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YANG-MILLS INDUCED CONNECTIONS

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Abstract. Let G and H be compact connected Lie groups with biinvariant Riemannian metrics g and h respectively, ϕ a group isomorphism of G onto H, and $E := \phi^{-1}TH$ the induced bundle by ϕ over the base manifold G of the tangent bundle TH of H. Let ∇ and ${}^H\nabla$ be the Levi-Civita connections for the metrics g and h respectively, $\tilde{\nabla}$ the induced connection by the map ϕ and ${}^H\nabla$. Then, a necessary and sufficient condition for $\tilde{\nabla}$ in the bundle $(\phi^{-1}TH, G, \pi)$ to be a Yang- Mills connection is the fact that the Levi-Civita connection ∇ in the tangent bundle over (G, g) is a Yang- Mills connection. As an application, we get the following: Let ψ be an automorphism of a compact connected semisimple Lie group G with the canonical metric g (the metric which is induced by the Killing form of the Lie algebra of G), ∇ the Levi-Civita connection for g. Then, the induced connection $\tilde{\nabla}$, by ψ and ∇, is a Yang-Mills connection in the bundle $(\phi^{-1}TG, G, \pi)$ over the base manifold (G, g).

1. Introduction

Let (M, g), (N, h) be two Riemannian manifolds. Let $\phi : M \longrightarrow N$ be a smooth map. Let $E := \phi^{-1}TN$ be the induced bundle by ϕ over M of the tangent bundle TN of N (cf.[8]). We denote by $\Gamma(E)$, the space of all smooth sections V of E. We denote by ∇, ${}^N\nabla$ the Levi-Civita connections of (M, g), (N, h), respectively. Then we give the induced connection $\tilde{\nabla}$ on E by

$$(1.1)\quad (\tilde{\nabla}_X V)_x := \frac{d}{dt}\, {}^N P_{\phi(\gamma(t))}{}^{-1} V_{\gamma(t)}|_{t=0}, \quad X \in \Gamma(TM),\ V \in \Gamma(E),$$

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where $x \in M$, $\gamma(t)$ is a curve through x at $t = 0$ whose tangent vector at x is X_x, and ${}^N P_{\phi(\gamma(t))} : T_{\phi(x)}N \longrightarrow T_{\phi(\gamma(t))}N$ is the parallel displacement along a curve $\phi(\gamma(s))$, $0 \leq s \leq t$, given by the Levi-Civita connection ${}^N\nabla$ of (N, h) (cf. [8, p.126]). It is interesting to study conditions for the induced connection $\tilde{\nabla}$ in the bundle $(E := \phi^{-1}TN, M, \pi)$ over the base manifold (M, g) to be a Yang-Mills connection.

The problem of finding metrics and connections which are critical points of some functional plays an important role in global analysis and Riemannian geometry. A Yang-Mills connection is a critical point of the Yang-Mills functional

$$\mathcal{YM}(D) = \frac{1}{2}\int_M \|R^D\|^2 \, v_g \tag{1.2}$$

on the space $\mathfrak{C}_E$ of all connections in a smooth vector bundle E over a closed (compact and connected) Riemannian manifold (M, g), where R^D is the curvature of $D \in \mathfrak{C}_E$. Equivalently, D is a Yang-Mills connection if it satisfies the Yang-Mills equation (cf. [1, 5, 6])

$$\delta_D R^D = 0, \tag{1.3}$$

(the Euler-Lagrange equations of the variational principle associated with (1.2)). If D is a connection in a vector bundle E with bundle metric h over a Riemannian manifold (M, g), then the connection D^* given by

(1.4)
$$h({D^*}_X s, t) = X(h(s,t)) - h(s, D_X t), \quad (X \in \mathfrak{X}(M)) \text{ and } s, t \in \Gamma(E)$$

is referred to *conjugate* (cf. [1, 5]) to D. A connection D in E is a Yang-Mills connection if only if (cf. [1, 5, 6])

$$(\delta_D R^D)(X)s = -\sum_{i=1}^{n} (D^*_{e_i} R^D)(e_i, X)s = 0 \tag{1.5}$$

for arbitrary given $X \in \mathfrak{X}(M)$, where $s \in \Gamma(E)$ and $\{e_i\}_{i=1}^n$ is a local orthonormal frame on(M, g).

