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ON LOCAL SPECTRAL PROPERTIES OF GENERALIZED SCALAR OPERATORS

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Abstract. In this paper, we prove that if $T \in L(X)$ is a generalized scalar operator then $\operatorname{Ker} T^p$ is the quasi-nilpotent part of T for some positive integer $p \in \mathbb{N}$. Moreover, we prove that a generalized scalar operator with finite spectrum is algebraic. In particular, a quasi-nilpotent generalized scalar operator is nilpotent.

1. Introduction

We first recall some basic notions and results from local spectra theory. Let X be a complex Banach space and $L(X)$ denotes the Banach algebra of all bounded linear operators of X itself, equipped with the usual operator norm. For $T \in L(X)$, TX and $\operatorname{Ker} T$ will denote the range and kernel, respectively. Given an operator $T \in L(X)$, $\sigma_p(T)$, $\sigma(T)$ and $\rho(T)$ denotes the point spectrum, the spectrum and resolvent set of T and let $\operatorname{Lat}(T)$ stand for the collection of all $T-$invariant closed linear subspaces of X, and for an $Y \in \operatorname{Lat}(T)$, $T|Y$ denotes the restriction of T on Y. An operator $T \in L(X)$ is called *decomposable* if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there are $T-$invariant closed linear subspaces Y and Z of X such that

$$Y + Z = X, \quad \sigma(T|Y) \subseteq U \quad \text{and} \quad \sigma(T|Z) \subseteq V.$$

This simple definition is equivalent to the original notion of decomposability, as introduced by Foiás in 1963 and discussed in the classical book by Colojoarvă and Foiás [8]. The class of decomposable operators contains all normal operators on Hilbert spaces and, more generally, all

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spectral operators in the sense of Dunford on Banach spaces. Moreover, a simple application of the Riesz functional calculus shows that all operators with totally disconnected spectrum are decomposable. In particular, all compact and all algebraic operators are decomposable.

We shall also need some closely related notions. An operator $T \in L(X)$ is said to have *Bishop's property* (β) if for every open subset U of $\mathbb{C}$ and for every sequence of analytic functions $f_n : U \to X$ for which $(T-\lambda)f_n(\lambda)$ converges uniformly to zero on each compact subset of U, it follows that also $f_n(\lambda) \to 0$ as $n \to \infty$, locally uniformly on U. Obviously, property (β) implies that T has the *single-valued extension property*(abbreviated to SVEP), which means that for every open $U \subseteq \mathbb{C}$, the only analytic solution $f : U \to X$ of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the constant $f \equiv 0$. An operator $T \in L(X)$ is said to have the *decomposition property* (δ) if given an arbitrary open covering $\{U_1, U_2\}$ of $\mathbb{C}$, every $x \in X$ has a decomposition $x = u_1 + u_2$, where $u_1, u_2 \in X$ satisfy $u_k = (T-\lambda)f_k(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \overline{U_k}$ and some analytic function $f_k : \mathbb{C} \setminus \overline{U_k} \to X$ for $k = 1, 2$.

Albrecht and Eschmeier show that (β) characterizes restrictions of decomposable operators to closed invariant subspaces, and that quotients of decomposable operators are determined by the decomposition property (δ). It has been observed in [4] that an operator $T \in L(X)$ is decomposable if and only if it has both properties (β) and (δ). Albrecht and Eschmeier further show that the properties (β) and (δ) are completely dual to each other in the sense that an operator $T \in L(X)$ satisfies (β) if and only if the adjoint operator T^* on the dual space X^* satisfies (δ) and that the corresponding statement remains valid if both properties are interchanged; see [3] and [5].

Given an arbitrary operator $T \in L(X)$, let $\sigma_T(x) \subseteq \mathbb{C}$ denote the *local spectrum* of T at the point $x \in X$, that is, the complement of the set $\rho_T(x)$ of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ in $\mathbb{C}$ and analytic function $f : U \to X$ such that $(T-\mu)f(\mu) = x$ holds for all $\mu \in U$. For every closed subset F of $\mathbb{C}$, let $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ denote the corresponding *analytic spectral subspace* of T. These linear subspaces, while generally not closed, play a fundamental role in the spectral theory of operators on Banach spaces.

For each closed $F \subseteq \mathbb{C}$, let the *local spectral subspace* $\mathcal{X}_T(F)$ consist of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ with $(T-\mu)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus F$. It is easy to see that $X_T(F)$ is a $T-$invariant linear subspace of X and also hyperinvariant for T. Clearly, $\mathcal{X}_T(F)$ is a linear subspace contained in $X_T(F)$. Moreover, the identity

$\mathcal{X}_T(F) = X_T(F)$ holds for all closed sets $F \subseteq \mathbb{C}$ precisely when T has SVEP. Obviously, property (δ) means precisely that $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open covering $\{U, V\}$ of $\mathbb{C}$.

