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ON THE QUALITATIVE BEHAVIOR OF DISCRETE VOLTERRA EQUATIONS

Sung Kyu Choi*, Yoon Hoe Goo**, Dong Man Im***, and Namjip Koo****

Abstract. We investigate the boundedness and asymptotic almost periodicity of solutions for discrete Volterra equations.

1. Introduction

Consider the nonlinear Volterra integral equation

$$x(t) = f(t) - \int_0^t A(t-s)G(s, x(s))ds \tag{1.1}$$

where $A(t)$ is the appropriate 2×2 matrix and $x, f(t)$ and $G(t, x)$ are the appropriate two-dimensional column vectors. More generally, we also consider the system of equations

$$\begin{aligned} x_1(t) &= f_1(t) - \int_0^t a_1(t-s)g_1(s, x_1(s))ds \\ &\quad - \int_0^t a_2(t-s)g_2(s, x_2(s))ds, \\ x_2(t) &= f_2(t) - \int_0^t a_2(t-s)g_1(s, x_1(s))ds \\ &\quad - \int_0^t a_1(t-s)g_2(s, x_2(s))ds. \end{aligned} \tag{1.2}$$

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Correspondence should be addressed to Namjip Koo, njkoo@cnu.ac.kr.
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Equations of the form (1.1) arise from the following diffusion problem:

$$\begin{aligned} u_t &= u_{yy},\ y>0,\ t>0, \\ \lim_{y\to\infty} u(t,y) &= 0,\ t>0, \\ u_y(t,0) &= g(t,u(t,0)),\ t>0. \end{aligned} \tag{1.3}$$

System (1.2) arises from a similar diffusion problem on a finite interval:

$$\begin{aligned} u_t &= u_{yy},\ 0<y<L,\ t>0, \\ u_y(t,0) &= g_1(t,u(t,0)),\ t>0, \\ u_y(t,L) &= -g_2(t,u(t,L)),\ t>0, \end{aligned} \tag{1.4}$$

where $L>0$.

The study of (1.3) and (1.4) was inspired by a theory of superfluidity of liquid helium(see [7] and [8]).

Burton and Furumochi [1] studied the existence of periodic solutions of

$$x(t) = a(t) - \int_0^t D(t,s,x(s))ds \tag{1.5}$$

and

$$x(t) = p(t) - \int_{-\infty}^t P(t,s,x(s))ds \tag{1.6}$$

by using techniques on limiting equations, Liapunov functions, the theory of minimal solutions, and contraction mappings. Also, they studied the existence of almost periodic and asymptotically almost periodic solutions of the integral equation (1.5) and (1.6) in [2]. Furthermore, corresponding to Volterra equations (1.5) and (1.6), Furumochi [5] investigated the existence of periodic solutions of the difference equations

$$x(n+1) = a(n) - \sum_{k=0}^{n} D(n,k,x(k)) \tag{1.7}$$

and

$$x(n+1) = p(n) - \sum_{k=-\infty}^{n} P(n,k,x(k)) \tag{1.8}$$

which is the discrete analogue of equations (1.5) and (1.6), respectively.

In this paper we investigate some qualitative properties in (1.7) and (1.8), that is, the boundedness and asymptotic almost periodicity.

For the asymptotic property of Volterra difference systems, see [3] and [4].

2. Main results

We denote by $\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-$, respectively, the set of integers, the set of nonnegative integers, and the set of nonpositive integers. Let $\mathbb{R}^d$ denote d-dimensional Euclidean space with the Euclidean norm $|\cdot|$.

Definition 2.1. A continuous function $f : \mathbb{Z}\times\mathbb{R}^d \to \mathbb{R}^d$ is said to be *almost periodic in* $n \in \mathbb{Z}$ *uniformly for* $x \in \mathbb{R}^d$ if for every $\epsilon > 0$ and every compact set $K \subset \mathbb{R}^d$, there corresponds an integer $N = N(\epsilon, K) > 0$ such that among N consecutive integers there is one, here denoted p, such that

$$|f(n + p, x) - f(n, x)| < \epsilon$$

for all $n \in \mathbb{Z}$ uniformly for $x \in \mathbb{R}^n$.

Definition 2.2. A function $\phi : \mathbb{Z} \to \mathbb{R}^d$ is called *asymptotically almost periodic* if it is a sum of an almost periodic function ϕ_1, and a function ϕ_2 defined on $\mathbb{Z}$ which tends to zero as $n \to \infty$, that is, $\phi(n) = \phi_1(n) + \phi_2(n)$, $n \in \mathbb{Z}$.

Note that the decomposition $\phi = \phi_1 + \phi_2$ in Definition 2.2 is unique [9].

