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ON HILBERT 2-CLASS FIELD TOWERS OF REAL QUADRATIC FUNCTION FIELDS

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Abstract. In this paper we prove that real quadratic function field F over $\mathbb{F}_q(T)$ has infinite 2-class field tower if the 4-rank of narrow ideal class group of F is equal to or greater than 4 when $q \equiv 3 \bmod 4$.

1. Introduction and statement of main result

Let $\mathrm{k} = \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ of q elements and $\mathbb{A} = \mathbb{F}_q[T]$. Let ∞ be the prime of k associated to $(1/T)$. For a finite separable extension F of k, let $\mathcal{O}_F$ be the integral closure of $\mathbb{A}$ in F and H_F be the Hilbert class field of F with respect to $\mathcal{O}_F$ ([5]). Let ℓ be a prime number. Let $F_1^{(\ell)}$ be the Hilbert ℓ-class field of $F_0^{(\ell)} = F$ (i.e., $F_1^{(\ell)}$ is the maximal ℓ-extension of F inside H_F) and inductively, $F_{n+1}^{(\ell)}$ be the Hilbert ℓ-class field of $F_n^{(\ell)}$ for $n \geq 1$. Then we obtain a sequence of fields $F_0^{(\ell)} = F \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots$, which is called the *Hilbert ℓ-class field tower of F*. We say that the Hilbert ℓ-class field tower of F is infinite if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$. For any multiplicative abelian group A, write $r_\ell(A) = \dim_{\mathbb{F}_\ell}(A/A^\ell)$, which is called the the ℓ-rank of A. In [6], Schoof has proved that the Hilbert ℓ-class field tower of F is infinite if

$$r_\ell(\mathcal{C}(F)) \geq 2 + 2\sqrt{r_\ell(\mathcal{O}_F^*) + 1}, \tag{1.1}$$

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where $\mathcal{C}(F)$ and $\mathcal{O}_F^*$ are the ideal class group and the group of units of $\mathcal{O}_F$, respectively. This is a function field analog of the theorem of Golod-Shafarevich.

Assume that q is odd. Throughout the paper, by a *real quadratic function field*, we always mean a quadratic extension F of k in which ∞ splits. Any real quadratic function field F can be written uniquely as $F = \mathrm{k}(\sqrt{D})$, where $D \in \mathbb{A}$ is a nonconstant square-free monic polynomial of even degree. In this paper, we study the infiniteness of Hilbert 2-class field tower of such a real quadratic function field F. Since $\mathcal{O}_F^* \cong \mathbb{F}_q^* \times \mathbb{Z}$, $r_2(\mathcal{O}_F^*) = 2$, so the Hilbert 2-class field tower of F is infinite if $r_2(\mathcal{C}(F)) \geq 6$ by Schoof's theorem. Let $\mathcal{C}^+(F)$ be the narrow ideal class group of $\mathcal{O}_F$ (cf. §2.1). Write $r_4(\mathcal{C}^+(F)) = r_2(\mathcal{C}^+(F)^2)$, which is called the 4-rank of $\mathcal{C}^+(F)$. The main result of this paper is the following theorem.

THEOREM 1.1. *Assume that $q \equiv 3 \bmod 4$. Let F be a real quadratic function field over* k. *If $r_4(\mathcal{C}^+(F)) \geq 4$, then F has infinite Hilbert 2-class field tower.*

In classical case, Lemmermeyer [3] has proved a similar result for the real quadratic number field F. Our method is elementary since we only use the Rédei matrix M_F^+ associated to F and this method also works for real quadratic number field case.

