J. Appl. Math. & Informatics Vol. **28**(2010), No. 5 - 6, pp. 1185-1202
Website: http://www.kcam.biz

DYNAMICS OF A SINGLE SPECIES POPULATION IN A POLLUTED ENVIRONMENT

A. K. PAL AND G. P. SAMANTA*

Abstract. In this paper, we have studied the dynamical behaviour such as boundedness, local and global stabilities, bifurcation of a single species population affected by environmental toxicant and population toxicant. We have also studied the effect of discrete delay of the environmental toxicant on the instantaneous growth rates of the population biomass and population toxicant due to incubation period. The length of delay preserving the stability is also estimated. Computer simulations are carried out to illustrate our analytical findings.

AMS Mathematics Subject Classification : 93C15.
Key words and phrases : Single species, Environmental toxicant, Population toxicant, stability, Hopf bifurcation, time-delay.

1. Introduction

Today, the most endangering problem to the society is the change in environment caused by pollution, affecting the long term survival of species, human life style and biodiversity of the habitat. The question of the effects of pollutants and toxicants on ecological communities is of tremendous interest from both environmental and conservational points of view. Therefore it is becoming utmost important to study the effects of toxicant on the population and the assessment of the risk to populations. In recent years, many countries have already realized that the pollution of the environment is a very urgent problem since the change in environment caused by pollution, affecting the long term survival of species, human life style and biodiversity of the habitat. Therefore the research of the effects of toxicant on the population and the assessment of the risk to populations are becoming more important.

Received December 1, 2009. Revised March 3, 2010. Accepted April 28, 2010. *Corresponding author.

Acid rain results from certain kinds of air pollution that mix with precipitation, such as rain or fog, then falls to earth as an acidic solution and its major components are oxides of sulfur and nitrogen that are mainly the by-products of coal-burning power plants, copper melting, factory and automobile emissions. These oxides are chemically changed in the atmosphere and return to the earth as rain, snow, fog or dust. In the United States, the mostly recognized form of acid rain results from sulfur dioxide emissions, which are converted into sulfuric acid in the atmosphere. When this is mixed with precipitation and falls to earth, the effect is precisely like pouring a diluted acid solution on everything it touches. In lakes also, this acidification process can change ecological structures. In this way toxic substances are invaded into the ecological communities [14,15]. By using mathematical models, Hallam and Clark [10], Hallam et. al. [11,12], Hallam and De Luna [13], De Luna and Hallam [2], Freedman and Shukla [5], Ghosh et. al. [7], Li et. al. [17], Wang [22] and many others studied the effects of toxic substances on various ecosystems. Recently Pal and Samanta [21] have analyzed the dynamical behaviours of a single species growth model with time delay under the influence of environmental and population toxicants where population toxicant is subject to exogeneous toxicant input.

Bunomo et. al. [1] viewed the internal toxicant as drifted by the living population and then, by balance arguments, they obtained a partial differential equation system consisting into two reaction diffusion equations coupled with a first order convection equation, and the corresponding ordinary differential equation system was derived as well. This model is the most realistic by now but the analysis of it is too difficult that they only used some analytical and numerical approaches.

In recent times, it is well understood that many of the processes, both natural and manmade, in ecology, medicine et cetra involve time-delays. Time-delays occur so often, in almost every situation, that to ignore them to ignore reality. Now it is beyond doubt that in an improved analysis, the effect of time-delay due to the reaction time or the activation period has to be taken into account. Time delay is also used to model the gestation lag, the incubation time for a infectious vector etc. Detailed arguments on the importance and usefulness of time-delays in realistic models may be found in the classical books of Macdonald [19], Gopalswami [8], Kuang [16].Thus "Ordinary Differential Equation", which is the heart of the Mathematical ecology, should be replaced by "Delay Differential Equation". In general, delay-differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate.

In the present work, we have investigated the dynamical behaviours and the effect of discrete delay of the model made by Bunomo et. al. [1] and Li et. al. [17]. Here we have studied the boundedness, local and global stabilities of the non-trivial equilibrium point and bifurcation analysis of this system. We have also studied the effect of discrete time delay of the environmental toxicant on the instantaneous growth rates of the population biomass and population toxicant.

The rest of the paper is structured as follows: In section 2, we present a brief sketch of the construction of the model, which may indicate the ecological relevance of it and also discussed the boundedness of the system (2.3). In section 3, we find out the necessary and sufficient conditions for the existence of the equilibrium points of the system (2.3) for zero exogeneous and non zero constant exogeneous toxicant input into the environmental toxicant and study its stability. Computer simulations of some solutions of the system (2.3) are also presented in this section. The occurrence of Hopf bifurcation is shown in section 4. It seems also reasonable to assume that the effect of the environmental toxicant on the population growth will not be instantaneous, but mediated by some discrete time lag τ required for incubation. The effect of discrete time-delay on system (2.3) is studied in section 5. In section 6, computer simulation of variety of numerical solutions of the system with delay is presented. In section 7, we calculate the length of delay for which the system preserves stability. Section 8 contains the general discussions of the paper.

2. The Basic Mathematical Model

The model we analyze in this paper describes the effect of toxicant on a single species. Here we take [17],

$X(t)$: Concentration of the population biomass.
$Y(t)$: Concentration of the toxicant in the environment.
$Z(t)$: Concentration of the toxicant in the population.

The model satisfies the following assumptions:

(A1) There is a given toxicant in the environment and the living organisms absorb into their bodies part of this toxicant so that the dynamics of the population is affected by this toxicant.

(A2) For the growth rate of population we assume that the birth rate is $b_0 - fX$ and the death rate is $d_0 + \alpha Y$, where b_0, f, d_0, α are assumed to be positive constants.

We consider the model

$$
\begin{aligned}
\frac{dX}{dt} &= X\left(b_0 - d_0 - \alpha Y - fX\right) \\
\frac{dY}{dt} &= kZ - \left(r + m + b_0 - fX\right)Y + u\left(t\right) \\
\frac{dZ}{dt} &= -kZX + \left(r + d_0 + \alpha Y\right)YX - hZ
\end{aligned}
\tag{2.1}
$$

with initial data $X\left(0\right) \geq 0, Y\left(0\right) \geq 0, Z\left(0\right) \geq 0$

To explain the parameters, we note that α is the depletion rate of toxicant in the environment due to organismal pollutant concentration, k is the depletion rate of toxicant in the environment due to its intake made by the population, r is the depletion rate of toxicant in the population due to egestion, m is the depletion rate of toxicant in the population due to metabolization processes, h is the depletion rate of the toxicant in the environment and $u(t) \geq 0,\ \forall\ t \geq 0$, is the exogeneous toxicant input rate which is assumed to be bounded [17]. Here $b_0, f, d_0, \alpha, k, r, m, h$ are assumed to be positive constants.