Lemma 2.3. [9] *A function* $\phi : \mathbb{Z} \to \mathbb{R}^d$ *is asymptotically almost periodic if and if for any integer sequence* $(\tau_k^{'})$ *with* $\tau_k^{'} \to \infty$ *as* $k \to \infty$, *there exists a subsequence* $(\tau_k) \subset (\tau_k^{'})$ *for which* $\phi(n + \tau_k)$ *converges uniformly for* $n \in \mathbb{Z}$ *as* $k \to \infty$.

We consider the systems of Volterra difference equations

$$x(n + 1) = a(n) - \sum_{k=0}^{n} D(n, k, x(k)),\ n \in \mathbb{Z}^+ \tag{2.1}$$

and

$$x(n + 1) = p(n) - \sum_{k=-\infty}^{n} P(n, k, x(k)),\ n \in \mathbb{Z}, \tag{2.2}$$

where

$$\begin{aligned} &a : \mathbb{Z}^+ \to \mathbb{R}^d,\\ &p : \mathbb{Z} \to \mathbb{R}^d,\\ &D : \Delta^+ \times \mathbb{R}^d \to \mathbb{R}^d,\\ &P : \Delta \times \mathbb{R}^d \to \mathbb{R}^d, \end{aligned}$$

and where

$$\Delta^+ = \{(n,k) : 0 \le k \le n,\ n,k \in \mathbb{Z}^+\},$$
$$\Delta = \{(n,k) : k \le n,\ n,k \in \mathbb{Z}\},$$

and $D(n,k,x), P(n,k,x)$ are continuous in x for any fixed (n,k). Moreover, we assume that

$$q(n) = a(n) - p(n) \to 0 \ as\ n \to \infty \quad \text{and } p(n) \text{ is almost periodic,} \tag{2.3}$$

$$Q(n,k,x) = D(n,k,x) - P(n,k,x), \tag{2.4}$$

$P(n,k,x)$ is almost periodic in $n \in \mathbb{Z}$ uniformly for $(k,x) \in \mathbb{Z} \times \mathbb{R}^d$, and for any $J > 0$ there are functions

$$P_J : \Delta \to \mathbb{R}^+ = [0, \infty)$$

and

$$Q_J : \Delta^+ \to \mathbb{R}^+$$

such that

$P_J(n,k)$ is almost periodic in n,
$|P(n,k,x)| \le P_J(n,k)$ if $(n,k,x) \in \Delta \times X_J,\ X_J = \{x \in \mathbb{R}^d : |X| \le J\}$,
$|Q(n,k,x)| \le Q_J(n,k)$ if $(n,k,x) \in \Delta^+ \times X_J$,

$$\sum_{k=-\infty}^{n} P_J(n+\nu,k) \to 0 \text{ uniformly for } n \in \mathbb{Z} \text{ as } \nu \to \infty \tag{2.5}$$

and

$$\sum_{k=0}^{n} Q_J(n,k) \to 0 \ as\ n \to \infty. \tag{2.6}$$

In view of almost periodicity of p and P, for any sequence $(n_k^{'}) \in \mathbb{Z}$ with $n_k^{'} \to \infty$ as $k \to \infty$, there exists a subsequence $(n_k) \subset (n_k^{'})$ such that

$$p(n+n_k) \to e(n) \tag{2.7}$$

and

$$P(n+n_k,k,x) \to E(n,k,x) \tag{2.8}$$

uniformly on $\mathbb{Z} \times K$ for any compact set $K \subset \mathbb{R}^d$, $e(n)$ and $E(n,k,x)$ are also almost periodic in n and almost periodic in n uniformly for $(k,x) \in \mathbb{Z} \times \mathbb{R}^d$, respectively. Thus we can define the *hull*:

Definition 2.4. $H(p,P) = \{(e,E)$: (2.7) and (2.8) hold for some sequence

$$(n_k) \subset \mathbb{Z} \text{ with } n_k \to \infty \text{ as } k \to \infty\}. \tag{2.9}$$

Definition 2.5. If $(e,E) \in H(p,P)$, then the equation

$$x(n+1) = e(n) - \sum_{k=-\infty}^{n} E(n,k,x(k)),\ n \in \mathbb{Z} \tag{2.10}$$

is called the *limiting equation* of (2.2).

First, we obtain the existence of bounded solutions of Eq. (2.1).