2. Preliminaries

2.1. Narrow ideal class group $\mathcal{C}^+(F)$

Let $\mathrm{k}_\infty = \mathbb{F}_q((1/T))$ be the completion of k at ∞. Let $sgn : \mathrm{k}_\infty^* \to \mathbb{F}_q^*$ be the sign function satisfying $sgn(1/T) = 1$ and define $s(x) = sgn(x)^{\frac{q-1}{2}}$ for any $x \in \mathrm{k}_\infty^*$. Let F be a real quadratic function field over k. Let ∞_1 and ∞_2 be primes of F lying above ∞. Define a homomorphism

$$\mathbf{s} : F^* \to \{\pm 1\} \times \{\pm 1\},\ x \mapsto (s_1(x), s_2(x)),$$

where $s_i(x) = s(\eta_i(x))$ and η_i is the embedding of F into k_∞ associated to ∞_i for $i = 1, 2$. An element $x \in F^*$ is said to be *positive* if $\mathbf{s}(x) = (1, 1)$. Put $F^+ = \mathrm{Ker}(\mathbf{s})$, which is the subgroup of F^* consisting of all positive elements of F^*. Let $I(F)$ be the group of fractional ideals of $\mathcal{O}_F$ and $P^+(F)$ be the subgroup of $I(F)$ consisting of principal ideals generated by an element of F^+. The narrow ideal class group $\mathcal{C}^+(F)$ of $\mathcal{O}_F$ is defined as $\mathcal{C}^+(F) = I(F)/P^+(F)$.

2.2. 4-rank of $\mathcal{C}^+(F)$ and Rédei matrix M_F^+

Consider a real quadratic function field $F = \mathrm{k}(\sqrt{D})$ with $D = P_1 \cdots P_t$, where P_i is a monic irreducible polynomial in $\mathbb{A}$ for $1 \le i \le t$. By genus theory, $r_2(\mathcal{C}^+(F)) = t-1$. Let $s = s(D)$ denote the number of the P_i with odd degree. Since $\deg(D)$ is even, s is even. From now on we always assume that $2 \nmid \deg(P_i)$ for $1 \le i \le s$ and $2 | \deg(P_i)$ for $s+1 \le i \le t$. For $1 \le i \ne j \le t$, let $e_{ij} \in \mathbb{F}_2$ be defined by $(-1)^{e_{ij}} = (\frac{\bar{P}_i}{P_j})$, where $\bar{P}_i = (-1)^{\deg(P_i)} P_i$ and e_{ii} is defined to satisfy $\sum_{i=1}^t e_{ij} = 0$. Let $M'_F = (e_{ij})_{1 \le i,j \le t}$. We associate a matrix M_F^+ to F defined as follows: If there is an ideal $\mathfrak{a}$ of F such that $\mathfrak{a}^{1-\sigma} = \alpha \mathcal{O}_F$ with $N_{F/\mathrm{k}}(\alpha) \in \mathbb{F}_q^{*2} \setminus \mathbb{F}_q^{*4}$, where σ is the generator of $\mathrm{Gal}(F/\mathrm{k})$, then M_F^+ is defined as the $t \times (t+1)$ matrix obtained from M'_F by adjoining $(e_{1A}\ e_{2A}\ \cdots\ e_{tA})^t$ to the last column, where $A \in \mathbb{A}$ is the monic polynomial with $N_{F/\mathrm{k}}(\mathfrak{a}) = (A)$ and $e_{iA} \in \mathbb{F}_2$ is defined to satisfy $(-1)^{e_{iA}} = (\frac{P_i}{A})$, and $M_F^+ = M'_F$ otherwise. We remark that if $q \equiv 3 \bmod 4$, we always have $M_F^+ = M'_F$. Then $r_4(\mathcal{C}^+(F))$ satisfies the following equality ([1])

$$r_4(\mathcal{C}^+(F)) = t - 1 - \mathrm{rank}(M_F^+). \tag{2.1}$$

2.3. Martinet's inequality

Let E and K be finite (geometric) separable extensions of k such that E/K is a cyclic extension of degree ℓ with $\Delta = \mathrm{Gal}(E/K)$, where ℓ is a prime number not dividing q. Then $H^0(\Delta, \mathcal{O}_E^*)$ and $H^1(\Delta, \mathcal{O}_E^*)$ are elementary abelian ℓ-groups with

$$\frac{|H^0(\Delta, \mathcal{O}_E^*)|}{|H^1(\Delta, \mathcal{O}_E^*)|} = \ell^{-1} \prod_{\mathfrak{p}_\infty \in S_\infty(K)} |\Delta_{\mathfrak{p}_\infty}|,$$

where $S_\infty(K)$ is the set of primes of K lying above ∞ and $\Delta_{\mathfrak{p}_\infty}$ denotes the decomposition group of $\mathfrak{p}_\infty$ in Δ. Note that $\Delta_{\mathfrak{p}_\infty} = \Delta$ if $\mathfrak{p}_\infty$ ramifies or inerts in E and $\Delta_{\mathfrak{p}_\infty} = \{1\}$ otherwise. Following the arguments in [4, §2], we get the following.