We can see that if $b_0 - d_0 - \alpha Y \leq 0$, $X(t)$ will be extinct after sometime, so we suppose

$$p = b_0 - d_0 > 0,\ Y(t) < \frac{b_0 - d_0}{\alpha},\ \forall\ t \geq 0 \tag{2.2}$$

The model we have just specified has nine parameters, which makes the analysis difficult. To reduce the number of parameters and to determine which combinations of parameters control the behaviour of the system, we nondimensionlize system (2.1), so we choose

$$x = fX, \quad y = \alpha Y \quad \text{and} \quad z = k\alpha Z$$

Then the system (2.1) take the form (after some simplification)

$$\begin{aligned}\frac{dx}{dt} &= x(p - y - x)\\ \frac{dy}{dt} &= z - (q - x)y + \gamma(t)\\ \frac{dz}{dt} &= -axz + a(d + y)xy - hz\end{aligned} \tag{2.3}$$

with initial data

$$x(0) \geq 0, \quad y(0) \geq 0, \quad z(0) \geq 0 \tag{2.4}$$

where $p = b_0 - d_0,\ q = r + m + b_0,\ a = \frac{k}{f},\ d = r + d_0,\ \gamma(t) = \alpha u(t) \geq 0\ \forall\ t \geq 0$. Therefore, from (2.2), we have

$$y(t) < p,\ \forall\ t \geq 0 \tag{2.5}$$

Proposition 2.1. *Each component of the solution of system (2.3) subject to (2.4) are non-negative and bounded for all $t > 0$.*

Proof. Since the right hand side of system (2.1) is completely continuous and locally Lipschitzian on C, the solution $(x(t), y(t), z(t))$ of (2.3) with initial conditions (2.4) exists and is unique on $[0, \beta)$, where $0 < \beta \leq +\infty$ [9, Chapter 2]. Now, from the first equation of system (2.3), we have

$$x(t) = x(0)\exp\int_0^t \{p - y(s) - x(s)\}ds \geq 0,\ \forall\ t \geq 0.$$

Next, we show that $y(t) \geq 0$ for all $t \in [0, \beta)$, where $0 < \beta \leq +\infty$. Otherwise, there exists a $t_1 \in [0, \beta)$ such that $y(t_1) = 0,\ \dot{y}(t_1) < 0$ and $y(t) \geq 0$ for all $t \in [0, t_1]$. Hence there must have $z(t) \geq 0$ for all $t \in [0, t_1]$. If this statement is not true, then there exists a $t_2 \in [0, t_1)$ such that $z(t_2) = 0$, $\dot{z}(t_2) < 0$ and $z(t) \geq 0$ for all $t \in [0, t_2]$. From the third equation of (2.3), we have:

$$\dot{z}(t_2) = a\,(d + y(t_2))\,x(t_2)y(t_2) \geq 0,$$

which is a contradiction with $\dot{z}(t_2) < 0$. So $z(t) \geq 0$ for all $t \in [0, t_1]$. Now from the second equation of (2.3), we have:

$$\dot{y}(t_1) = z(t_1) + \gamma\,(t_1) \geq 0 \quad as \quad \gamma\,(t) \geq 0 \;\; \forall \;\; t \geq 0,$$

which is a contradiction with $\dot{y}(t_1) < 0$. So $y(t) \geq 0$, $\forall\ t \geq 0$ and hence $z(t) \geq 0$, $\forall\ t \geq 0$. Therefore,

$$x(t) \geq 0, \quad y(t) \geq 0, \quad z(t) \geq 0, \ \forall\ t \geq 0. \tag{2.6}$$

From the first equation of (2.3) and (2.6), we have:

$$\frac{dx}{dt} = x\,(p - y - x) \leq p\left(x - \frac{x^2}{p}\right)$$

Therefore, by a standard comparison theorem, we have,

$$\limsup_{t\to\infty} x\,(t) \leq p. \tag{2.7}$$

From the third equation of (2.3),(2.5),(2.6) and (2.7), we have:

$$\frac{dz}{dt} < a\,(d + p)\,px - hz \Rightarrow \limsup_{t\to\infty} z(t) \leq \frac{a(d + p)p^2}{h} \tag{2.8}$$

From (2.5)-(2.8), we conclude that each component of the solution of system (2.3) subject to (2.4) are non-negative and bounded for all $t > 0$. This completes the proof of the proposition. $\square$

3. Stability behaviour of the model

Case I : Zero exogeneous input $(\gamma\,(t) = 0)$

If $\gamma\,(t) = 0$, then the model (2.3) has two non-negative equilibria in xyz-plane, namely $E_0\,(0, 0, 0)$ and $E_1\,(p, 0, 0)$. It is noted here that the other equilibrium point (x^*, y^*, z^*) is not feasible, since

$$x^* = \frac{hq}{h - am},\ y^* = \frac{-h(r + m + b_0) - apm}{h - am},\ p > 0 \text{ by (2.2)},$$

are opposite in sign. The variational matrices of system (2.3) at E_0 and E_1 are respectively

$$V(E_0) = \begin{pmatrix} p & 0 & 0 \\ 0 & -q & 1 \\ 0 & 0 & -h \end{pmatrix}, \quad V(E_1) = \begin{pmatrix} -p & -p & 0 \\ 0 & -(q - p) & 1 \\ 0 & adp & -(ap + h) \end{pmatrix}$$

It is obvious that E_0 is unstable (hyperbolic saddle) in the direction orthogonal to yz-plane. The characteristic equation of $V\,(E_1)$ is $(p + \lambda)\left(\lambda^2 + B\lambda + C\right) = 0$, where $B = ap + h + q - p = ap + h + d_0 + r + m > 0$ and $C = (ap + h)\,(q - p) - apd = ap(q - p - d) + h(q - p) = apm + h(d_0 + r + m) > 0$, since $p > 0$ by (2.2).

The eigenvalues are $\lambda_1 = -p$ and $\lambda_{2,3} = \frac{-B \pm \sqrt{B^2-4C}}{2}$. Since $B > 0$, $C > 0$, therefore the signs of the real parts of λ_2, λ_3 are negative. This implies that E_1 is locally asymptotically stable.

Case II : Non-zero exogeneous input $(\gamma(t) = Q > 0)$

When $(\gamma(t) = Q > 0)$, the model (2.3) has two non-negative equilibria, $E_2\left(0, \frac{Q}{q}, 0\right)$ and $\overline{E}(\overline{x}, \overline{y}, \overline{z})$. The variational matrix of system (2.3) at E_2 is given by

$$V(E_2) = \begin{pmatrix} p - \frac{Q}{q} & 0 & 0 \\ \frac{Q}{q} & -q & 1 \\ \frac{aQ}{q}\left(d + \frac{Q}{q}\right) & 0 & -h \end{pmatrix}$$

The characteristic equation of $V(E_2)$ is $\left(p - \frac{Q}{q} - \lambda\right)(q+\lambda)(h+\lambda) = 0$. So, if E_2 exists, then it is asymptotically stable if and only if $pq < Q$ and if $pq > Q$, E_2 is unstable in the direction to yz plane.

