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CONVEXITY AND SEMICONTINUITY OF FUZZY MAPPINGS USING THE SUPPORT FUNCTION

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Abstract. Since Goetschel and Voxman [5] proposed a linear order on fuzzy numbers, several authors studied the concept of semicontinuity and convexity of fuzzy mappings defined through the order. Since the order is only defined for fuzzy numbers on $\mathbb{R}$, it is natural to find a new order for normal fuzzy sets on $\mathbb{R}^n$ in order to study the concept of semicontinuity and convexity of fuzzy mappings on normal fuzzy sets. In this paper, we introduce a new order "$\preceq_s$" for normal fuzzy sets on $\mathbb{R}^n$ with respect to the support function. We define the semicontinuity and convexity of fuzzy mappings with this order. Some issues which are related with semicontinuity and convexity of fuzzy mappings will be discussed..

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1. Introduction

Since Zadeh [22] first proposed the idea of a fuzzy set, thousands of papers on fuzzy sets and related themes have appeared. The idea of fuzzy convexity goes back to Zadeh [22]. Nguyen [11] exploited the idea of fuzzy numbers without using normality, which was introduced by Puri and Ralescu [12]. Level sets were first comprehensively used by Mizumoto and Tanaka [9]. Support functions have long been used in the theory of convex analysis. The first application to fuzzy set theory was seen in Puri and Ralescu's paper [13]. The function space metric was discussed in Gottwald [6]. Kloeden [8] first intorduced the sendograph metric and it was used for fuzzy numbers by Goestschel and Voxman [4]. It is recorded that d_∞ metric was first used by Heilpern [7]. Puri and Ralescu [13] exploited its metric properties in a metric space context, and introduced the isometrical embedding.

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Since Goetschel and Voxman [5] proposed a linear ordering $\preceq$ on fuzzy numbers which is called "fuzzy-max" order, the concept of convex fuzzy mappings defined through the "fuzzy-max" order were studied by several authors, including Furukawa [3], Nanda [10], Syau [14, 15], Syau and Lee [16], Wu and Xu [18], Yan and Xu [19], and Wang and Wu [17]. The concept of upper and lower semicontinuity of fuzzy mapping based on the Hausdorff separation was introduced by Diamond and Kloeden [2]. Recently, Bao and Wu [1] introduced a new concept of upper and lower semicontinuity of fuzzy mappings through the "fuzzy-max" order on fuzzy numbers.

Since the "fuzzy-max" order is only defined for fuzzy numbers on $\mathbb{R}$, we need to find new order for normal fuzzy sets on $\mathbb{R}^n$ in order to study the concept of semicontinuity and convexity of fuzzy mappings on normal fuzzy sets on $\mathbb{R}^n$. In this paper, we introduce a new order on the set of normal fuzzy sets with respect to the support function. And we study the concept of semicontinuity and convexity of fuzzy mapping on fuzzy normal sets. Since the new order is different from "fuzzy-max" order(Example 1), we define convexity and semicontinuity of fuzzy mappings on fuzzy normal sets through the order in section 3. We will show some properties of convexity and semicontinuity of fuzzy mappings and some other issues will be discussed in section 4.

2. Preliminaries

In this section, we briefly recall some of the basic notations in the theory of fuzzy sets.

A fuzzy subset of $\mathbb{R}^n$ is defined in terms of a membership function which assigns to each point $x \in \mathbb{R}^n$ a grade of membership in the fuzzy set. Such a membership function

$$u : \mathbb{R}^n \to I = [0, 1]$$

is used synonomously to denote the corresponding fuzzy set. Denote by $\mathcal{F}^n$ the set of all fuzzy sets on $\mathbb{R}^n$.

For each $\alpha \in (0, 1]$ the α-level set $[u]^\alpha$ of a fuzzy set u is the subset of points $x \in \mathbb{R}^n$ with membership grade $u(x)$ of at least α, that is,

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}.$$

The support $[u]^0$ of a fuzzy set is then defined as the closure of the union of all its level sets, that is

$$[u]^0 = \overline{\bigcup_{\alpha \in (0,1]} [u]^\alpha}.$$

We recall the definition of a normal fuzzy set.