Proposition 2.1. *Let E and K be finite (geometric) separable extensions of* k *such that E/K is a cyclic extension of degree ℓ, where ℓ is a prime number not dividing q. Let $\gamma_{E/K}$ be the number of prime ideals of $\mathcal{O}_K$ that ramify in E and $\rho_{E/K}$ be the number of primes $\mathfrak{p}_\infty$ in $S_\infty(K)$ that ramify or inert in E. If $\gamma_{E/K}$ satisfies the inequality*

$$\gamma_{E/K} \ge |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{\ell |S_\infty(K)| + (1-\ell)\rho_{E/K} + 1}, \tag{2.2}$$

then the Hilbert ℓ-class field tower of E is infinite.

The inequality (2.2) is called the Martinet's inequality for E/K. Let F be a real quadratic function field over k. We remark that if there exists an extension E of F which has infinite Hilbert 2-class field tower and $F \subset E \subset F_1^{(2)}$, then F also has infinite Hilbert 2-class field tower.

COROLLARY 2.2. *Let $F = \mathrm{k}(\sqrt{D})$ be a real quadratic function field over* k. *If D has a nonconstant monic divisor D_1 of even degree satisfying $(\frac{D_1}{Q_j}) = 1$ for monic irreducible divisors Q_j $(1 \le j \le 5)$ of D, then F has infinite Hilbert 2-class field tower.*

Proof. Put $K = \mathrm{k}(\sqrt{D_1})$, which is a real quadratic extension of k in which Q_1, Q_2, Q_3, Q_4 and Q_5 split. Let $E = KF$. Applying Proposition 2.1 on E/K with $\gamma_{E/K} = 10$ and $(|S_\infty(K)|, \rho_{E/K}) = (2, 0)$, we see that E has infinite Hilbert 2-class field tower, so F also has infinite Hilbert 2-class field tower. □

COROLLARY 2.3. *Let $F = \mathrm{k}(\sqrt{D})$ be a real quadratic function field over* k. *If D has a two distinct nonconstant monic divisors D_1 and D_2 of even degrees satisfying $(\frac{D_1}{Q_j}) = (\frac{D_2}{Q_j}) = 1$ for monic irreducible divisors Q_j $(1 \le j \le 4)$ of D, then F has infinite Hilbert 2-class field tower.*

Proof. Put $K = \mathrm{k}(\sqrt{D_1}, \sqrt{D_2})$, which is a real biquadratic extension of k in which Q_1, Q_2, Q_3 and Q_4 split completely. Let $E = KF$. Applying Proposition 2.1 on E/K with $\gamma_{E/K} \ge 16$ and $(|S_\infty(K)|, \rho_{E/K}) = (4, 0)$, we see that E has infinite Hilbert 2-class field tower, so F also has infinite Hilbert 2-class field tower. □

COROLLARY 2.4. *Let $F = \mathrm{k}(\sqrt{D})$ be a real quadratic function field over* k. *If D has a two distinct nonconstant monic divisors D_1 and D_2 of even degrees satisfying $(\frac{D_1}{Q_j}) = (\frac{D_2}{Q_j}) = 1$ for monic irreducible divisors Q_j $(1 \le j \le 3)$ of D and there is a monic irreducible divisor Q of D which is different from Q_1, Q_2, Q_3 and $Q \nmid D_1 D_2$, then F has infinite Hilbert 2-class field tower.*