Lemma 3.1. *The unique interior equilibrium point $\overline{E}(\overline{x}, \overline{y}, \overline{z})$ of system (2.3) exists if and only if the following two conditions are satisfied*

$$(i)\ am > h \quad \text{and} \quad (ii)\ pq > Q$$

If these conditions are satisfied, then $\overline{x}, \overline{y}, \overline{z}$ are given by

$$\overline{x} = \frac{A - hq - aQ + p(am-h)}{2(am-h)}, \quad \overline{y} = p - \overline{x}, \quad \overline{z} = (q - \overline{x})(p - \overline{x}) - Q$$

where

$$A = \sqrt{\{hq + aQ - p(am-h)\}^2 + 4h(am-h)(pq-Q)} \quad \text{and} \quad m = q - p - d.$$

Local Stability of $\overline{E}$

The variational matrix at $\overline{E}$ is given by

$$V\left(\overline{E}\right) = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

where

$$m_{11} = -\overline{x}, \quad m_{12} = -\overline{x}, \quad m_{21} = \overline{y}, \quad m_{22} = -(q - \overline{x}), \quad m_{23} = 1,$$
$$m_{31} = -a\overline{z} + a\overline{y}(d + \overline{y}), \quad m_{32} = a\overline{x}(d + 2\overline{y}), \quad m_{33} = -(a\overline{x} + h)$$

The characteristic equation is

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$$

where

$$A_1 = -m_{11} - m_{22} - m_{33} = -\mathrm{tr}\left[V\left(\overline{E}\right)\right],$$
$$A_2 = m_{11}m_{22} + m_{11}m_{33} + m_{22}m_{33} - m_{23}m_{32} - m_{12}m_{21},$$
$$A_3 = -\det\left[V\left(\overline{E}\right)\right] = m_{11}m_{23}m_{32} + m_{12}m_{21}m_{33} - m_{11}m_{22}m_{33} - m_{12}m_{23}m_{31}.$$

By the Routh-Hurwitz criterion, it follows that all eigenvalues of characteristic equation have negative real part if and only if

$$A_1 > 0, \quad A_3 > 0, \quad A_1 A_2 - A_3 > 0.$$

Proposition 3.1. *$\overline{E}$ is locally asymptotically stable if and only if the above inequalities are satisfied.*

Global Stability of $\overline{E}$

$\overline{E}$ is not always globally asymptotically stable. In the following, we able to write down conditions, which guarantee the global stability of $\overline{E}$.

Theorem 3.1. *Since $x(t)$, $y(t)$, $z(t)$ are bounded, let $m_1 \leq x(t) \leq M_1$, $m_2 \leq y(t) \leq M_2$, $m_3 \leq z(t) \leq M_3$ where m_i, M_i (i=1,2,3) are some positive constants. If the following inequalities hold :*

$$\begin{array}{l} (M_2 - 1)^2 < (q - \overline{x}) \\ \{aM_2(d + M_2) - am_3\}^2 < (a\overline{x} + h) \\ (1 + ad\overline{x} + 2a\overline{x}M_2)^2 < (q - \overline{x})(h + a\overline{x}) \end{array} \tag{3.1}$$

then $\overline{E}$ is globally asymptotically stable.

Proof. We consider the following positive definite function about $\overline{E}$

$$L(x, y, z) = \left(x - \overline{x} - \overline{x} ln\frac{x}{\overline{x}}\right) + \frac{1}{2}(y - \overline{y})^2 + \frac{1}{2}(z - \overline{z})^2$$

Differentiating with respect to t along the solution of (2.3), we get (after some simple calculations)

$$\begin{aligned} \frac{dL}{dt} &= -(x - \overline{x})^2 + (y - 1)(x - \overline{x})(y - \overline{y}) - (q - \overline{x})(y - \overline{y})^2 \\ &\quad + (ay(d + y) - az)(x - \overline{x})(z - \overline{z}) - (a\overline{x} + h)(z - \overline{z})^2 \\ &\quad + (1 + ad\overline{x} + a\overline{x}(y + \overline{y}))(y - \overline{y})(z - \overline{z}) \\ &\leq -(x - \overline{x})^2 + (M_2 - 1)(x - \overline{x})(y - \overline{y}) - (q - \overline{x})(y - \overline{y})^2 \\ &\quad - (a\overline{x} + h)(z - \overline{z})^2 + (aM_2(d + M_2) - am_3)(x - \overline{x})(y - \overline{y}) \\ &\quad + (1 + ad\overline{x} + 2aM_2\overline{x})(y - \overline{y})(z - \overline{z}) \\ &= -a_{11}(x - \overline{x})^2 - a_{22}(y - \overline{y})^2 - a_{33}(z - \overline{z})^2 + a_{12}(x - \overline{x})(y - \overline{y}) \\ &\quad + a_{13}(x - \overline{x})(z - \overline{z}) + a_{23}(y - \overline{y})(z - \overline{z}) \end{aligned}$$

where

$$a_{11} = 1, \quad a_{22} = q - \overline{x}, \quad a_{33} = a\overline{x} + h, \quad a_{13} = M_2 - 1,$$

$$a_{13} = (aM_2(d + M_2) - am_3), \quad a_{23} = 1 + ad\overline{x} + 2aM_2\overline{x}$$

A sufficient condition for $\frac{dL}{dt}$ to be negative definite is

$$a_{12}^2 - a_{11}a_{22} < 0, \quad a_{13}^2 - a_{11}a_{33} < 0, \quad a_{22}^2 - a_{22}a_{33} < 0 \tag{3.2}$$

Now it is easy to see that if conditions (3.1) hold then (3.2) will hold automatically. Hence L is a Lyapunov functions with respect to all solutions in the interior of the positive orthant, proving the theorem. $\square$

We choose the parameters of the system (2.3) as $p = 2.75$ $q = 3.05$, $a = 4.1429$, $d = 0.7$, $h = 0.2$, $Q = 0.6$, and $(x(0), y(0), z(0)) = (1, 1, 5)$. Then the conditions of Proposition 3.1 is satisfied as $A_1 = 5.1630 > 0$, $A_3 = 1.2982 > 0$, $\Delta = 0.1712 > 0$ and consequently $\overline{E}\,(\overline{x}, \overline{y}, \overline{z}) = (0.4617, 1.7883, 4.0284)$ is locally asymptotically stable. The phase diagram is shown in Fig. 1(a). The xy-plane and xz-plane projections of the solution are shown in Fig. 1(b) and Fig. 1(c) respectively. Clearly the solution is a stable spiral converging to $\overline{E}$. Fig. 1(d) shows that x, y and z populations approach to their steady-state values $\overline{x}, \overline{y}$ and $\overline{z}$ respectively in finite time.