Let G and H be compact connected Lie groups with biinvariant Riemannian metrics g and h, $\phi : G \longrightarrow H$ a group isomorphism, $E := \phi^{-1}TN$ the induced bundle by ϕ over G of the tangent bundle TH of H. Let ∇ and ${}^H\nabla$ be the Levi-Civita connections of (G, g) and (H, h), respectively. Then we get the fact that the induced connection $\tilde{\nabla}$, by ϕ and ${}^H\nabla$, is a metric connection in the bundle $(\phi^{-1}TH, G, \pi)$ with bundle metric h. Moreover, we have the main result

Theorem 1.1. *Let G and H compact connected Lie groups, g and h biinvariant Riemannian metrics on G and H respectively. Let ϕ be a group isomorphism of G onto H, ∇ and ${}^H\nabla$ the Levi-Civita connections for the metric g and h respectively. Then, a necessary and sufficient condition for the induced connection $\tilde{\nabla}$ in the induced bundle $(\phi^{-1}TH, G, \pi)$ over the base manifold (G, g) to be a Yang-Mills connection is the fact that the Levi-Civita connection ∇ for g on (G, g) is a Yang-Mills connection.*

Now, let G be a compact connected semisimple Lie group, $\mathfrak{g}$ the Lie algebra of G, g the canonical metric (cf.[7, p.194]) (the biinvariant Riemannian metric which is induced from the Killing form of $\mathfrak{g}$), $\psi : G \to G$ a group automorphism. Let ∇ be the Levi-Civita connection for the metric g. Then, we get that the Levi-Civita connection ∇ in the tangent bundle TG over (G, g) is a Yang-Mills connection (cf. Proposition 3.4). By virtue of this fact and Theorem 3.2, we obtain

Theorem 1.2. *Let ψ be an automorphism of a compact connected semisimple Lie group G. Let $\tilde{\nabla}$ be the induced connection by ψ and ∇ the Levi-Civita connection for the canonical metric g on (G, g). Then, the induced connection $\tilde{\nabla}$ in the induced bundle $(\psi^{-1}TG, G, \pi)$ over the base manifold (G, g) is a Yang-Mills connection.*

2. Yang-Mills connections in vector bundles over a Riemannian manifold

Let E be a vector bundle, with bundle metric h, over an n-dimensional closed (compact and connected) Riemannian manifold (M, g). Let $D \in \mathfrak{C}_E$ and ∇ the Levi-Civita connection of (M, g). The pair (D, ∇) induces a connection in product bundles $\bigwedge^p TM^* \otimes E$, also denoted by D. Set $A^p(E) := \Gamma(\bigwedge^p TM^* \otimes E)$. We consider the differential operator

$$d_D : A^p(E) \longrightarrow A^{p+1}(E),$$

$$(d_D\varphi)(X_1, X_2, \cdots, X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}(D_{X_i}\varphi)(X_1, \cdots, \widehat{X}_i, \cdots, X_{p+1}),$$

$$\varphi \in A^p(E),\ X_i \in \mathfrak{X}(M)\ (i = 1, 2, \cdots, p+1),$$

which are defined by

$$d_D(\omega \otimes \xi) := d\omega \otimes \xi + (-1)^p \omega \wedge D\xi,$$
$$D_X(\omega \otimes \xi) := (\nabla_X \omega) \otimes \xi + \omega \otimes D_X \xi,$$

for $\omega \in \Gamma(\bigwedge^p TM^*)$, $\xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$.