Proof. Put $K = \mathrm{k}(\sqrt{D_1}, \sqrt{D_2})$, which is a real biquadratic extension of k in which Q_1, Q_2 and Q_3 split completely. Let $E = KF$. Since Q splits in at least one quadratic subfield of K, we have $\gamma_{E/K} \ge 14$. Applying Proposition 2.1 on E/K with $\gamma_{E/K} \ge 14$ and $(|S_\infty(K)|, \rho_{E/K}) = (4, 0)$, we see that E has infinite Hilbert 2-class field tower, so F also has infinite Hilbert 2-class field tower. □

3. Proof of Theorem 1.1

Consider a real quadratic function field $F = \mathrm{k}(\sqrt{D})$ with $D = P_1 \cdots P_t$, where P_i is a monic irreducible polynomial in $\mathbb{A}$ for $1 \le i \le t$. Recall that $s = s(D)$ is the number of the P_i with odd degree and we assume that $2 \nmid \deg(P_i)$ for $1 \le i \le s$ and $2 | \deg(P_i)$ for $s+1 \le i \le t$. Assume that $q \equiv 3 \bmod 4$. In this section, we are going to prove the infiniteness of Hilbert 2-class field tower of F under the condition $r_4(\mathcal{C}^+(F)) \ge 4$. Note that $r_2(\mathcal{C}(F)) = t-1$ if $s = 0$ and $t-2$ if $s \ge 2$. Hence, if $r_2(\mathcal{C}^+(F)) \ge 6$ when $s = 0$ or $r_2(\mathcal{C}^+(F)) \ge 7$ when $s \ge 2$, then $r_2(\mathcal{C}(F)) \ge 6$, so F has infinite Hilbert 2-class field tower. If $r_2(\mathcal{C}^+(F)) = r_4(\mathcal{C}^+(F))$, then $\operatorname{rank} M_F^+ = 0$, i.e., $M_F^+ = O$, so $e_{12} = e_{21}$. Thus the case $r_2(\mathcal{C}^+(F)) = r_4(\mathcal{C}^+(F))$ with $s \ge 2$ can't occur by the quadratic reciprocity law. Thus we only need to consider the cases

$$(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = \begin{cases} (4,4),(5,4),(5,5) & \text{if } s = 0, \\ (5,4),(6,4),(6,5) & \text{if } s \ge 2. \end{cases}$$

Let $\mathbf{r}_i(F)$ denote the i-th row of M_F^+ and $\mathbf{0}$ denote the zero one in $\mathbb{F}_2^t$.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (4,4)$ with $D = P_1P_2P_3P_4P_5$ and $s = 0$. Since $M_F^+ = O$, $(\frac{P_1}{P_i}) = (\frac{P_2}{P_i}) = 1$ for $3 \le i \le 5$, so P_3, P_4 and P_5 split completely in $K = \mathrm{k}(\sqrt{P_1}, \sqrt{P_2})$. Let $E = KF$. Since $\mathbb{F}_q^* = \mathbb{F}_q^* \cap N_{F/\mathrm{k}}(F^*)$, $\mathbb{F}_q^*$ is contained in $\mathcal{O}_K^* \cap N_{E/K}(E^*)$ and so $(\mathcal{O}_K^* : \mathcal{O}_K^* \cap N_{E/K}(E^*)) \le 2^3$. Since $(\frac{P_1}{P_2}) = 1$, the ideal class number $h(\mathcal{O}_K)$ of $\mathcal{O}_K$ is even. Since $\gamma_{E/K} = 12$, by the ambiguous class number formula ([2, Lemma 2.2]), $r_2(\mathcal{C}(E)) \ge 9$. Since $r_2(\mathcal{O}_E^*) = 8$, by Schoof's theorem, E has infinite Hilbert 2-class field tower. Since $F \subset E \subset F_1^{(2)}$, F also has infinite Hilbert 2-class field tower.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (5,5)$ with $D = P_1P_2P_3P_4P_5P_6$ and $s = 0$. Since $M_F^+ = O$, $(\frac{P_6}{P_i}) = 1$ for $1 \le i \le 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (5,4)$ with $D = P_1P_2P_3P_4P_5P_6$ and $s \in \{0,2,4,6\}$. In this case $\operatorname{rank} M_F^+ = 1$, so at least one row of M_F^+ is nonzero and the other ones are multiple of this row. Assume first $s = 0$. Since at least two rows of M_F^+ are equal, we may assume $\mathbf{r}_5(F) = \mathbf{r}_6(F)$. Then $e_{5j} = e_{6j}$ for $1 \le j \le 4$, so P_1, P_2, P_3 and P_4 split in $K = \mathrm{k}(\sqrt{P_5P_6})$. Let $E = KF$. Since $\mathbb{F}_q^* = \mathbb{F}_q^* \cap N_{F/\mathrm{k}}(F^*)$, $\mathbb{F}_q^*$ is contained in $\mathcal{O}_K^* \cap N_{E/K}(E^*)$ and so $(\mathcal{O}_K^* : \mathcal{O}_K^* \cap N_{E/K}(E^*)) \le 2$. Since $\gamma_{E/K} = 8$ and $r_2(\mathcal{C}(K)) = 1$, by the ambiguous class number formula, $r_2(\mathcal{C}(E)) \ge 7$. Since $r_2(\mathcal{O}_E^*) = 4$, by (1.1), E has infinite Hilbert 2-class