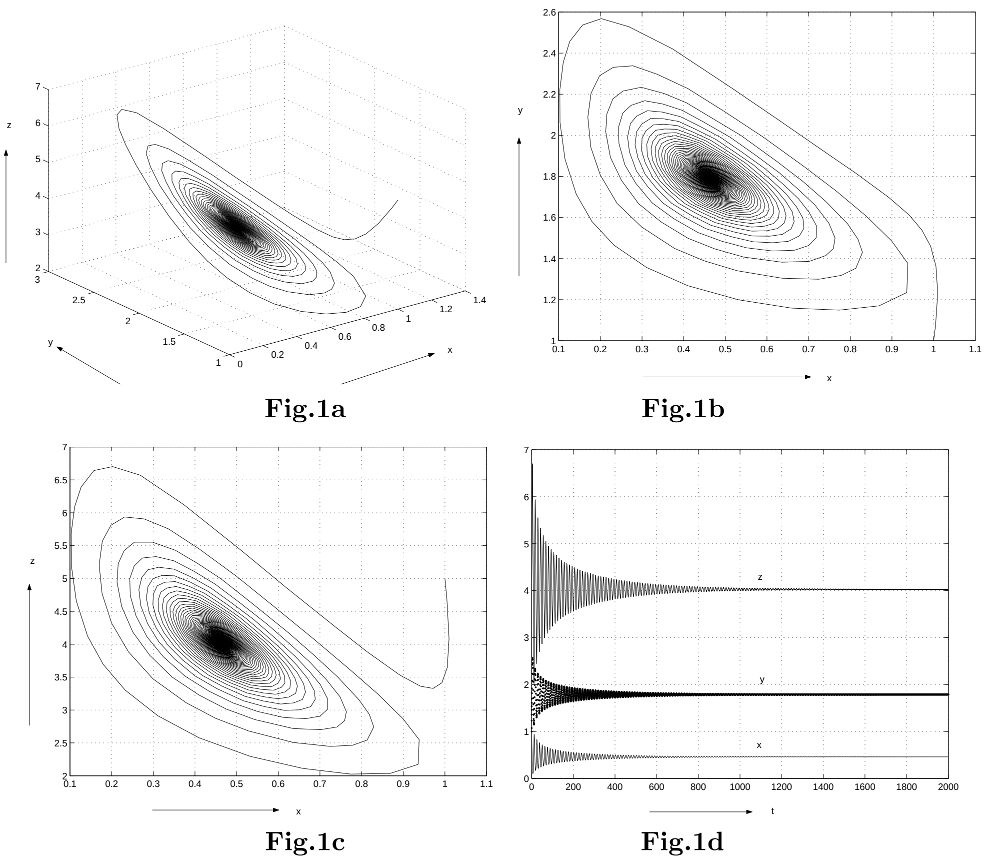

Fig. 1. *Here* $x(0) = 1.0$, $y(0) = 1.0$, $z(0) = 5.0$ *and* $p = 2.75$, $q = 3.05$, $a = 4.1429$, $d = 0.7$, $h = 0.2$, $Q = 0.6$. (a) *Phase portrait of the system* (2.3) *showing that* $\overline{E}$ *is locally asymptotically stable.* (b) *xy-plane projection of the solution.* (c) *xz-plane projection of the solution.* (d) *Top curve depicts* $z(t)$*, middle one depicts* $y(t)$ *and the bottom one depicts* $x(t)$*. Clearly the* x, y, z *populations approach to their equilibrium values in finite time.*

4. Bifurcation Analysis

It is really difficult to establish an analytical criterion for the existence of Hopf bifurcation for the model (2.3) by using either Hopf bifurcation Theorem [20] or Liu's criterion [18]. Let us choose the parameters as $p = 2.75,\ a = 4.1429,\ d = 0.7,\ h = 0.2,\ Q = 0.6$. If $q = 3.05$, then we have seen that $\overline{E}$ is locally asymptotically stable. Now if we decrease the value of the parameter q, keeping other parameters fixed, the stability behaviour of the system (2.3) changes at the bifurcation value $q^* = 3.02$. For $q = 2.98 < q^*$, we see that $\overline{E}\,(0.4528, 1.7972, 3.9419)$ is unstable and there is a limit cycle near $\overline{E}$ which is shown in Fig. 2.

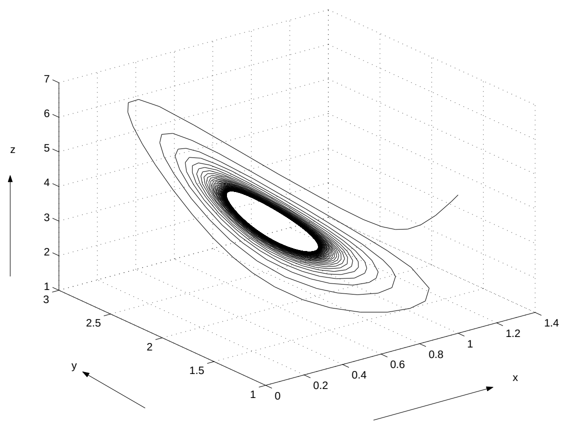

Fig.2

Fig. 2. *Here* $x(0) = 1.0,\ y(0) = 1.0,\ z(0) = 5.0$ *and* $p = 2.75,\ q = 2.98,\ a = 4.1429,\ d = 0.7,\ h = 0.2,\ Q = 0.6$. *Phase portrait of the system (2.3) showing a periodic orbit near* $\overline{E}$.

5. Model with discrete delay

It is already mentioned that time-delay is an important factor in biological system. It is also reasonable to assume that the effect of the environmental toxicant on the population growth will not be instantaneous, but mediated by some discrete time lag τ required for incubation. As a starting point of this section, we consider the following generalization of the model (2.3) involving discrete time-delay :

$$\begin{aligned}
&\frac{dx}{dt} = x\,(p - y\,(t-\tau) - x)\\
&\frac{dy}{dt} = z - (q - x)\,y + Q\\
&\frac{dz}{dt} = -axz + a\,(d + y\,(t-\tau))\,xy - hz\\
&x\,(0) \geq 0,\quad y\,(\theta) \geq 0,\quad z\,(0) \geq 0,\quad \theta \in [-\tau, 0]
\end{aligned} \tag{5.1}$$

All parameters are the same as in system (2.3) except that the positive constant τ represents the activation period or reaction time of the toxicant in the environment.
The system (5.1) has the same equilibria as in the previous case. The main purpose of this section is to study the stability behaviour of $\overline{E}\left(\overline{x},\overline{y},\overline{z}\right)$ in the presence of discrete delay ($\tau \neq 0$). We linearize system (5.1) by using the following transformation :

$$x = \overline{x} + x_1, \quad y = \overline{y} + y_1, \quad z = \overline{z} + z_1$$