An operator $T \in L(X)$ is said to have *Dunford's property (C)* if $X_T(F)$ is closed for every closed $F \subseteq \mathbb{C}$. It is well known that Bishop's property (β) implies Dunford's property (C), and property (C) implies the single-valued extension property. Note that neither of the implications may be reversed in general, see more details [8] and [18].

If $A \subseteq \mathbb{C}$ then the *algebraic spectral subspace* $E_T(A)$ is the largest subspaces of X on which all the restrictions of $T - \lambda$, $\lambda \in \mathbb{C} \setminus A$, are surjective. Thus $E_T(A)$ is the largest subspace of X with this surjectivity property; this space need not be closed in general. An operator $T \in L(X)$ on a Banach space X is said to be *admissible* if, for each closed $F \subseteq \mathbb{C}$, the algebraic spectral subspace $E_T(F)$ is closed. This concept was introduced in [19], but the relevance of the condition was recognized earlier [12]. Laursen proved in [12] that if $E_T(F)$ is closed then $E_T(F) = X_T(F)$, and that an admissible operator cannot have non-trivial divisible subspaces, that is, $E_T(\phi) = \{0\}$. Moreover, T has the SVEP if and only if $X_T(\phi) = \{0\}$; for more information we refer to [8] and [15].

Given an operator $T \in L(X)$, the *quasi-nilpotent part* of T is the set

$$H_0(T) := \{x \in X \, : \, \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

The systematic investigation of these spaces was initiated by Mbekhta in [16] and [17] after an earlier work of Vrbová [24]. It is clear that $H_0(T)$ is a linear subspace of X and in fact hyperinvariant under T, generally, $H_0(T)$ is not closed. It follows from Theorem 1.5 [24] that T is quasi-nilpotent if and only if $H_0(T) = X$. Moreover, if T is invertible then $H_0(T) = \{0\}$. It is well known that

$$\operatorname{Ker} T^n \subseteq N^\infty(T) \subseteq H_0(T) \subseteq X_T(\{0\})$$

for all positive integer $n \in \mathbb{N}$, where $N^\infty(T) := \bigcap_{n=1}^{\infty} \operatorname{Ker} T^n$ is the hyperkernel of T. As shown by Schmoeger [21], the quasi-nilpotent part of an operator play a significant role in the local spectral and Fredholm theory of operators on Banach spaces.

2. Local spectral properties of generalized scalar operator

We denote by $C^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. An operator $T \in L(X)$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to L(X)$ satisfying $\Phi(1) = I$, the identity operator on X, and $\Phi(z) = T$ where z denotes the identity function on $\mathbb{C}$, and $C^\infty(\mathbb{C})$ denotes the Frécht algebra of all infinitely differentiable complex-valued function on $\mathbb{C}$. Such a continuous function Φ is in fact an operator valued distribution and it is called a spectral distribution for T.

Theorem 2.1. [9] *If $T \in L(X)$ is a generalized scalar operator, then*

$$E_T(F) = X_T(F) = \bigcap_{\lambda \in \mathbb{C}\setminus F} (T-\lambda)^p X,$$

for all sufficiently large integers p and for all closed $F \subseteq \mathbb{C}$.

In the special case of a normal operator on a Hilbert space p can be taken to be 1 by a theorem of Pták and Vrbová [20]. Since generalized scalar operators have SVEP, Curtis and Neumann shows that generalized scalar operators have no divisible subspace different from zero, see [9].

Given an operator $T \in L(X)$ on a Banach space X and an element $x \in X$,

$$r_T(x) := \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$$

is called the *local spectral radius* of T at x. It is well known that

$$\max\{|\lambda| : \lambda \in \sigma_T(x)\} \le r_T(x) \quad \text{for all} \quad x \in X,$$

but for operators without SVEP, this inequality may well strict, see [10]. It is well known that if T has SVEP and a non-zero element $x \in X$, then the compact set $\sigma_T(x)$ is non-empty and the local spectral radius formula

$$r_T(x) = \max\{|\lambda| \,:\, \lambda \in \sigma_T(x)\}$$

holds, and the spectral radius $r(T) = \max\{r_T(x) \,:\, x \in X\}$.