field tower. Since $F \subset E \subset F_1^{(2)}$, F also has infinite Hilbert 2-class field tower.

Assume that $s = 2$ or 4. If $\mathbf{r}_i(F) = \mathbf{0}$ for some $s+1 \le i \le 6$, say $\mathbf{r}_6(F) = \mathbf{0}$, then $(\frac{P_6}{P_j}) = 1$ for $1 \le j \le 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Now we may assume that $\mathbf{r}_i(F) \neq \mathbf{0}$ for $s+1 \le i \le 6$, so they are all equal. If $\mathbf{r}_i(F) = \mathbf{r}_j(F) = \mathbf{0}$ for some $1 \le i \neq j \le s$, say $\mathbf{r}_1(F) = \mathbf{r}_2(F) = \mathbf{0}$, then $e_{12} = e_{21} = 0$ which is a contradiction. Thus at most one row of M_F^+ is zero. Then all rows of M_F^+ are nonzero and they are all equal. If $e_{12} = 1$, then all rows of M_F^+ are $(1\ 0\ 1\ 1\ 1\ 1)$, but then $e_{26} = 1 \neq 0 = e_{62}$ which is a contradiction. If $e_{12} = 0$, then all rows of M_F^+ are $(0\ 1\ 0\ 0\ 0\ 0)$, but then $e_{26} = 0 \neq 1 = e_{62}$ which is a contradiction.

Consider the case $s = 6$. If $\mathbf{r}_i(F) = \mathbf{r}_j(F) = \mathbf{0}$ for some $1 \le i \neq j \le 6$, say $\mathbf{r}_1(F) = \mathbf{r}_2(F) = \mathbf{0}$, then $e_{12} = e_{21} = 0$ which is a contradiction. Thus at most one row of M_F^+ is zero. Then all rows of M_F^+ are nonzero and they are all equal, so we can get a contradiction as above. Thus this case can't occur.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (6,4)$ with $D = P_1 \cdots P_7$ and $s \in \{2,4,6\}$. In this case, rank $M_F^+ = 2$, so two rows of M_F^+ are independent over $\mathbb{F}_2$ and the others are $\mathbb{F}_2$-linear combinations of these two rows. If $\mathbf{r}_7(F) = \mathbf{0}$, then $(\frac{P_7}{P_j}) = 1$ for $1 \le j \le 6$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Thus we may assume $\mathbf{r}_7(F) \neq \mathbf{0}$. Consider first the case $s = 2$. At least two of $\mathbf{r}_3(F), \mathbf{r}_4(F), \mathbf{r}_5(F), \mathbf{r}_6(F)$ and $\mathbf{r}_7(F)$ are equal, say $\mathbf{r}_6(F) = \mathbf{r}_7(F)$, then $(\frac{P_6P_7}{P_j}) = 1$ for $1 \le j \le 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2.