Then linear system is given by

$$\frac{du}{dt} = Au\left(t\right) + Bu\left(t-\tau\right) \tag{5.2}$$

where

$$u\left(t\right) = \left[x_1 \quad y_1 \quad z_1\right]^T, \quad A = \left(a_{ij}\right)_{3\times 3}, \quad B = \left(b_{ij}\right)_{3\times 3}$$

and

$$a_{11} = -\overline{x}, \quad a_{21} = \overline{y}, \quad a_{22} = -\left(q - \overline{x}\right), \quad a_{23} = 1, \quad a_{31} = -a\overline{z} + a\left(d + \overline{y}\right)\overline{y},$$

$$a_{32} = a\left(d + \overline{y}\right)\overline{x}, \quad a_{33} = -\left(a\overline{x} + h\right)$$

and all other $a_{ij} = 0$; $b_{12} = -\overline{x}$, $b_{32} = a\overline{xy}$ and other $b_{ij} = 0$. We took for the solution of the model (5.2) of the form $u\left(t\right) = \rho e^{\lambda t}$, $0 \neq \rho \in \Re$. This leads to the following characteristic equation :

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + \left(a_4\lambda + a_5\right)e^{-\lambda\tau} = 0 \tag{5.3}$$

where

$$a_1 = -a_{11} - a_{22} - a_{33}, \quad a_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32},$$

$$a_3 = a_{11}a_{32}a_{23} - a_{11}a_{22}a_{33}, \quad a_4 = -a_{23}b_{32} - a_{21}b_{12},$$

$$a_5 = a_{11}a_{23}b_{32} + a_{21}a_{33}b_{12} - a_{23}a_{31}b_{12}$$

It is well known that the signs of the real parts of the solutions of (5.3) characterize the stability behaviour of $\overline{E}$. Therefore, substituting $\lambda = \xi + i\eta$ in (5.3) we obtain real and imaginary parts, respectively as

$$\xi^3 - 3\xi\eta^2 + a_1\xi^2 - a_1\eta^2 + a_2\xi + a_3 + e^{-\xi\tau}\left\{\left(a_4\xi + a_5\right)\cos\eta\tau + a_4\eta\sin\eta\tau\right\} = 0 \tag{5.4}$$

$$3\xi^2\eta - \eta^3 + 2a_1\xi\eta + a_2\eta + e^{-\xi\tau}\left\{a_4\eta\cos\eta\tau - \left(a_4\xi + a_5\right)\sin\eta\tau\right\} = 0 \tag{5.5}$$

A necessary condition for a stability change of $\overline{E}$ is that the characteristic equation (5.3) should have purely imaginary solutions. Hence to obtain the stability criterion, we set $\xi = 0$ in (5.4) and (5.5). Then we have,

$$a_1\eta^2 - a_3 = a_5\cos\eta\tau + a_4\eta\sin\eta\tau \tag{5.6}$$

$$-\eta^3 + a_2\eta = a_5\sin\eta\tau - a_4\eta\cos\eta\tau \tag{5.7}$$

Eliminating τ by squaring and adding (5.6) and (5.7), we get the equation for determining η as

$$\eta^6 + d_1\eta^4 + d_2\eta^2 + d_3 = 0 \tag{5.8}$$

where

$$d_1 = a_1^2 - 2a_2, \quad d_2 = a_2^2 - 2a_1a_3 - a_4^2, \quad d_3 = a_3^2 - a_5^2$$

Substituting $\eta^2 = \sigma$ in (5.8), we get a cubic equation given by

$$h(\sigma) = \sigma^3 + d_1\sigma^2 + d_2\sigma + d_3 = 0 \tag{5.9}$$

Since $d_3 = a_3^2 - a_5^2 > 0$ for the parameter values given in previous case, we assume that $d_3 \geq 0$ and have the following claim.

Claim 1: If

$$d_3 \geq 0 \tag{5.10}$$

and

$$d_2 > 0 \tag{5.11}$$

then equation (5.9) has no positive real roots. In fact, notice that

$$\frac{dh(\sigma)}{d\sigma} = 3\sigma^2 + 2d_1\sigma + d_2$$

Set,

$$3\sigma^2 + 2d_1\sigma + d_2 = 0 \tag{5.12}$$

Then the roots of equation (5.12) can be expressed as

$$\sigma_{1,2} = \frac{-d_1 \pm \sqrt{d_1^2 - 3d_2}}{3} \tag{5.13}$$

If $d_2 > 0$, then $d_1^2 - 3d_2 < d_1^2$; that is $\sqrt{d_1^2 - 3d_2} < d_1$. Hence neither σ_1 nor σ_2 is positive. Thus equation (5.12) does not have positive roots. Since $h(0) = d_3 \geq 0$, it follows that the equation (5.9) has no positive roots.

Claim 1 thus implies that there is no η such that $i\eta$ is an eigen value of the characteristic equation (5.3).Therefore, the real parts of all the eigen values of (5.3) are negative for all delay $\tau \geq 0$. Summarizing the above analysis, we have the following proposition :

Proposition 5.1. *Suppose that*

$$(i) \quad a_1 > 0, \quad a_3 + a_5 > 0, \quad a_1(a_2 + a_4) - (a_3 + a_5) > 0$$

$$(ii) \quad d_3 \geq 0 \quad and \quad d_2 > 0$$

Then the equilibrium point $\overline{E}$ of the delay model (5.1) is absolutely stable; that is $\overline{E}$ is asymptotically stable for all $\tau \geq 0$.

Remark 1 : Proposition 5.1 indicates that if the parameters satisfy the conditions (i) and (ii), then the steady state of the delay model (5.1) is asymptotically stable for all delay values; that is, independent of the delay. However, we should point out that if the conditions (condition (ii)) in proposition 5.1 are not satisfied, then the stability of the steady state depends on the delay value and the delay could even induce oscillation.

For example, if **(a)** $d_3 < 0$, then from equation (5.9) we have $h(0) < 0$ and $\lim_{\sigma\to\infty} h(\sigma) = \infty$. Thus equation (5.9) has at least one positive root, say σ_0. Consequently, equation (5.8) has at least one positive root, denoted by η_0.

If **(b)** $d_2 < 0$, then $\sqrt{d_1^2 - 3d_2} > d_1$. By (5.13), $\sigma_1 = \frac{1}{3}\left(-d_1 + \sqrt{d_1^2 - 3d_2}\right) > 0$. It follows that equation (5.9), hence equation (5.8), has a positive root η_0. This implies that the characteristic equation (5.3) has a pair of purely imaginary roots $\pm i\eta_0$. Let $\lambda(\tau) = \xi(\tau) + i\eta(\tau)$ be the eigenvalues of equation (5.3) such that $\xi(\tau_0) = 0, \quad \eta(\tau_0) = \eta_0$. From (5.6) and (5.7) we have,

$$\tau_j = \frac{1}{\eta_0}\arccos\left(\frac{a_4\eta_0^4 + (a_1a_5 - a_2a_4)\,\eta_0^2 - a_3a_5}{a_5^2 + a_4\eta_0^2}\right) + \frac{2j\pi}{\eta_0}, \quad j = 0, 1, 2, ...,$$

Also, we can verify that the following transversality conditions :

$$\frac{d}{d\tau}\mathrm{Re}(\lambda(\tau)) \mid_{\tau=\tau_0} = \frac{d}{d\tau}\xi(\tau) \mid_{\tau=\tau_0} > 0$$

that is

$$f(\eta_0) = \eta^2[3\eta^4 + 2(a_1^2 - 2a_2)\eta^2 + (a_2^2 - 2a_1a_3 - a_4^2)] > 0 \tag{5.14}$$

holds.