Let δ_D be the formal adjoint of d_D with respect to the L^2-inner product

$$(\varphi, \psi) = \int_M < \varphi, \psi > v_g$$

for $\varphi, \psi \in A^p(E)$. Here $< \ , \ >$ is the bundle metric in $\bigwedge^p TM^* \otimes E$ induced by the pair (g, h) and v_g is the canonical volume form on (M, g). The following identity is elementary, yet crucial (cf. [1, 2])

$$\delta_D \varphi = (-1)^{p+1}(*^{-1} \cdot d_{D^*} \cdot *)(\varphi) = (-1)^{np+1}(* \cdot d_{D^*} \cdot *)(\varphi) \tag{2.1}$$

for any $\varphi \in A^{p+1}(E)$. Here, $* : A^q(E) \longrightarrow A^{n-q}(E), (0 \leq q \leq n)$, is the Hodge operator with respect to g. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on (M, g). Note that (2.1) may also be written as (cf. [1])

$$(\delta_D \varphi)(X_1, \cdots, X_p) = -\sum_{i=1}^n (D^*_{e_i} \varphi)(e_i, X_1, \cdots, X_p). \tag{2.2}$$

The connections D, $D^* \in \mathfrak{C}_E$ naturally induce connections, denoted by the same symbols, in $\mathrm{End}(E)$ $(:= E \otimes E^*)$. Then, a straightforward argument shows that D, $D^* \in \mathfrak{C}_{\mathrm{End}(E)}$ are conjugate connections. Thus, we find from (1.3) and (2.2) that a connection D in E is a Yang-Mills connection if and only if (cf. [1, 5, 6])

$$(\delta_D R^D)(X)s = -\sum_{i=1}^n (D^*_{e_i} R^D)(e_i, X)s = 0 \tag{2.3}$$

for arbitrary given $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$.

3. Yang-Mills induced connections

3.1. Let us denote by ∇, ${}^N\nabla$, the Levi-Civita connections on Riemannian manifolds (M, g), (N, h) respectively. Then for a C^∞-map ϕ of M into N, we can define the *induced connection* $\tilde{\nabla}$ in the induced bundle $E = \phi^{-1}TN = \cup_{x \in M} T_{\phi(x)}N$ over the base manifold (M, g) as follows:

For $X \in \mathfrak{X}(M)$, $V \in \Gamma(\phi^{-1}TN)$, define $\tilde{\nabla}_X V \in \Gamma(\phi^{-1}TN)$ by

$$(\tilde{\nabla}_X V)(x) = {}^N\nabla_{\phi_* X} V := \frac{d}{dt}|_{t=0} {}^N P^{-1}_{\phi \circ \sigma_t} V(\sigma(t)), \quad x \in M, \tag{3.1}$$

where $t \longmapsto \sigma(t) \in M$ is a smooth curve in M satisfying $\sigma(0) = x$, $\sigma'(0) = X_x \in T_x M$, and σ_t is a curve given by $\sigma_t(s) := \sigma(s)$, $0 \leq$

$s \leq t$, *i.e.*, the restriction of σ to the part between x and $\sigma(t)$. Here ${}^N P_{\phi\circ\sigma_t} : T_{\phi(x)}N \to T_{\phi(\sigma(t))}N$ is the parallel transport along the curve $\phi \circ \sigma_t$ with respect to the Levi-Civita connection ${}^N\nabla$ on (N, h), and ϕ_* is the differential map of ϕ. Then, since ${}^N\nabla$ is torsion free, the following Lemma ([8; Lemma 1.16, p.219]) is obtained.