Definition 1. ([2]) Let $u \in \mathcal{F}^n$. Then u is a *normal fuzzy set* if it satisfies following assumptions;

(1) u maps $\mathbb{R}^n$ onto I;
(2) $[u]^0$ is a bounded subset of $\mathbb{R}^n$;
(3) u is upper semicontinuous;
(4) u is fuzzy convex.

We denote $\mathcal{E}^n$ the space of all normal fuzzy subsets u of $\mathcal{F}^n$

Note. If $n = 1$ then a normal fuzzy subset $u \in \mathcal{E}^\infty$ is a fuzzy number.

The space $\mathcal{E}^n$ can be endowed with an inner composition law which is the extension of the Minkowski addition between sets, and an external composition which is the product by a scalar. These two laws are compatible with the ones obtained by applying Zadeh's extension principle. Thus for all $u, v \in \mathcal{E}^n$ and $\lambda \in \mathbb{R}$, $u + v$ and λu can be defined as the unique fuzzy sets so that for all $\alpha \in [0, 1]$

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha = \{x + y | x \in [u]^\alpha, y \in [v]^\alpha\}$$
$$[\lambda u]^\alpha = \lambda[u]^\alpha = \{\lambda x | x \in [u]^\alpha\}$$

In order to introduce the support function of fuzzy sets in $\mathcal{E}^n$, we recall the concept of support function of a nonempty compact convex subset of $\mathbb{R}^n$. Let A be a nonempty subset of $\mathbb{R}^n$. The support fucntion of A is defined by

$$s(p, A) = \sup\{< p, a >: a \in A\}, \quad \text{for all} \quad p \in \mathbb{R}^n.$$

We denote by $\mathcal{K_C}^n$ the set of all nonempty compact convex subset of $\mathbb{R}^n$. The support function $s(p, A)$ is uniquely paired to subset A in $\mathcal{K_C}^n$. It also preserves set addition and nonnegative scalar multiplication. That is, for all $p \in \mathbb{R}^n$ and for all $t \geq 0$,

$$s(p, A + B) = s(p, A) + s(p, B), \quad s(p, tA) = ts(p, A)$$

Moreover for all $A, B \in \mathcal{K_C}^n$,

$$d_H(A, B) = \sup\{\|s(p, A) - s(p, B)\| : p \in S^{n-1}\}$$

Definition 2. ([2]) Let $u \in \mathcal{E}^n$. The *support function* of u is $s_u : I \times S^{n-1} \to \mathbb{R}$ defined by

$$s_u(\alpha, p) = s(p, [u]^\alpha) = \sup\{< p, a >: a \in [u]^\alpha\}$$

for $(\alpha, p) \in I \times S^{n-1}$, where $s(\cdot, [u]^\alpha)$ is the support function of $[u]^\alpha$.

Moreover, for $u, v \in \mathcal{E}^n$ and $\lambda \geq 0$

$$u = v \quad \text{if and only if} \quad s_u = s_v$$

Since the support function on $\mathcal{K_C}^n$ uniquely characterizes the elements of $\mathcal{K_C}^n$,

$$s_{u+v} = s_u + s_v, s_{\lambda u} = \lambda s_u.$$

Definition 3. ([7]) The supremum metric d_∞ on $\mathcal{E}^n$ is defined by

$$d_\infty(u,v) = \sup\{d_H([u]^\alpha,[v]^\alpha) : \alpha \in [0,1]\} \quad \text{for} \quad u,v \in \mathcal{E}^{\setminus}$$

where

$$d_H([u]^\alpha,[v]^\alpha) = \sup\{|s_u(\alpha,x) - s_v(\alpha,x)| : (\alpha,x) \in [0,1] \times S^{n-1}\}$$

We now define the continuity of mappings between metric spaces.

Definition 4. A fuzzy mapping $F : C \to \mathcal{E}^n$ is said to be continuous at $a_0 \in C$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_\infty(F(a), F(a_0)) < \varepsilon \quad \text{whenever} \quad a \in C \cap B_\delta(a_0).$$

F is continuous if it is continuous at each point of C.