By continuity, the real part of $\lambda(\tau)$ becomes positive when $\tau > \tau_0$ and the steady state becomes unstable. Moreover, a Hopf bifurcation occurs when τ passes through the critical value τ_0. The above analysis can be summarized into the following proposition.

Proposition 5.2. *Suppose that*

$$(i) \quad a_1 > 0, \quad a_3 + a_5 > 0, \quad a_1(a_2 + a_4) - (a_3 + a_5) > 0$$

If either (ii) $d_3 < 0$ or (iii) $d_3 \geq 0$ and $d_2 < 0$ is satisfied, then the steady state $\overline{E}$ of the delay model (5.1) is asymptotically stable when $0 \leq \tau < \tau_0$ and unstable when $\tau > \tau_0$, where

$$\tau_0 = \frac{1}{\eta_0}\arccos\left(\frac{a_4\eta_0^4 + (a_1a_5 - a_2a_4)\,\eta_0^2 - a_3a_5}{a_5^2 + a_4\eta_0^2}\right)$$

when $\tau = \tau_0$, a Hopf bifurcation occurs; that is, a family of periodic solutions bifurcates from $\overline{E}$ as τ passes through the critical value τ_0.

Proposition 5.2 indicates that the delay model could exhibit Hopf bifurcation at certain value of the delay if the parameters satisfy the conditions in (ii) and (iii).

6. Numerical Simulation for the model with delay

Analytical studies can never be completed without numerical verification of the results. In this section we present computer simulation of some solutions of the system (5.1). Beside verification of our analytical findings, these numerical solutions are very important from practical point of view.

It is mentioned before that the stability criteria in the absence of delay ($\tau = 0$) will not necessarily guarantee the stability of the system in presence of delay ($\tau \neq 0$). Let us choose the parameters of the system as $p = 2.75$ $q = 3.05,\ a = 4.1429,\ d = 0.7,\ h = 0.2,\ Q = 0.6$, and $(x(0), y(0), z(0)) = (1, 1, 5)$. It is already seen that for such choices of parameters $\overline{E}\,(\overline{x}, \overline{y}, \overline{z}) = (0.4617, 1.7883, 4.0284)$ is locally asymptotically stable in the absence of delay. Now for these choices of parameters, it is seen from Proposition 5.2 that there is a unique positive root of (5.9) given by $\sigma_o = \eta_0^2 = 0.2465$ for which $f\,(\eta_0) = 1.9627 > 0$ and $\tau = \tau^* = 0.0356$. Therefore by Proposition 5.2, $\overline{E}\,(\overline{x}, \overline{y}, \overline{z})$ losses its stability as τ passes through the critical value τ^*. We verify that for $\tau = 0.01 < \tau^*$, $\overline{E}$ is locally asymptotically stable, the phase portrait of the solution (presented in Fig. 3a) being stable spiral. Fig. 3b shows that for the above choices of parameters, x, y, z populations converge to their equilibrium values $\overline{x}$, $\overline{y}$, $\overline{z}$, respectively. Keeping other parameters fixed, if we take $\tau = 0.06 > \tau^*$, it is seen that $\overline{E}$ is unstable and there is a bifurcating periodic solution near $\overline{E}$ (see Fig. 4a). Fig. 4b depicts the oscillations of the populations in finite time.

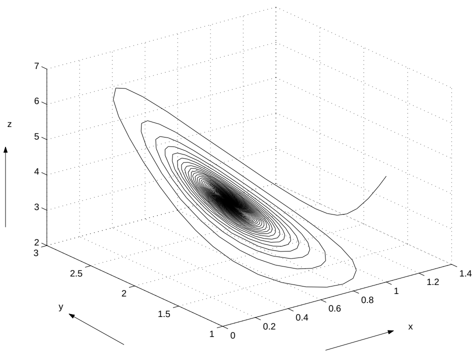

Fig.3a

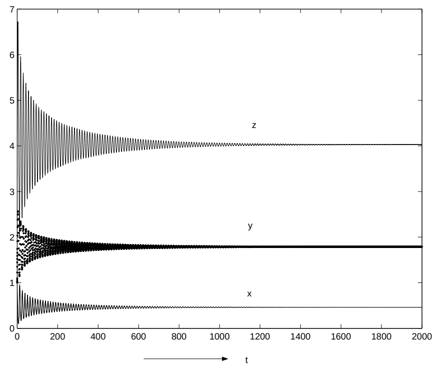

Fig.3b

Fig. 3. *Here* $x(0) = 1.0,\ y(0) = 1.0,\ z(0) = 5.0$ *and* $p = 2.75,\ q = 3.05,\ a = 4.1429,\ d = 0.7,\ h = 0.2,\ Q = 0.6,$ *and* $\tau = 0.01 < \tau^*$. *(a) Phase portrait of the system. (b) Stable behaviour of x, y, z in time.*

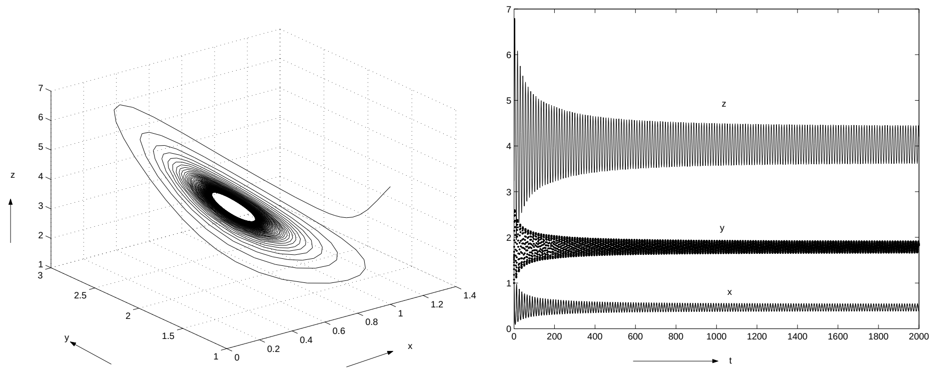

Fig.4a **Fig.4b**

Fig. 4. *Here all other parameter values are same as in Fig. 3 except* $\tau = 0.06 > \tau^*$.
(a) Phase portrait of the system (5.1) showing a limit cycle which grows out of $\overline{E}$.
(b) Oscillations of x, y, z in time.