Theorem 2.6. *For Eq. (2.1), we assume that for any* $J > 0$,

$$|D(n,k,x)| \le D_J(n,k),\ n,k \in \mathbb{Z}^+,\ x \in X_J, \tag{2.11}$$

where $D_J : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^+$,

$$\sum_{k=0}^{n} D_J(n,k) \text{ is bounded on } \mathbb{Z}, \tag{2.12}$$

$$|D(n,k,x) - D(n,k,y)| \le L_J(n,k)|x-y|, n,k \in \mathbb{Z}^+, x,y \in X_J, \tag{2.13}$$

where $L_J : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^+$. *Moreover, we suppose that*

$$\lambda_J = \sup_{n \in \mathbb{Z}^+} \sum_{k=0}^{\infty} L_J(n,k),\ \lambda = \sup_{J>0} \lambda_J < 1. \tag{2.14}$$

Then (2.1) has a unique $\mathbb{Z}^+$*-bounded solution.*

Proof. Let B be the set of bounded sequences $\xi : \mathbb{Z}^+ \to \mathbb{R}^d$ with the norm $\| \xi \| = \sup_{n \ge 0} | \xi(n) |$. Then B is a Banach space. Define T on B by

$$(T\xi)(n) = a(n) - \sum_{k=0}^{n} D(n,k,\xi(k)).$$

Then, by (2.11) and (2.12), T maps B into B. We claim that T is a contraction. For any $\xi_1, \xi_2 \in B$, we have

$$|(T\xi_1)(n) - (T\xi_2)(n)| \le \sum_{k=0}^{n} L_J(n,k) \,|\, \xi_1(k) - \xi_2(k) \,|\le \lambda_J \,\|\, \xi_1 - \xi_2 \,\|$$

by (2.11), (2.12), and (2.14). Hence, by (2.14), T is a contraction. Therefore there exists a unique fixed point of (2.1) and it is a $\mathbb{Z}^+$-bounded solution of (2.1). □

Now, we obtain an asymptotic behavior of a $\mathbb{Z}^+$-bounded solution of Eq. (2.1) as follows.

THEOREM 2.7. *Under the assumptions (2.3) $\sim$ (2.6), we suppose that Eq. (2.1) has a $\mathbb{Z}^+$-bounded solution $x(n)$. For any sequence $(n_k) \subset \mathbb{Z}^+$ with $n_k \to \infty$ $k \to \infty$, let, for any $n \in \mathbb{Z}$,*

$$x_k(n) = \begin{cases} x_0 & \text{if } n < -n_k, \\ x(n+n_k) & \text{if } n \geq -n_k. \end{cases}$$

Then $x_k(n)$ converges to a $\mathbb{Z}$-bounded solution $y(n)$ of Eq. (2.10).

Proof. Let $x(n)$ denote again the $\mathbb{Z}$-extension of the solution $x(n)$ obtained by defining $x(n) = x_0$ for $n < 0$. Then the sequence $(x_k(n))$ is obtained by an (n_k)-translation to the left of $(x(n))$. It is clear that $(x_k(n))$ is uniformly bounded on $\mathbb{Z}$. Thus we way assume that $(x_k(n))$ converges to a bounded sequence $(y(n))$ by taking a subsequence if necessary.

Now, we show that $y(n)$ satisfies (2.10) on $\mathbb{Z}$, i.e.,

$$y(n) = e(n) - \sum_{k=-\infty}^{n} E(n,k,y(k)), n \in \mathbb{Z} \tag{2.15}$$

From (2.1), (2.3) and (2.4), we have

$$\begin{aligned} x(n+n_k) = x_k(n) &= a(n+n_k) - \sum_{k=0}^{n+n_k} D(n+n_k,k,x(k)) \\ &= p(n+n_k) + q(n+n_k) - \sum_{k=-n_k}^{n} P(n+n_k,k+n_k,x_k(n)) \\ &- \sum_{k=0}^{n+n_k} Q(n+n_k,k,x(k)). \end{aligned} \tag{2.16}$$

Let $J > 0$ be a number with $|x| = \sup_{n \in Z} |x(n)| \leq J$. From (2.5), for any $\epsilon > 0$, there exists an $N > 0$ such that

$$\sum_{k=-\infty}^{n} P_J(n+N,k) < \epsilon, \;\; n \in \mathbb{Z}. \tag{2.17}$$

Note that

$$P(n+n_k) \to e(n) \;\; \text{as } k \to \infty. \tag{2.18}$$

By (2.3), we have

$$q(n+n_k) \to 0 \text{ as } k \to \infty \tag{2.19}$$

for any $n \in \mathbb{Z}$. In view of (2.6), we obtain

$$\begin{aligned}&\limsup_{k\to\infty} \Big|\sum_{k=0}^{n+n_k} Q(n+n_k,k,x(k))\Big| \\ &\le \limsup_{k\to\infty} \sum_{k=0}^{n+n_k} Q_J(n+n_k,k) = 0. \end{aligned} \tag{2.20}$$