Lemma 2.2. [15] *If $T \in L(X)$ has the single-valued extension property, then*

$$X_T(\{0\}) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

LEMMA 2.3. [15] *Let $T \in L(X)$ be an operator with Bishop's property (β) on a Banach space X. Then for every $x \in X$*

$$r_T(x) = \lim_{n\to\infty} ||T^n x||^{\frac{1}{n}}.$$

For all $x \in X$ and for all non-negative integer k and n, we have

$$\|T^n(T^k x)\|^{\frac{1}{n}} = (\|T^{n+k}x\|^{\frac{1}{n+k}})^{\frac{n+k}{n}}.$$

It follows from Lemma 2.3 that $\lim_{n\to\infty} \|T^{n+k}x\|^{\frac{1}{n+m}} = r_T(x)$, and hence $r_T(x) = r_T(T^k x)$ for all $x \in X$.

THEOREM 2.4. *Let $T \in L(X)$ be a generalized scalar operator and $x_0 \in X$. If $r_T(x_0) = 0$, then there exists positive integer $p \in \mathbb{N}$ such that $T^p x_0 = 0$. Moreover, $Ker T^p$ is the quasi-nilpotent part of T, and*

$$H_0(T) = X_T(\{0\}) = E_T(\{0\}) = Ker\, T^p = \bigcap_{\lambda\neq 0} (T-\lambda)^p X,$$

Proof. If $T \in L(X)$ is generalized scalar then, by Theorem 2.1,

$$X_T(F) = E_T(F) = \bigcap_{\lambda\in\mathbb{C}\setminus F} (T-\lambda)^p X$$

holds for all closed set $F \subseteq \mathbb{C}$ and for all sufficiently large integers p. By Lemma 2.3 we have,

$$X_T(\{0\}) = H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\},$$

since T has the single-valued extension property. According to Theorem 1.2 of [25], a generalized scalar operator has no divisible linear subspace different from $\{0\}$. At first, we will show that

$$\mathrm{Ker}(T-\lambda)^p = X_T(\{\lambda\}) = E_T(\{\lambda\}) \quad \text{for all} \quad \lambda \in F.$$

For each $\lambda \in F$, $(T-\mu)E_T(\{\lambda\}) = E_T(\{\lambda\})$ for all $\mu \neq \lambda$, and hence

$$E_T(\{\lambda\}) = (T-\mu)^n E_T(\{\lambda\}) \subseteq (T-\mu)^n X$$

for all positive integer $n \in \mathbb{N}$. By Theorem 2.1, we obtain

$$\begin{aligned}(T-\lambda)^p E_T(\{\lambda\}) &\subseteq (T-\lambda)^p[\bigcap_{\mu\neq\lambda}(T-\mu)^p X] \\ &\subseteq \bigcap_{\mu\in\mathbb{C}}(T-\mu)^p X \\ &= X_T(\phi) \\ &= \{0\},\end{aligned}$$

since T has SVEP. It follows that $E_T(\{\lambda\}) \subseteq \operatorname{Ker}(T-\lambda)^p$ for all $\lambda \in F$. On the other hand, by Proposition 1.2.16 of [15], we have,

$$\operatorname{Ker}(T-\lambda)^k \subseteq X_T(\{\lambda\}) \subseteq E_T(\{\lambda\})$$

for all $\lambda \in F$ and $k \in \mathbb{N}$. This means that

$$\operatorname{Ker}(T-\lambda)^p = X_T(\{\lambda\}) = E_T(\{\lambda\}) \quad \text{for all} \quad \lambda \in F,$$

and therefore we have,

$$\operatorname{Ker} T^p = X_T(\{0\}) = E_T(\{0\}).$$

Finally, it follows from Lemma 2.3 and Theorem 2.1 that

$$X_T(\{0\}) = \bigcap_{\lambda \neq 0} (T-\lambda)^p X = \{x \in X : r_T(x) = 0\}.$$

This completes the proof. □

Recall that an operator $T \in L(X)$ on a complex Banach space X is said to be *algebraic* if $p(T) = 0$ for some non-zero complex polynomial p.