7. Estimation of the length of delay to preserve stability

We consider the system (2.3) and the space of all real valued continuous functions defined on $[-\tau, \infty]$ satisfying the initial conditions on $[-\tau, 0]$. We linearize the system (2.3) about its interior equilibrium $\overline{E}(\overline{x}, \overline{y}, \overline{z})$ and get

$$\begin{aligned} \frac{dx_1}{dt} &= -\overline{x}x_1 - \overline{x}y_1(t-\tau) \\ \frac{dy_1}{dt} &= \overline{y}x_1 - (q-\overline{x})\,y_1 + z_1 \\ \frac{dz_1}{dt} &= \{-a\overline{z} + a(d+\overline{y})\,\overline{y}\}\,x_1 + a(d+\overline{y})\,\overline{x}y_1 + a\overline{x}\overline{y}y_1(t-\tau) - (a\overline{x}+h)\,z_1 \end{aligned} \tag{7.1}$$

where

$$x(t) = \overline{x} + x_1(t), \quad y(t) = \overline{y} + y_1(t) \quad \text{and} \quad z(t) = \overline{z} + z_1(t)$$

Taking laplace transform of the system given by (7.1), we get,

$$\begin{aligned} (s+\overline{x})\,\overline{x_1}(s) &= -\overline{x}e^{-s\tau}\overline{y_1}(s) - -\overline{x}e^{-s\tau}k_1(s) + x_1(0) \\ (s+q-\overline{x})\,\overline{y_1}(s) &= \overline{y}\overline{x_1}(s) + \overline{z_1}(s) + y_1(0) \\ (s+a\overline{x}+h)\,\overline{z_1}(s) &= \{-a\overline{z} + a(d+\overline{y})\,\overline{x}\}\,\overline{x_1}(s) + a(d+\overline{y})\,\overline{x}\overline{y_1}(s) \\ &\quad + a\overline{x}\overline{y}e^{-s\tau}\overline{y_1}(s) + a\overline{x}\overline{y}e^{-s\tau}k_1(s) + z_1(0) \end{aligned} \tag{7.2}$$

where

$$k_1(s) = \int_{-\tau}^{0} e^{-st}y_1(t)\,dt$$

and $\overline{x_1}(s)$, $\overline{y_1}(s)$ and $\overline{z_1}(s)$ are the laplace transform of $x_1(t)$, $y_1(t)$ and $z_1(t)$ respectively.

From [6] and using "Nyquist criterion" (see Appendix), it can be shown that the conditions for local asymptotic stability of $\overline{E}\left(\overline{x},\overline{y},\overline{z}\right)$ are given by [3]

$$\mathrm{Im}H\left(i\eta_0\right) \quad > \quad 0 \tag{7.3}$$

$$\mathrm{Re}H\left(i\eta_0\right) \quad = \quad 0 \tag{7.4}$$

where $H\left(s\right) = s^3 + a_1 s^2 + a_2 s + a_3 + e^{-s\tau}\left(a_4 s + a_5\right)$ and η_0 is the smallest positive root of equation (7.4).

We have already shown that $\overline{E}\left(\overline{x},\overline{y},\overline{z}\right)$ is stable in absence of delay. Hence, by continuity, all eigenvalues will continue to have negative real parts for sufficiently small $\tau > 0$ provided one can guarantee that no eigenvalues with positive real parts bifurcates from infinity as τ increases from zero. This can be proved by using Butler's lemma [3].
In this case, (7.3) and (7.4) gives

$$a_2\eta_0 - \eta_0^3 \quad > \quad a_5 \sin\left(\eta_0\tau\right) - a_4\eta_0\cos\left(\eta_0\tau\right) \tag{7.5}$$

$$a_3 - a_1\eta_0^2 \quad = \quad -a_5\cos\left(\eta_0\tau\right) - a_4\eta_0\sin\left(\eta_0\tau\right) \tag{7.6}$$

(7.5) and (7.6), if satisfied simultaneously, are sufficient conditions to guarantee stability. We shall utilize them to get an estimate on the length of delay. Our aim is to find an upper bound η_+ on η_0, independent of τ so that (7.5) holds for all values of η, $0 \le \eta \le \eta_+$ and hence in particular at $\eta = \eta_0$.
We rewrite (7.6) as

$$a_1\eta_0^2 \quad = \quad a_3 + a_5\cos\left(\eta_0\tau\right) + a_4\eta_0\sin\left(\eta_0\tau\right) \tag{7.7}$$

Maximizing $a_3 + a_5\cos\left(\eta_0\tau\right) + a_4\eta_0\sin\left(\eta_0\tau\right)$

subject to $\left|\sin\left(\eta_0\tau\right)\right| \quad \le \quad 1, \quad \left|\cos\left(\eta_0\tau\right)\right| \quad \le \quad 1$

We obtain

$$a_1\eta_0^2 \quad \le \quad a_3 + \left|a_5\right| + \left|a_4\right|\eta_0 \tag{7.8}$$

Hence, if

$$\eta_+ \quad = \quad \frac{1}{2a_1}\left[\left|a_4\right| + \sqrt{a_4^2 + 4a_1\left(a_3 + \left|a_5\right|\right)}\right] \tag{7.9}$$

then clearly from (7.8) we have $\eta_0 \quad \le \quad \eta_+$
From the inequality (7.5) we get

$$\eta_0^2 \quad < \quad a_2 + a_4\cos\left(\eta_0\tau\right) - \frac{a_5}{\eta_0}\sin\left(\eta_0\tau\right) \tag{7.10}$$

As $\overline{E}\left(\overline{x},\overline{y},\overline{z}\right)$ is locally asymptotically stable for $\tau = 0$, therefore, for sufficiently small $\tau > 0$, (7.10) will continue to hold. Substituting (7.7) in (7.10) and rearranging we get,

$$\left(a_5 - a_1a_4\right)\left[\cos\left(\eta_0\tau\right) - 1\right] + \left(a_4\eta_0 + \frac{a_1a_5}{\eta_0}\right)\sin\left(\eta_0\tau\right) < a_1a_2 + a_1a_4 - a_3 - a_5 \tag{7.11}$$

Using the bounds

$$(a_5 - a_1a_4)\left[1 - \cos(\eta_0\tau)\right] = (a_5 - a_1a_4)\, 2\sin^2\left(\frac{\eta_0\tau}{2}\right) \leq \frac{1}{2}\left|a_5 - a_1a_4\right| \eta_+^2\tau^2$$

and

$$\left(a_4\eta_0 + \frac{a_1a_5}{\eta_0}\right)\sin(\eta_0\tau) \leq \left(a_4\eta_+^2 + a_1a_5\right)\tau$$

So we obtain from (7.11),

$$k_1\tau^2 + k_2\tau < k_3$$

where

$$k_1 = \frac{1}{2}\left|a_5 - a_1a_4\right| \eta_+^2 \tag{7.12}$$

$$k_2 = \left(a_4\eta_+^2 + a_1a_5\right) \tag{7.13}$$

$$k_3 = a_1a_2 + a_1a_4 - a_3 - a_5 \tag{7.14}$$

Hence, if

$$\tau_+ = \frac{1}{2k_2}\left[-k_2 + \sqrt{k_2^2 + 4k_1k_2}\right] \tag{7.15}$$

then stability is preserved for $0 \leq \tau < \tau_+$.