LEMMA 3.1. *For any* C^∞*-map* $\phi : (M, g) \to (N, h)$ *and* $X, Y \in \mathfrak{X}(M)$, *we have*

$$\tilde{\nabla}_X(\phi_* Y) - \tilde{\nabla}_Y(\phi_* X) - \phi_*([X, Y]) = 0. \tag{3.2}$$

3.2. Let G be a compact connected Lie group, g a biinvariant Riemannian metric on G, and $\mathfrak{g}$ the Lie algebra of G. Here, $\mathfrak{g}$ is identified with the algebra of all left invariant vector fields on G. Then, the Levi-Civita connection ∇ for the metric g is given as follows (cf. [4, Theorem 13.1]):

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad (X, Y \in \mathfrak{g}). \tag{3.3}$$

3.3. Let G and H be compact connected Lie groups, g and h biinvariant Riemannian metrics on G and H respectively, ∇ and ${}^H\nabla$ the Levi-Civita connections for the metrics g and h on G and H respectively. Let $\{X_i\}_{i=1}^n$ (respectively $\{Y_\alpha\}_{\alpha=1}^n$) be an orthonormal basis of the Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{h}$) with respect to the metric g (resp. h). Let ϕ be an isomorphism of G onto H, $\phi^{-1}TH := \cup_{x\in G} T_{\phi(x)}H$ the induced bundle with fibre metric h over the base manifold (G, g), $\tilde{\nabla}$ the induced connection in the bundle $\phi^{-1}TH$ by ϕ and ${}^H\nabla$. Then, for $X \in \mathfrak{X}(M)$ and $s, t \in \Gamma(\phi^{-1}TH)$,

$$X(h(s,t)) = (\tilde{\nabla}_X h)(s,t) + h(\tilde{\nabla}_X s, t) + h(s, \tilde{\nabla}_X t) = h(\tilde{\nabla}_X s, t) + h(s, \tilde{\nabla}^*_X t),$$

and so $\tilde{\nabla} = \tilde{\nabla}^*$, *i.e.*, $\tilde{\nabla}$ is a metric connection. From this fact and (1.5), we have

$$\begin{aligned} (\delta_{\tilde{\nabla}} R^{\tilde{\nabla}})(X_j)\phi_* X_k &= -\sum\nolimits_{i=1}^n (\tilde{\nabla}_{X_i} R^{\tilde{\nabla}})(X_i, X_j)\phi_* X_k \\ &= -\sum\nolimits_{i=1}^n \{\tilde{\nabla}_{X_i}(R^{\tilde{\nabla}}(X_i, X_j)\phi_* X_k) - R^{\tilde{\nabla}}(\nabla_{X_i} X_i, X_j)\phi_* X_k \\ &\qquad - R^{\tilde{\nabla}}(X_i, \nabla_{X_i} X_j)\phi_* X_k - R^{\tilde{\nabla}}(X_i, X_j)\tilde{\nabla}_{X_i}(\phi_* X_k)\}. \end{aligned} \tag{3.4}$$

For the orthonormal bases $\{X_i\}_i$ and $\{Y_\alpha\}_\alpha$, we put

$$\phi_* X_k =: \sum\nolimits_\alpha {\phi_k}^\alpha Y_\alpha, \ [X_i, X_j] =: \sum\nolimits_k {D_{ij}}^k X_k, \ [Y_\alpha, Y_\beta] =: \sum\nolimits_\gamma {C_{\alpha\beta}}^\gamma Y_\gamma. \tag{3.5}$$

Each ${\phi_k}^\alpha$ appeared in (3.5) is constant. We get from this fact, (3.3) and (3.5)

$$\tilde{\nabla}_{X_i}(\phi_* X_k) = \frac{1}{2}\sum\nolimits_{\alpha,\beta,\gamma} {\phi_i}^\alpha {\phi_k}^\beta {C_{\alpha\beta}}^\gamma Y_\gamma. \tag{3.6}$$

By the help of (3.2), (3.3), (3.5) and (3.6), we obtain

$$\sum\nolimits_\gamma \Big(\sum\nolimits_{\alpha,\gamma} {\phi_i}^\alpha {\phi_j}^\beta {C_{\alpha\beta}}^\gamma - \sum_k {\phi_k}^\gamma {D_{ij}}^k\Big) Y_\gamma = 0. \tag{3.7}$$