We recall the definitions of upper and lower semicontinuous real-valued functions.

Definition 5. A real-valued function $f : S \to \mathbb{R}^n$ is said to be

(1) upper semicontinuous at $x_0 \in S$ if given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$f(x) < f(x_0) + \varepsilon \quad \text{whenever} \quad x \in S \cap B_\delta(x_0).$$

f is upper semicontinuous on S if it is upper semicontinuous at each point of S.

(2) lower semicontinuous at $x_0 \in S$ if given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$f(x) > f(x_0) - \varepsilon \quad \text{whenever} \quad x \in S \cap B_\delta(x_0).$$

f is lower semicontinuous on S if it is lower semicontinuous at each point of S.

We now recall two important results concerning convex functions.

Theorem 1. ([20]) *Let C be a non-empty convex subset of $\mathbb{R}^n$, and let $f : C \to \mathbb{R}^{\nVdash}$ be a lower semicontinuous function. If for all $a,b \in C$, there exists $\lambda \in (0,1)$ such that*

$$f(\lambda a + (1-\lambda)b) \le \lambda f(a) + (1-\lambda) f(b),$$

then f is a convex function on C.

Theorem 2. ([21]) *Let C be a non-empty convex subset of $\mathbb{R}^n$, and let $f : C \to \mathbb{R}^{\nVdash}$ be an upper semicontinuous function. If there exists $\lambda \in (0,1)$ such that*

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b), \quad \forall a, b \in C,$$

then f is a convex function on C.

3. Semicontinuity and convexity of fuzzy mappings on $\mathcal{E}^n$

Goetschel and Voxman [5] proposed a linear ordering "$\preceq$" on fuzzy numbers which is called "fuzzy-max" order and several authors studied the concept of convex and semicontinuous fuzzy mappings defined through the "fuzzy-max" order. Since "fuzzy-max" order is only defined on $\mathcal{E}^1$, it is natural to find a new order on $\mathcal{E}^n$ in order to study the concept of semicontinuity and convexity of fuzzy mapping on $\mathcal{E}^n$. Since the support function uniquely characterizes the element of $\mathcal{E}^n$, we define the order on $\mathcal{E}^n$ with respect to the order of support function.

Definition 6. Let $u, v \in \mathcal{E}^n$ be normal fuzzy subsets whose support functions are s_u and s_v, respectively. Then we say that $u \preceq_s v$ if

$$s_u(\alpha, x) \leq s_v(\alpha, x) \text{ for each } (\alpha, x) \in [0,1] \times S^{n-1}.$$

We see that $u = v$ if $u \preceq_s v$ and $u \preceq_s v$. Moreover $u \prec_s v$ if $u \preceq_s v$ and there exists $(\alpha_0, x_0) \in [0,1] \times S^{n-1}$ such that $s_u(\alpha_0, x_0) < s_v(\alpha_0, x_0)$.

The following example shows that the order "$\preceq_s$" is different to "fuzzy-max" order.

Example 1. *Let $u, v \in \mathcal{E}^\infty$ have level set*

$$[u]^\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)], \quad [v]^\alpha = [\underline{v}(\alpha), \overline{v}(\alpha)], \quad \forall \alpha \in [0,1].$$

Then, as $S^0 = \{-1, 1\}$, the support function s_u, s_v are given by

$$S_u(\alpha, -1) = -\underline{u}(\alpha), \quad S_u(\alpha, 1) = \overline{u}(\alpha),$$

and

$$S_v(\alpha, -1) = -\underline{v}(\alpha), \quad S_v(\alpha, 1) = \overline{v}(\alpha), \quad \textit{for all} \quad \alpha \in [0,1]$$

Hence if $u \preceq_s v$ then

$$\underline{u}(\alpha) \geq \underline{v}(\alpha) \quad \textit{and} \quad \overline{u}(\alpha) \leq \overline{v}(\alpha), \quad \textit{for all} \quad \alpha \in [0,1]$$

This does not mean $u \preceq v$. In fact $u \preceq_s v$ means that $[v]^\alpha$ contains $[u]^\alpha$ for each $\alpha \in [0,1]$.