8. Discussion

In this paper, we have studied the effects of environmental toxicant and population toxicant on single species population model governed by modified logistic equation [4]. It is shown (in proposition 2.1) that the nondimensionalized system (2.3) is bounded, which in turn, implies that the system is ecologically well behaved. Criterion for the long-time survival (Persistence) and extinction of the population of the system are presented in Li et. al. [17]. In deterministic situation, theoretical ecologists are usually guided by an implicit assumption that most toxicant models we observe in nature correspond to stable equilibria of the models. From this viewpoint, we have presented the stability analysis of the most important equilibrium point $\overline{E}\left(\overline{x}, \overline{y}, \overline{z}\right)$. The stability criteria given in Proposition 3.1 and Theorem 3.1 are the conditions for stable coexistence of the population biomass, the environmental toxicant and the population toxicant.

It is mentioned by several researchers that the effect of time-delay must be taken into account to have a ecologically useful mathematical model [8,16,19]. It seems also reasonable to assume that the effect of the environmental toxicant on the population growth will not be instantaneous, but mediated by some discrete time lag τ required for incubation. From this viewpoint, we have formulated model (5.1) where the delay may be looked upon as the reaction time of the environment toxicant on the population biomass and the population toxicant. Then a rigorous analysis leads us to Proposition 5.1 and Proposition 5.2 which mentions that the stability criteria in absence of delay are no longer enough to guarantee the stability in the presence of delay, rather there is a value τ^* of the delay τ such that the system is stable for $\tau < \tau^*$ and becomes unstable for $\tau > \tau^*$.

All our important mathematical findings without and with time-delay are numerically verified and graphical representation of a variety of solutions of system (2.3) and (5.1) are depicted using MATLAB. Our analytical and numerical studies show that, using the delay τ as control, it is possible to break the stable (spiral) behaviour of the system and drive it to an unstable (cyclic) state. Also it is possible to keep the levels of single species population and toxicants (environment and population) at a required state using the above control.

Finally, our model can be generalized in obvious ways to food chains and competitive systems.

Appendix

Nyquist Criterion : If L be the length of a curve encircling the right half-plane, the curve $\overline{x}_1\,(L)$ will encircle the origin a number of times equal to the difference between the number of poles and the number of zeroes of $\overline{x}_1\,(L)$ in the right half-plane.

References

1. B. Buonomo, A. D. Liddo and I. Sgura, *A diffusive-connective model for the dynamics of population-toxicant intentions : some analytical and numerical results*, Math. Biosci., **157** (1999), 37-64.
2. J. T. De Lena and T. G. Hallam, *Effects of toxicants on population : A qualitative approach IV. Resources-Consumer-toxicant models*, Ecol.Model., **35** (1987), 249-273.
3. L. H. Erbe, V. S. H. Rao and H. Freedman, *Three-species food chain models with mutual interference and time delays*, Math. Biosci., **80** (1986), 57-80.
4. H. I. Freedman, Deterministic mathematical models in populations ecology, Marcel Dekker, New York, 1980.
5. H. I. Freedman and J. B. Shukla, *Models for the effect of toxicant in single species and predator-prey system*, J. Math. Biol., **30** (1991), 15-30.
6. H. Freedman and V. S. H. Rao, *The trade-off between mutual interference and tome lags in predator-prey systems*, Bull. Math. Biol., **45** (1983), 991-1004.
7. M. Ghosh, P. Chandra and P. Sinha, *A Mathematical model to study the effect of toxicant chemicals on a prey-predator type fishery*, J. biol. Syst., **10** (2002), 97-105.
8. K. Gopalsamy, *Stability and Oscillation in Delay-Differential Equations of Population Dynamics*, Kluwer, Dordrecht, 1992.
9. J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.
10. T. G. Hallam and C. E. Clark, *Non-autonomous logistic equations as models of populations in a deteriorating environment*, J. Theor. Biol., **93** (1982), 303-311.
11. T. G. Hallam, C. E. Clark and G. S. Jordan, *Effects of toxicants on populations : a qualitative approach II. First order kinetics*, J. Math. Biol., **18** (1983a), 25-37.
12. T. G. Hallam, C. E. Clark and R. R. Lassiter, *Effects of toxicants on populations : a qualitative approach I. Equilibrium environmental exposure*, Ecol. Model., **18** (1983b), 291-304.
13. T. G. Hallam and J. T. De Lena, *Effects of toxicants on populations : a qualitative approach III. Environmentaland food chain pathways*, J. Theor. Biol., **109** (1984), 411-429.
14. G. Howells, *Acid Rain and Acid Water*, Ellis Horwood, 1990.

15. J. R. E. Jones, *Fish and River Pollution*, Butterworths, Washington, 1964.
16. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993.
17. Z. Li, Z. Shuai and K. Wang, *Persistence and Extinction of single population in a polluted environment*, Electronic Journal of Differential Equations, **108** (2004), 1-5.
18. W. Liu, *Criterion of Hopf bifurcations without using eigenvalues*, J. Math. Anal. Appl., **182** (1994), 250-256.
19. N. Macdonald, *Time delays in Biological Models*, Spring, Heidelberg, 1978.
20. J. D. Murray, *Mathematical Biology*, Spinger-Verlag, Newyork, 1993.
21. A.K. Pal and G.P. Samanta, *A single species population in a polluted environment*, Int. J. Biomath. (to appear).
22. J. Wang and K. Wang, *Analysis of a single species with diffusion in a polluted environment*, Electronic Journal of Differential Equations, **112** (2006), 1-11.

A. K. Pal received his MSc in Applied Mathematics from Calcutta University in 2000. Now he is working as a senior lecturer at St. Thomas' College of Engineering & Technology, Calcutta, West Bengal. His research interest is complexity and numerical analysis for population dynamical system. He is doing Ph.D. under guidance of Dr.G.P.Samanta

St. Thomas' College of Engineering & Technology, Kolkata, 700023, India.
e-mail: akpal_2002@yahoo.co.in

G. P. Samanta got his MSc and Ph.D. in Applied Mathematics from Calcutta University in 1985 and 1991 respectively. He was a Premchand Raychand Scholar of Calcutta University and received Mouat Medal at the convocation of Calcutta University in 1996. His areas of research are Mathematical Ecology and Operations Research. He is now working as Professor and Head of the Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah, 711103, India.

Bengal Engineering and Science University, Shibpur, Howrah, 711103, India.
e-mail: g_p_samanta@yahoo.co.uk