From (3.3) and (3.7), we get

$$\tilde{\nabla}_{X_i}(\phi_* X_k) = \phi_*(\nabla_{X_i} X_k). \tag{3.8}$$

Since

$$R^{\tilde{\nabla}}(X_i, X_j)\phi_* X_k = \tilde{\nabla}_{X_i}\tilde{\nabla}_{X_j}(\phi_* X_k) - \tilde{\nabla}_{X_j}\tilde{\nabla}_{X_i}(\phi_* X_k) - \tilde{\nabla}_{[X_i,X_j]}(\phi_* X_k),$$

we have from (3.8)

$$R^{\tilde{\nabla}}(X_i, X_j)\phi_* X_k = \phi_*(R^{\nabla}(X_i, X_j) X_k). \tag{3.9}$$

By virtue of (3.4) and (3.9), we obtain

$$(\delta_{\tilde{\nabla}} R^{\tilde{\nabla}})(X_j)\phi_* X_k = \phi_*((\delta_\nabla R^\nabla)(X_j) X_k). \tag{3.10}$$

Thus, we get from (3.10)

THEOREM 3.2. *Let G and H compact connected Lie groups, g and h bi-invariant Riemannian metrics on G and H respectively. Let ϕ be a group isomorphism of G onto H, ∇ and ${}^H\nabla$ the Levi-Civita connections for the metric g and h respectively. Let $\tilde{\nabla}$ be the induced connection by ϕ and ${}^H\nabla$. Then, a necessary and sufficient condition for the connection $\tilde{\nabla}$ to be a Yang-Mills connection is the fact that ∇ is a Yang-Mills connection.*

Let G be an n-dimensional compact connected semisimple Lie group. Then, minus the Killing form of its Lie algebra $\mathfrak{g}$ (the set of all left invariant vector fields on G) is said to be the *canonical metric* on the Lie group G. Let g be the canonical metric on the Lie group G. Then, g is bi-invariant on G. Let $\{X_i\}_{i=1}^n$ be an orthonormal basis of the semisimple Lie algebra $\mathfrak{g}$ with respect to the canonical metric g. Let $\{\theta^j\}_{j=1}^n$ be the dual basis of the basis $\{X_i\}_{i=1}^n$. Then each θ^j is left invariant, that is, $L_x^*(\theta^j) = \theta^j$ $(x \in G)$. From (3.3), the Levi-Civita connection ∇ for the metric g is given by

$$\theta^l(\nabla_{X_i} X_j) = \frac{1}{2}\, {C_{ij}}^l. \tag{3.11}$$

where $C_{ij}{}^l := \theta^l([X_i, X_j])$ for the orthonormal frame $\{X_i\}_{i=1}^n$. By virtue of (3.3) and properties of the Killing form on the semisimple Lie algebra $\mathfrak{g}$, we have for $X, Y, Z \in \mathfrak{g}$ (cf. [2, 3, 7])

$$(3.12) \quad g([X,Y],Z) + g(Y,[X,Z]) = 0, \quad R^{\nabla}(X,Y) = -\frac{1}{4}\, ad([X,Y]),$$

where ad is the adjoint representation of the semisimple Lie algebra $\mathfrak{g}$. From (3.12) and the definition of the Killing form B of the semisimple Lie algebra $\mathfrak{g}$ such that $B(X,Y) := \text{Trace}\,(ad(X)\ ad(Y)) \quad (X, Y \in \mathfrak{g})$, we get for $Y, Z \in \mathfrak{g}$ (cf. [2, 3, 7])

$$(3.13) \qquad \sum_{i=1}^{n} g(R^{\nabla}(X_i, Y)Z, X_i) = \frac{1}{4}\, g(Y,Z),$$

that is, the Riemannian manifold (G, g) is an Einstein manifold of Ricci curvature $\frac{1}{4}$. From the fact $g(\nabla_{X_i}X_j, X_l) + g(X_j, \nabla_{X_i}X_l) = 0$, (3.3) and (3.11), we have