Example 2. *For $\varepsilon > 0$, let $\widetilde{\varepsilon} \in \mathcal{E}^n$ where*

$$\widetilde{\varepsilon}(x) = \begin{cases} 1 & \text{if } |x| \le \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Then the support function of $\widetilde{\varepsilon}$ is a constant function. In fact, $s_{\widetilde{\varepsilon}} = \varepsilon$.

Note. From Example 2 we can get a normal fuzzy set corresponding to a positive real number, but such normal fuzzy sets are restricted. Thus we give Definition 7.

Definition 7. A normal fuzzy set u in $\mathcal{E}^n$ is called a *positive normal fuzzy set* if $0 \in \mathrm{Int}[u]^1$. Denote by

$$\mathcal{E}^+ = \{u \in \mathcal{E}^n | \quad u \text{ is a positive normal fuzzy set}\}$$

Lemma 1. *A positive normal fuzzy set has a positive support function.*

Proof. If $u \in \mathcal{E}^+$ then $0 \in \mathrm{Int}[u]^1$. Hence there exists $\varepsilon > 0$ such that $B_\varepsilon(0)$ is contained in $\mathrm{Int}[u]^1$. Since $[u]^1$ is closed, $\overline{B}_\varepsilon(0)$ is also contained in $[u]^1$. Thus $s_u(\alpha, x) \ge s_{\widetilde{\varepsilon}}(\alpha, x) = \varepsilon$ by Definition 7 This completes the proof. □

We now define semicontinuity and convexity of a fuzzy mappings in the sense of the order "$\preceq_s$" by using Definition 6 and 7.

Definition 8. A fuzzy mapping $F : C \to \mathcal{E}^n$ is said to be

(1) upper semicontinuous at $a_0 \in C$ if for all $u \in \mathcal{E}^+$, there exists $\delta > 0$ such that

$$F(a) \preceq_s F(a_0) + u \quad \text{whenever} \quad a \in C \cap B_\delta(a_0).$$

F is upper semiconinuous if it is upper semicontinuous at each point of C.

(2) lower semiconinuous at $a_0 \in C$ if for all $u \in \mathcal{E}^+$, there exists $\delta > 0$ such that

$$F(a_0) \preceq_s F(a) + u \quad \text{whenever} \quad a \in C \cap B_\delta(a_0).$$

F is lower semiconinuous if it is lower semicontinuous at each point of C.

Definition 9. Let C be a non-empty convex subset of $\mathbb{R}^n$. A fuzzy mapping $F : C \to \mathcal{E}^n$ is said to be convex if for every $\lambda \in [0, 1]$ and $a, b \in C$,

$$F(\lambda a + (1 - \lambda)b) \preceq_s \lambda F(a) + (1 - \lambda)F(b).$$

4. The properties of fuzzy mappings on $\mathcal{E}^n$

In Section 3 we defined a new order on $\mathcal{E}^n$ and introduced the concepts of semicontinuity and convexity of fuzzy mappings on $\mathcal{E}^n$. In this section we will find the properties of fuzzy mappings on $\mathcal{E}^n$.

Theorem 3. *Let $F : C \to \mathcal{E}^n$ be a fuzzy mapping and $\widetilde{\varepsilon}$ be in $\mathcal{E}^n$ defined by*

$$\widetilde{\varepsilon}(x) = \begin{cases} 1 & \text{if } |x| \le \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) *F is upper semicontinuous at a_0 in C if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$F(a) \preceq_s F(a_0) + \widetilde{\varepsilon} \quad \text{whenever} \quad a \in C \cap B_\delta(a_0).$$

(2) *F is lower semicontinuous at a_0 in C if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$F(a_0) \preceq_s F(a) + \widetilde{\varepsilon} \quad \text{whenever} \quad a \in C \cap B_\delta(a_0).$$

Proof. (1) Suppose that F is upper semicontinuous at a_0 in C. Let $\varepsilon > 0$ be given. Since $\widetilde{\varepsilon}$ is in $\mathcal{E}^+$, there exists $\delta > 0$ such that

$$F(a_0) \preceq_s F(a) + \widetilde{\varepsilon} \quad \text{whenever} \quad a \in C \cap B_\delta(a_0).$$