Assume $s = 4$. If two of $\mathbf{r}_1(F), \mathbf{r}_2(F), \mathbf{r}_3(F)$ and $\mathbf{r}_4(F)$ are equal, say $\mathbf{r}_1(F) = \mathbf{r}_2(F)$, then $(\frac{P_1P_2}{P_j}) = 1$ for $3 \le j \le 7$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Hence, we may assume that $\mathbf{r}_1(F), \mathbf{r}_2(F), \mathbf{r}_3(F)$ and $\mathbf{r}_4(F)$ are all distinct (†). If $\mathbf{r}_i(F) = \mathbf{0}$ for $i = 5$ or 6, say $\mathbf{r}_6(F) = \mathbf{0}$, then $(\frac{P_6}{P_j}) = 1$ for $1 \le j \le 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. If two of $\mathbf{r}_5(F), \mathbf{r}_6(F)$ and $\mathbf{r}_7(F)$ are equal, say $\mathbf{r}_6(F) = \mathbf{r}_7(F)$, then $(\frac{P_6P_7}{P_j}) = 1$ for $1 \le j \le 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Thus, we may assume that $\mathbf{r}_5(F), \mathbf{r}_6(F)$ and $\mathbf{r}_7(F)$ are all distinct and nonzero (‡). From (†) and (‡), without loss of generality, we may assume that $\mathbf{r}_5(F), \mathbf{r}_6(F)$ are linearly independent over $\mathbb{F}_2$ and $\mathbf{r}_1(F) = \mathbf{0}$, $\mathbf{r}_2(F) = \mathbf{r}_5(F)$, $\mathbf{r}_3(F) = \mathbf{r}_6(F)$, $\mathbf{r}_4(F) = \mathbf{r}_7(F) = \mathbf{r}_5(F) + \mathbf{r}_6(F)$. But, since $\mathbf{r}_1(F) = \mathbf{0}$, we have $e_{21} = e_{31} = e_{41} = 1$ and $e_{51} = e_{61} = e_{71} = 0$, which is a contradiction.

Assume $s = 6$. Since $\mathbf{r}_7(F) \neq \mathbf{0}$, at least two rows of M_F^+ except the 7-th row are equal, say $\mathbf{r}_1(F) = \mathbf{r}_2(F)$. Then $(\frac{P_1P_2}{P_j}) = 1$ for $3 \leq j \leq 7$, so F has infinite Hilbert 2-class field tower by Corollary 2.2.

• Case $(r_2(\mathcal{C}^+(F)), r_4(\mathcal{C}^+(F))) = (6, 5)$ with $D = P_1 \cdots P_7$ and $s \in \{2, 4, 6\}$. In this case, rank $M_F^+ = 1$, so at least one row of M_F^+ is nonzero and the other ones are multiple of this row. If $\mathbf{r}_7(F) = \mathbf{0}$, then $(\frac{P_7}{P_j}) = 1$ for $1 \leq j \leq 6$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. Thus we may assume $\mathbf{r}_7(F) \neq \mathbf{0}$ and the other rows of M_F^+ are multiple of $\mathbf{r}_7(F)$.

Assume that $s = 2$ or 4. If $\mathbf{r}_i(F) = \mathbf{0}$ for $i = 5$ or 6, say $\mathbf{r}_6(F) = \mathbf{0}$, then $(\frac{P_6}{P_j}) = 1$ for $1 \leq j \leq 5$, so F has infinite Hilbert 2-class field tower by Corollary 2.2. We may assume that $\mathbf{r}_i(F) \neq \mathbf{0}$ for $i = 5, 6$, so $\mathbf{r}_5(F) = \mathbf{r}_6(F) = \mathbf{r}_7(F)$. Then $(\frac{P_5P_6}{P_j}) = (\frac{P_5P_7}{P_j}) = 1$ for $1 \leq j \leq 4$, so F has infinite Hilbert 2-class field tower by Corollary 2.4.

Consider the case $s = 6$. At least three rows of M_F^+ except the 7-th row are equal, say $\mathbf{r}_4(F) = \mathbf{r}_5(F) = \mathbf{r}_6(F)$. Then $(\frac{P_4P_5}{P_j}) = (\frac{P_4P_6}{P_j}) = 1$ for $1 \leq j \leq 3$, so F has infinite Hilbert 2-class field tower by Corollary 2.4. We complete the proof of Theorem 1.1.

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