$$(3.14) \qquad C_{ij}{}^k = -C_{ik}{}^j = -C_{kj}{}^i.$$

By virtue of (3.12), (3.13) and (3.14), we get

$$(3.15) \qquad \sum_{i,l=1}^{n} C_{il}{}^k C_{il}{}^j = \delta_{kj}.$$

Putting $R^{\nabla}(X_i, X_j)X_k =: \sum_t R_{ijk}{}^t X_t$, we have from (3.11)

$$(3.16) \qquad R_{ijk}{}^t = \frac{1}{4}\sum_s (C_{jk}{}^s C_{is}{}^t - C_{ik}{}^s C_{js}{}^t - 2C_{ij}{}^s C_{sk}{}^t).$$

From (2.3), we get

$$(3.17)$$
$$(\delta_{\nabla}R^{\nabla})(X_j)X_k = -\sum\{\nabla_{X_i}(R^{\nabla}(X_i,X_j)X_k) - R^{\nabla}(\nabla_{X_i}X_i, X_j)X_k - R^{\nabla}(X_i, \nabla_{X_i}X_j)X_k - R^{\nabla}(X_i,X_j)\nabla_{X_i}X_k\}.$$

By the helf of (3.11),(3.14)-(3.17), we have

$$(3.18) \qquad (\delta_{\nabla}R^{\nabla})(X_j)X_k = -\frac{1}{2}\sum_l (C_{jk}{}^l - 2\sum_{i,t} C_{ij}{}^t C_{tk}{}^s C_{si}{}^l)X_l.$$

On the other hand, we get

LEMMA 3.3.

$$2\sum_{i,s,t=1}^{n} C_{ij}{}^{t} C_{tk}{}^{s} C_{si}{}^{l} = C_{jk}{}^{l}.$$

Proof. By virtue of (3.12), (3.14) and (3.15),

$$\begin{aligned}
&\sum_{i,s,t} C_{ij}{}^{t} C_{tk}{}^{s} C_{si}{}^{l} \\
&\qquad = \sum_{i,s,t} g([[[X_i, X_j], X_k], X_i], X_l) \\
&\qquad = \sum_{i,s,t} g([[X_i, X_j], X_k], [X_i, X_l]) \\
&\qquad = -\sum_{i,s,t} g([[X_j, X_k], X_i] + [[X_k, X_i], X_j], [X_i, X_l]) \\
&\qquad = -\sum_{i,s,t} (C_{jk}{}^{t} C_{ti}{}^{s} C_{il}{}^{s} + C_{ki}{}^{t} C_{tj}{}^{s} C_{il}{}^{s}) \\
&\qquad = C_{jk}{}^{l} - \sum_{i,s,t} C_{tj} i C_{ks}{}^{t} C_{sl}{}^{i} = C_{jk}{}^{l} - \sum_{i,s,t} C_{ij}{}^{t} C_{tk}{}^{s} C_{si}{}^{l}.
\end{aligned}$$

Thus, the proof of this Lemma is completed. □

By virtue of (3.18) and Lemma 3.3, we obtain

PROPOSITION 3.4. *Let G be a compact connected semisimple Lie group, g the canonical metric on G, ∇ the Levi-Civita connection for the metric g. Then, ∇ is a Yang-Mills connection in the tangent bundle over the base manifold (G, g).*

By the help of Theorem 3.2 and Proposition 3.4, we get

THEOREM 3.5. *Let G be a compact connected semisimple Lie group, g the canonical metric on G, ∇ the Levi-Civita connection for g. Let ϕ be an automorphism of G, $\phi^{-1}TG := \cup_{x\in G} T_{\phi(x)}G$ the induced bundle, $\tilde{\nabla}$ the induced connection by ϕ and ∇. Then, $\tilde{\nabla}$ is a Yang-Mills connection in the bundle $\phi^{-1}TG$ over the base manifold (G, g).*

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