Conversely, if u in $\mathcal{E}^+$ then there exist $\varepsilon > 0$ such that

$$s_u(\alpha, x) \ge s_{\widetilde{\varepsilon}}(\alpha, x) = \varepsilon$$

by Lemma 1. Hence there exists $\delta > 0$ such that

$$F(a_0) \preceq_s F(a) + \widetilde{\varepsilon} \preceq_s F(a) + u \quad \text{whenever} \quad a \in C \cap B_\delta(a_0).$$

It completes the proof.

(2) It is similar to (1).

□

By using Theorem 3 we will show the semicontinuity of a fuzzy mapping. First of all, we wlll see the the relationship between a fuzzy mapping and a support function.

Theorem 4. *Let $F : C \to \mathcal{E}^n$ be a fuzzy mapping, and for each $a \in C$ let $s_{F(a)}$ be a support function of $F(a)$. Then the following conditions are equivalent;*

(1) *F is upper semicontinuous at $a_0 \in C$.*
(2) *$s_{F(a)}$ is upper semicontinuous at a_0 uniformly in $(\alpha, x) \in [0, 1] \times S^{n-1}$.*

Proof. If (1) holds then by Theorem 3, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$F(a) \preceq_s F(a_0) + \widetilde{\varepsilon} \quad \text{whenever} \quad a \in C \cap B_\delta(a_0).$$

By Definition 6, for each $(\alpha, x) \in [0,1] \times S^{n-1}$,

$$s_{F(a)}(\alpha, x) \le s_{F(a_0)}(\alpha, x) + s_{\widetilde{\varepsilon}}(\alpha, x).$$

It means that for each $a \in C \cap B_\delta(a_0)$,

$$s_{F(a)}(\alpha, x) \le s_{F(a_0)}(\alpha, x) + \varepsilon, \quad \forall (\alpha, x) \in [0,1] \times S^{n-1}.$$

Hence $s_{F(a)}$ is upper semicontinuous at a_0 uniformly in $(\alpha, x) \in [0,1] \times S^{n-1}$. Since it holds in the reverse order, this completes the proof. □

Theorem 5. *Let $F : C \to \mathcal{E}^n$ be a fuzzy mapping, and for each $a \in C$ let $s_{F(a)}$ be a support function of $F(a)$. Then the following conditions are equivalent;*

(1) *F is lower semicontinuous at $a_0 \in C$.*
(2) *$s_{F(a)}$ is lower semicontinuous at a_0 uniformly in $(\alpha, x) \in [0,1] \times S^{n-1}$.*

Proof. It is similar to Theorem 4. □

Theorem 6. *Let C be a non-empty convex subset of $\mathbb{R}^n$ and let $F : C \to \mathcal{E}^n$ be a fuzzy mapping. Then F is convex on C if and only if for each $(\alpha, x) \in [0,1] \times S^{n-1}$, $s_{F(a)}$ is convex with respect to a on C.*

Proof. Assume that for each $(\alpha, x) \in [0,1] \times S^{n-1}$, $s_{F(a)}$ is convex with respect to a on C. Let $(\alpha, x) \in [0,1] \times S^{n-1}$ be given. Then

$$s_{F(\lambda a + (1-\lambda) b)}(\alpha, x) \le \lambda s_{F(a)}(\alpha, x) + (1-\lambda) s_{F(b)}(\alpha, x)$$

for all $a, b \in C$ and $\lambda \in [0,1]$. Then

$$F(\lambda a + (1-\lambda) b) \preceq_s \lambda F(a) + (1-\lambda) F(b)$$

for all $a, b \in C$ and $\lambda \in [0,1]$. Hence F is convex on C.