Theorem 2.5. *A generalized scalar operator with finite spectrum is algebraic. In particular, a quasi-nilpotent generalized scalar operator is nilpotent.*

Proof. Suppose that $\sigma(T)$ is a finite set, say $\sigma(T) = \{\lambda_1, \cdots, \lambda_n\}$. Then, by Theorem 2.1, for each closed $F \subseteq \mathbb{C}$

$$X_T(F) = E_T(F) = \bigcap_{\mu \in \mathbb{C} \setminus F} (T-\mu)^m X$$

for some positive integer $m \in \mathbb{N}$. Thus, for each $\lambda_k \in \sigma(T)$, there exists positive integer $m_k \in \mathbb{N}$ such that

$$E_T(\{\lambda_k\}) = X_T(\{\lambda_k\}) = \operatorname{Ker}(T-\lambda_k)^{m_k} \quad \text{for all} \;\; k = 1, 2, \cdots, n.$$

Thus we have

$$X_T(\sigma(T)) = E_T(\sigma(T)) = \bigcap_{\mu \in \mathbb{C} \setminus \sigma(T)} (T-\mu)^m X = X,$$

since $(T-\mu)X = X$ for all $\mu \in \rho(T) = \mathbb{C} \setminus \sigma(T)$ and every positive integer $m \in \mathbb{N}$. It follows from Theorem 1 of [22] that

$$\begin{aligned} X &= X_T(\sigma(T)) = X_T(\{\lambda_1\}) \oplus X_T(\{\lambda_2\}) \oplus \cdots \oplus X_T(\{\lambda_n\}) \\ &= \operatorname{Ker}(T-\lambda_1)^{m_1} \oplus \operatorname{Ker}(T-\lambda_2)^{m_2} \oplus \cdots \oplus \operatorname{Ker}(T-\lambda_n)^{m_n} \end{aligned}$$

holds as an algebraic direct sum. Consequently, if p denotes the complex polynomial given by

$$p(\lambda) := (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_n)^{m_n} \quad \text{for all } \lambda \in \mathbb{C},$$

then we conclude that $p(T) = 0$, and hence T is algebraic. □

A different proof of Theorem 2.5 is given Proposition 4.1 of [19].

In the following, let X and Y be complex Banach spaces over complex field $\mathbb{C}$ and let $L(X, Y)$ denote the space of all continuous linear operators from X to Y. For given operator $T \in L(X)$ and $S \in L(Y)$, we consider the corresponding commutator $C(S, T) : L(X, Y) \to L(X, Y)$ defined by

$$C(S, T)(A) := SA - AT \quad \text{for all } A \in L(X, Y).$$

Obviously, for all $n \in \mathbb{N}$ and $A \in L(X, Y)$ we have

$$\begin{aligned} C(S, T)^n(A) \colon &= C(S, T)^{n-1}(SA - AT) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A T^k. \end{aligned}$$

An operator $A \in L(X, Y)$ is said to *intertwine* S and T *asymptotically* if

$$\lim_{n \to \infty} \| C(S, T)^n(A) \|^{\frac{1}{n}} = 0.$$

This condition has been investigated by Colojoarvă and C. Foiás [8] and Vasilescu [23] in the context of decomposable operators.

Corollary 2.6. *Let $T \in L(X)$ and $S \in L(Y)$ be two generalized scalar operators. Then the following assertions are equivalent.*
(a) $A \in L(X, Y)$ intertwines S and T asymptotically.
(b) $A^* \in L(Y^*, X^*)$ intertwines T^* and S^* asymptotically.
(c) $AX_T(F) \subseteq Y_S(F)$ for every closed set $F \subseteq \mathbb{C}$.
(d) $A\mathcal{X}_T(F) \subseteq \mathcal{Y}_S(F)$ for every closed set $F \subseteq \mathbb{C}$.
(e) $\sigma_{C(S,T)}(A) = \{0\}$.
(f) $C(S, T)^p A = 0$ for some positive integer $p \in \mathbb{N}$.

Proof. It follows from Theorem 4.4.3 [8] that $C(S, T)$ is also generalized scalar. Since T and S are generalized scalar, T and S are decomposable. Hence T has property (δ) and S has property (C). Hence Theorem 2.4 [14] shows that $(a) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$. It is clear that if $C(S, T)^p(A) = 0$ for some $p \in \mathbb{N}$, then $A \in L(X, Y)$ intertwines S and

T asymptotically. If $A \in L(X,Y)$ intertwines S and T asymptotically, then, by Lemma 2.3,

$$r_{C(S,T)}(A) = \lim_{n\to\infty} \|C(S,T)^n(A)\|^{\frac{1}{n}} = 0.$$

It follows from Theorem 2.5 that $C(S,T)^p A = 0$ for some positive integer $p \in \mathbb{N}$. The final assertion follows immediately from

$$[C(S,T)^n(A)]^* = (-1)^n C(T^*,S^*)^n(A^*)$$

for all positive integer $n \in \mathbb{N}$. □

Corollary 2.7. *Every generalized scalar operator on a Banach space of dimension greater than 1 has a non-trivial closed invariant linear subspace.*

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