Finally, we have

$$\begin{aligned}&\limsup_{k\to\infty} \Big|\sum_{k=-n_k}^{n} P(n+n_k,k+n_k,x_k(n)) - \sum_{k=-\infty}^{n} E(n,k,y(k))\Big| \\ &\le \limsup_{k\to\infty} \sum_{k=n-N}^{n} |P(n+n_k,k+n_k,x_k(n)) - E(n,k,y(k))| \\ &+ \limsup_{k\to\infty} \sum_{k=-\infty}^{n-N} P_J(n+n_k,k+n_k) + \sum_{k=-\infty}^{n-N} |E(n,k,y(k))| \\ &\le \sum_{k=-\infty}^{n-N} |E(n,k,y(k))| + \epsilon \\ &\le \sup \sum_{k=-\infty}^{n-N} P_J(n,k) + \epsilon \le 2\epsilon \end{aligned}$$

by (2.17) and (2.5). Hence, letting $k \to \infty$ in (2.12), we obtain the result (2.15). □

THEOREM 2.8. *Suppose that (2.3) $\sim$ (2.6). Let $\xi = \psi + \mu$ be any asymptotically almost periodic function on $\mathbb{Z}^+$ with $|\xi| \le J$ for some $J > 0$. Then*

$$d(n) = \sum_{k=0}^{n} D(n,k,\xi(k)),\ n \in \mathbb{Z}^k$$

is asymptotically almost periodic function and its almost periodic part is

$$\pi(n) = \sum_{k=-\infty}^{n} P(n,k,\psi(k)),\ n \in \mathbb{Z}.$$

Proof. To prove that $\pi(n)$ is almost periodic we show that for any sequence $(n_k') \subset \mathbb{Z}$ there exists a subsequence $(n_k) \subset (n_k')$ such that

$(\pi(n+n_k))$ converges for $n \in \mathbb{Z}$. Let $(n_k^{'})$ be any sequence in $\mathbb{Z}$. Then, by taking a subsequence (n_k) of $(n_k^{'})$ if necessary, we have

$$P(n+n_k, k+n_k, x) \to E(n,k,x) \text{ as } k \to \infty \tag{2.21}$$

uniformly on $\Delta \times X_J$ and

$$\psi(n+n_k) \to \eta(n) \text{ as } k \to \infty \tag{2.22}$$

for $n \in \mathbb{Z}$. We claim that

$$\sum_{k=-\infty}^{n+n_k} P(n+n_k, k, \psi(k)) \to \sum_{k=-\infty}^{n} E(n,k,\eta(k)) \text{ as } k \to \infty \tag{2.23}$$

uniformly on $\mathbb{Z}$. From (2.5), for any $\varepsilon > 0$ there exists an $N_1 > 0$ such that

$$\sum_{k=-\infty}^{n-N_1} P_J(n,k) < \epsilon,\ n \in \mathbb{Z}. \tag{2.24}$$

In view of (2.21), there exists an $M_1 > 0$ such that for all $k \geq M_1$,

$$|P(n+n_k, k+n_k, k) - E(n,k,x)| < \frac{\epsilon}{N_1} \tag{2.25}$$

on $\Delta \times X_J$. Since $E(n,k,x)$ is continuous in $x \in X_J$, there exists a $\delta > 0$ such that

$$|E(n,k,x) - E(n,k,y)| < \frac{\epsilon}{N_1}, \tag{2.26}$$

whenever $|x-y| < \delta$ on X_J. Also, from (2.22), there exists an $M_2 > 0$ such that for all $k \geq M_2$,

$$|\psi(n+n_k) - \eta(n)| < \delta \text{ on } \mathbb{Z}. \tag{2.27}$$

Let $k \geq \max\{M_1, M_2\}$. Then, for any $n \in \mathbb{Z}$, we obtain

$$\begin{aligned}
&|\sum_{k=-\infty}^{n+n_k} P(n+n_k, k, \psi(k)) - \sum_{k=-\infty}^{n} E(n,k,\eta(k))| \\
&\leq |\sum_{k=n-N_1}^{n} |P(n+n_k, k+n_k, \psi(k+n_k)) - E(n,k,\eta(k))| \\
&+ \sum_{k=-\infty}^{n+n_k-N_1} |P(n+n_k, k, \psi(k))| + \sum_{k=-\infty}^{n-N_1} |E(n,k,\eta(k))|
\end{aligned}$$

$$
\begin{aligned}
&< \sum_{k=n-N_1}^{n} |P(n+n_k, k+n_k, \psi(k+n_k)) - E(n,k,\psi(k+n_k))| \\
&+ \sum_{k=n-N_1}^{n} |E(n,k,\psi(k+n_k)) - E(n,k,\eta(k))| + \epsilon \\
&+ \sum_{k=-\infty}^{n-N_1} |E(n,k,\eta(k))| < \sum_{k=-\infty}^{n-N_1} |E(n,k,\eta(k))| + 3\epsilon \\
&\le 4\epsilon
\end{aligned}
$$

by (2.24), (2.25) and (2.26). Thus (2.23) is satisfied. This implies that $\pi(n)$ is almost periodic.