Conversely, let F be convex on C. Then for every $a, b \in C$ and $\lambda \in [0,1]$, we have

$$F(\lambda a + (1-\lambda) b) \preceq_s \lambda F(a) + (1-\lambda) F(b)$$

From Definition 6, we have

$$s_{F(\lambda a + (1-\lambda) b)}(\alpha, x) \in [0,1] \times S^{n-1} \le \lambda s_{F(a)}(\alpha, x) + (1-\lambda) s_{F(b)}(\alpha, x)$$

for all $a, b \in C$ and $\lambda \in [0,1]$. Hence we can conclude that for each $(\alpha, x) \in [0,1] \times S^{n-1}$, $s_{F(a)}$ is convex with respect to a on C.

This completes the proof. □

Theorem 7. *Let C be a non-empty convex subset of $\mathbb{R}^n$, and let $F : C \to \mathcal{E}^n$ be a lower semicontinuous fuzzy mapping. If for all $a, b \in C$, there exists a $\lambda \in (0,1)$ such that*

$$F(\lambda a + (1-\lambda)b) \preceq_s \lambda F(a) + (1-\lambda)F(b) \tag{1}$$

then F is a convex fuzzy mapping on C.

Proof. (a) Since F is lower semicontinuous, by Theorem 5, we have $s_{F(a)}$ is lower semicontinuous at $a \in C$ uniformly in $(\alpha, x) \in [0,1] \times S^{n-1}$.

(b) In view of (1) and Definition 6, it can be written as for all $a, b \in C$, there exists a $\lambda \in (0,1)$ such that

$$s_{F(\lambda a+(1-\lambda)b)}(\alpha, x) \leq \lambda s_{F(a)}(\alpha, x) + (1-\lambda)s_{F(b)}(\alpha, x)$$

for all $(\alpha, x) \in [0,1] \times S^{n-1}$. Combining (a),(b) and Theorem 1, we have $s_{F(a)}$ is convex with respect to a on C. That is for all $a, b \in C$ and $\lambda \in [0,1]$

$$s_{F(\lambda a+(1-\lambda)b)}(\alpha, x) \leq \lambda s_{F(a)}(\alpha, x) + (1-\lambda)s_{F(b)}(\alpha, x)$$

This means

$$F(\lambda a + (1-\lambda)b) \preceq_s \lambda F(a) + (1-\lambda)F(b)$$

for for all $a, b \in C$ and $\lambda \in [0,1]$. Hence F is convex by Definition 9.

□

Similarly, by Theorems 2 and 4, we obtain an analogous result to Theorem 7 for the case of upper semicontinuous fuzzy mappings:

Theorem 8. *Let C be a non-empty convex subset of $\mathbb{R}^n$, and let $F : C \to \mathcal{E}^n$ be a upper semicontinuous fuzzy mapping. If there exists a $\lambda \in (0,1)$ such that*

$$F(\lambda a + (1-\lambda)b) \preceq_s \lambda F(a) + (1-\lambda)F(b) \quad \forall a, b \in C$$

then F is a convex fuzzy mapping on C.

A real valued function is continuous if and only if upper semicontinuous and lower semicontinuous. Thus we will show that it holds on fuzzy mappings in the sense of Definition 4 and 8.

Theorem 9. *Let $F : C \to \mathcal{E}^n$ be a fuzzy mapping. F is upper semicontinuous and lower semicontinuous if and only if F is continuous.*

Proof. Suppose that F is upper semicontinuous and lower semicontinuous. Let $a_0 \in C$ and $\varepsilon > 0$ be given. By Theorem 3 there exists $\delta > 0$ such that

$$F(a) \preceq_s F(a_0) + \widetilde{\varepsilon} \quad \text{and} \quad F(a_0) \preceq_s F(a) + \widetilde{\varepsilon}, \quad \forall a \in C \cap B_\delta(a_0).$$

This implies for all $a \in C \cap B_\delta(a_0)$

$$s_{F(a)}(\alpha, x) \leq s_{F(a_0)}(\alpha, x) + \varepsilon \quad \text{and} \quad s_{F(a_0)}(\alpha, x) \leq s_{F(a)}(\alpha, x) + \varepsilon$$

And so

$$|s_{F(a)}(\alpha, x) - s_{F(a_0)}(\alpha, x)| \leq \varepsilon, \quad \forall a \in C \cap B_\delta(a_0)$$

Hence

$$d_\infty(F(a), F(a_0)) \leq \varepsilon, \quad \forall a \in C \cap B_\delta(a_0)$$

By Definition 4, F is continuous at a_0.

