(x_1, t_1) \le 0, \quad d\log\Phi(x_1, t_1) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\log\Phi(x_1, t_1) \ge 0.$$

Rewriting these at (x_1, t_1), we have

$$0 = \frac{-2d\gamma^2}{(a^2-\gamma^2)} + \frac{de(u)}{e(u)} + \frac{2d\phi(u)}{(b-\phi(u))} \tag{2}$$

$$\begin{aligned} 0 \geq\ & \frac{-2\Delta\gamma^2}{(a^2-\gamma^2)} + \frac{-2|d\gamma^2|^2}{(a^2-\gamma^2)^2} + \frac{(\Delta - \frac{\partial}{\partial t})e(u)}{e(u)} \\ & -\frac{|de(u)|^2}{e(u)^2} + \frac{2(\Delta - \frac{\partial}{\partial t})\phi(u)}{(b-\phi(u))} + \frac{2|d\phi(u)|^2}{(b-\phi(u))^2} - \frac{1}{t}. \end{aligned} \tag{3}$$

Note that

$$d(\sum u_{i\alpha}^2) = 2\sum u_{i\alpha}u_{i\alpha\beta} \leq 2(\sum u_{i\alpha}^2)^{\frac{1}{2}}(\sum u_{i\alpha\beta}^2)^{\frac{1}{2}}. \tag{4}$$

Putting (4) to the Bochner type formula (2) , we get

$$(\Delta - \frac{\partial}{\partial t})e(u) \geq \frac{1}{2}\frac{|de(u)|^2}{e(u)} - 2e^2(u) - 2Ke(u). \tag{5}$$

By the Hessian comparison, we have for some constant $C_1 > 0$

$$(\Delta - \frac{\partial}{\partial t})(\phi(u)) \geq (\cos\rho)e(u), \qquad \text{and} \qquad \Delta\gamma^2 \leq C_1(1+K\gamma). \tag{6}$$

Putting (2),(5) and (6) to (3), we have

$$\begin{aligned} 0 \geq\ & -\frac{1}{t} + \frac{-2\Delta\gamma^2}{a^2-\gamma^2} - \frac{4|d\gamma^2|^2}{(a^2-\gamma^2)^2} - \frac{4|d\gamma^2||d\phi(u)|}{(a^2-\gamma^2)(b-\phi(u))^2} - 2e(u) \\ & -2K + \frac{2\cos\rho}{(b-\phi(u))}e(u). \end{aligned}$$

Letting $b_1 = \sup_{M\times[0,T]}\phi(u)$, we get that

$$\sup_{M\times(0,T]}(b-\phi(u)(x,t)) = b_1 \text{ and } (-2 + \frac{2\cos\rho}{b-\phi(u)}) > \frac{1-b}{b-\phi(u)} > 0.$$

Using $|d\phi(u)| \leq \sin\rho(u)\sqrt{e(u)}$, we have

$$\begin{aligned} 0 \geq\ & -\frac{1}{t} - 2K - \frac{2C_1(1+K\gamma)}{a^2-\gamma^2} - \frac{16\gamma^2}{(a^2-\gamma^2)^2} \\ & -\frac{8\gamma\sin\rho(u)}{(a^2-\gamma^2)(b-\phi(u))}\sqrt{e(u)} + \frac{1-b}{b-\phi(u)}e(u). \end{aligned}$$

As in [4],we have a constant $C_2 > 0$ such that

$$\begin{aligned} & e(u)(x_1,t_1) \\ \leq\ & C_2 \max\left\{ \frac{64\gamma^2\sin^2\rho(u)}{(1-b)^2(a^2-\gamma^2)^2}, \frac{1(b-\phi(u))}{t_1(1-b)} + \frac{2K(b-\phi(u))}{(1-b)} \right. \\ & \left. + \frac{2(1+K\gamma)(b-\phi(u))}{(1-b)(a^2-\gamma^2)} + \frac{16\gamma^2(b-\phi(u))}{(1-b)(a^2-\gamma^2)^2} \right\}. \end{aligned}$$

Note that $\sin^2\rho(u) = 1-\cos^2\rho(u) \le 2\phi(u) \le 2b_1$. If $(x,t)\in B_{\frac{a}{2}}(x_0)\times(0,T]$,

$$\begin{aligned}
&\left\{\frac{t\,(a^2-\gamma^2)^2 e(u)}{(b-\phi(u))^2}\right\}(x,t)\\
\le\ &\left\{\frac{t_1(a^2-\gamma^2)^2 e(u)}{(b-\phi(u))^2}\right\}(x_1,t_1)\\
\le\ & C_3\max\left\{\frac{128\gamma^2 t_1 b_1}{(1-b)^2(b-\phi(u))^2},\ \frac{(a^2-\gamma^2)^2}{(1-b)(b-\phi(u))}+\frac{2t_1\,K(a^2-\gamma^2)^2}{(1-b)(b-\phi(u))}\right.\\
&\left.+\frac{2t_1(1+K\gamma)(a^2-\gamma^2)}{(1-b)(b-\phi(u))}+\frac{16t_1\gamma^2}{(1-b)(b-\phi(u))}\right\}|_{(x_1,t_1)}\\
\le\ & C_4\max\left\{\frac{32a^2T}{b_1^2},\ \frac{9a^4}{16b_1}+\frac{9TK\,a^4}{16b_1}+\frac{3(1+aK)a^2\,T}{2b_1}+\frac{4T\,a^2}{b_1}\right\},
\end{aligned}$$

for some constants $C_3>0$ and $C_4>0$. Therefore we have for any $(x,t)\in B_{\frac{a}{2}}(x_0)\times(0,T]$,

$$e(u)(x,t)\le C\max\left\{\frac{b^2T}{\beta^2a^2t}\ ,\quad \frac{b^2}{\beta t}+\frac{Kb^2T}{\beta t}+\frac{(1+aK)b^2T}{a^2\beta t}+\frac{b^2T}{a^2\beta t}\right\}.$$

Letting $b=\frac{1+b_1}{2}$, we get $4\beta=(1-b_1)^2$ and

$$e(u)(x,t)\quad \le C\left(\frac{1+b_1}{1-b_1}\right)^2\left(\frac{T}{a^2t}+\frac{1}{t}+K\frac{T}{t}\right).$$

So we get the (i) of Main Theorem.

Since the lower bound of Ricci curvature of M is 0, we have for any $a>0$ and $T>0$, and for any $(x,t)\in B_a(x_0)\times(0,T]$,

$$e(u)(x,t)\le C\max\left\{\frac{T}{a^2t}+\frac{1}{t}\right\}.$$

As $a\to\infty$, the (ii) in Main Theorem is proved.

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