Now, we prove that $d(n) - \pi(n) \to 0$ as $n \to \infty$. For any $\epsilon > 0$, let $N_1 > 0$ be an integer as in (2.24). Note that there exists a $\delta > 0$ such that $(n,k,x), (n,k,y) \in \Delta \times X_J$ and $|x-y| < \delta$ imply

$$|P(n,k,x) - P(n,k,y)| < \frac{\epsilon}{N_1},$$

since $P(n,k,x)$ is continuous in x. Also, there exists an $N_2 > 0$ such that for all $n \ge N_2, |\xi(n) - \psi(n)| < \delta$. For any $n \ge N_1 + N_2$, we have

$$
\begin{aligned}
&\sum_{k=0}^{n} |P(n,k,\xi(k)) - P(n,k,\psi(k))| \\
&\le 2 \sum_{k=0}^{n-N_1} P_J(n,k) + \sum_{k=n-N_1}^{n} |P(n,k,\xi(k) - P(n,k,\psi(k))| \\
&< 3\epsilon. \qquad (2.28)
\end{aligned}
$$

Also, we obtain from (2.24)

$$| \sum_{k=-\infty}^{0} P(n,k,\psi(k))| \le \sum_{k=-\infty}^{0} P_J(n,k) < \epsilon. \qquad (2.29)$$

Furthermore,

$$
\begin{aligned}
|\sum_{k=0}^{n} Q(n,k,\xi(k))| \quad &\le \quad \sum_{k=0}^{n} Q_J(n,k), \ n \in \mathbb{Z}^+ \\
&\to \quad 0 \text{ as } n \to \infty, \qquad (2.30)
\end{aligned}
$$

by (2.6). Hence, by (2.28), (2.29) and (2.30), $d(n) - \pi(n) \to 0$ as $n \to \infty$. It follows that $d(n)$ is asymptotically almost periodic. This completes the proof. $\square$

References

[1] T. A. Burton and T. Furumochi, *Periodic solutions of Volterra equations and attractivity*, Dyn. Sys. Appl. **3** (1994), 583-598.
[2] T. A. Burton and T. Furumochi, *Almost periodic solutions of Volterra equations and attractivity*, J. Math. Anal. Appl. **198** (1996), 581-599.
[3] S. K. Choi, N.J. Koo and Y. H. Goo, *Asymptotic property of nonlinear Volterra difference systems*, Nonlinear Anal. **51** (2002), 321-337.
[4] S. K. Choi and N. J. Koo, *Asymptotic property of linear Volterra difference systems*, J. Math. Appl. **321** (2006), 206-272.
[5] T. Furumochi, *Periodic solutions of Volterra difference equations and attractivity*, Nonlinear Anal. **47** (2001), 4013-4024.
[6] T. Furumochi, T. Tsuchie and H. Wada, *Existence theorems and periodic solutions of neutral difference equations*, CUBO **8** (2006), 35-46.
[7] G. Gripenberg, *On the convergence of solutions of Volterra equations to almost periodic functions*, Quart. Appl. Math. **39** (1981), 363-373.
[8] R. K. Miller, *Almost periodic behavior of solutions of a nonlinear Volterra system*, Quart. Appl. Math. **28** (1971), 553-570.
[9] C. Zhang, *Almost Periodic Type Fuctions and Ergodicity*, Kluwer Academic Publishers, Dordrecht, 2003.

*
Department of Mathematics
Chungnam National University
Daejon 305-764, Republic of Korea
E-mail: sgchoi@cnu.ac.kr

**
Department of Mathematics
Hanseo University
Seosan, Chungnam 352-820, Republic of Korea
E-mail: yhgoo@hanseo.ac.kr

***
Department of Mathematics Education
Cheongju University
Cheongju 360-764, Republic of Korea
E-mail: dmim@cju.ac.kr

****
Department of Mathematics
Chungnam National University
Daejon 305-764, Republic of Korea
E-mail: njkoo@cnu.ac.kr