Conversely, suppose that F is continuous at a_0. Let $a_0 \in C$ and $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$d_\infty(F(a), F(a_0)) \leq \varepsilon, \quad \forall a \in C \cap B_\delta(a_0)$$

It means

$$F(a) \preceq_s F(a_0) + \widetilde{\varepsilon} \quad \text{and} \quad F(a_0) \preceq_s F(a) + \widetilde{\varepsilon}, \quad \forall a \in C \cap B_\delta(a_0).$$

This completes the proof.

□

We observe the property of a sequence of convex fuzzy mappings.

Theorem 10. *Let $\{F_n\}$ be a sequence of convex fuzzy mappings. If there exists a fuzzy map F such that $\{F_n\}$ converges to F, then F is convex.*

Proof. Let $a, b \in C$ and $\lambda \in [0,1]$, $\varepsilon > 0$ be given. Since $\{F_n\}$ converges to F, there exists $n \in \mathbb{N}$ such that

$$d_\infty(F_n(a), F(a)) < \frac{\varepsilon}{2}, \quad d_\infty(F_n(b), F(b)) < \frac{\varepsilon}{2}$$

and

$$d_\infty(F_n(\lambda a + (1-\lambda)b), F(\lambda a + (1-\lambda)b)) < \frac{\varepsilon}{2}$$

By Definition 3,

$$\begin{array}{c} |s_{F_n(a)}(\alpha, x) - s_{F(a)}(\alpha, x)| < \frac{\varepsilon}{2} \\ |s_{F_n(b)}(\alpha, x) - s_{F(b)}(\alpha, x)| < \frac{\varepsilon}{2} \\ |s_{F_n(\lambda a+(1-\lambda)b)}(\alpha, x) - s_{F(\lambda a+(1-\lambda)b)}(\alpha, x)| < \frac{\varepsilon}{2} \end{array} \tag{2}$$

for all $(x, \alpha) \in S^1 \times [0,1]$.

Since F_n is convex by assumption,

$$F_n(\lambda a + (1-\lambda)b)) \preceq_s \lambda F_n(a) + (1-\lambda)F_n(b).$$

Thus, by Theorem 6, for all $(\alpha, x) \in [0,1] \times S^{n-1}$

$$s_{F_n(\lambda a+(1-\lambda)b)}(\alpha, x) \leq \lambda s_{F_n(a)}(\alpha, x) + (1-\lambda)s_{F_n(b)}(\alpha, x). \tag{3}$$

Also, by (2) and (3), for all $(\alpha, x) \in [0,1] \times S^{n-1}$

$$\begin{aligned} s_{F(\lambda a+(1-\lambda)b)}(\alpha, x) - \frac{\varepsilon}{2} &< s_{F_n(\lambda a+(1-\lambda)b)}(\alpha, x) \\ &\leq \lambda s_{F_n(a)}(\alpha, x) + (1-\lambda)s_{F_n(b)}(\alpha, x) \\ &< \lambda s_{F(a)}(\alpha, x) + (1-\lambda)s_{F(b)}(\alpha, x) + \frac{\varepsilon}{2}. \end{aligned}$$

Therefore

$$s_{F(\lambda a+(1-\lambda)b)}(\alpha, x) < s_{F(a)}(\alpha, x) + (1-\lambda)s_{F(b)}(\alpha, x) + \varepsilon$$

for all $(\alpha, x) \in [0,1] \times S^{n-1}$.

Since ε is arbitrary,

$$s_{F(\lambda a+(1-\lambda)b)}(\alpha, x) \leq s_{F(a)}(\alpha, x) + (1-\lambda)s_{F(b)}(\alpha, x)$$

for all $(\alpha, x) \in [0,1] \times S^{n-1}$.

It means

$$F(\lambda a + (1-\lambda)b)) \preceq_s \lambda F(a) + (1-\lambda)F(b)$$

Thus F is convex. □

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