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퍼지 일반화된 위상 공간에서 FUZZY r-GENERALIZED ALMOST CONTINUITY에 관한 연구

Fuzzy r-Generalized Almost Continuity on Fuzzy Generalized Topological Spaces

민원근
Won Keun Min

강원대학교 수학과

요 약

본 논문에서는 fuzzy r-generalized almost continuity의 개념과 특성을 연구한다. 특히 fuzzy r-generalized regular open sets를 이용하여 fuzzy r-generalized almost continuity의 특성을 밝힌다.

Abstract

In this paper, we introduce the concept of fuzzy r-generalized almost continuous mapping and obtain some characterizations of such a mapping. In particular, we investigate characterizations for the fuzzy r-generalized almost continuity by using the concept of fuzzy r-generalized regular open sets.

Key Words: fuzzy generalized topological space, fuzzy r-generalized open set, fuzzy r-generalized continuous, fuzzy r-generalized almost continuous, fuzzy r-generalized regular open set.

1. Introduction

Let I be the unit interval $[0,1]$ of the real line. A member A of I^X is called a *fuzzy set* [6] X. By $\tilde{0}$ and $\tilde{1}$, we denote constant maps on X with value 0 and 1, respectively. For any $A \in I^X$, A^c denotes the complement $\tilde{1} - A$. All other notations are standard notations of fuzzy set theory.

A *fuzzy point* x_α in X is a fuzzy set x_α is defined as follows

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

A fuzzy point x_α is said to *belong to* a fuzzy set A in X, denoted by $x_\alpha \in A$, if $\alpha \le A(x)$ for $x \in X$.

A fuzzy set A in X is the union of all fuzzy points which belong to A.

Let $f: X \to Y$ be a mapping and $A \in I^X$ and $B \in I^Y$. Then $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} \sup A(z)_{z \in f^{-1}(y)}, & \text{if } f^{-1}(y) \neq 0,\ y \in Y \\ 0, & \text{otherwise}, \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x))$, $x \in X$.

A *Chang's fuzzy topological space* [1] is an ordered pair (X, T) is a non-empty set X and $T \subseteq I^X$ satisfying the following conditions:

(01) $0_X, 1_X \in T$.

(02) If $A, B \in T$, then $A \cap B \in T$.

(03) If $A_i \in \tau$, for all $i \in J$, then $\cup A_i \in \tau$.

A *smooth topological space* [5] is an ordered pair (X, T), where X is a non-empty set and $T: I^X \to I$ is a mapping satisfying the following conditions:

(01) $T(0_X) = T(1_X) = 1$.

(02) $T(A_1 \cap A_2) \ge T(A_1) \wedge T(A_2)$ for $A_1, A_2 \in I^X$.

(03) $T(\cup A_i) \ge \wedge T(A_i)$ for all $i \in J$, $A_i \in I^X$.

Then $T: I^X \to I$ is called a *smooth topology* on X. The number $T(A)$ is called the *degree of openness* of A.

A mapping $T^*: I^X \to I$ is called a *smooth cotopology* [2] iff the following three conditions are satisfied:

(C1) $T^*(0_X) = T^*(1_X) = 1$.

(C2) $T^*(A_1 \cup A_2) \ge T^*(A_1) \wedge T^*(A_2)$, $A_1, A_2 \in I^X$.

(C3) $T^*(\cap A_i) \ge \wedge T^*(A_i)$ for all $i \in J$ $A_i \in I^X$.

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A *fuzzy generalized topological space* (simply, FGTS) [3] is an ordered pair (X, T), where X is a non-empty set and $T: I^X \rightarrow I$ is a mapping satisfying the following conditions:

(GO1) $T(0_X)=1$.

(GO2) $T(\vee A_i) \geq \wedge T(A_i)$ for all $i \in J$, $A_i \in I^X$.

Then the mapping $T: I^X \rightarrow I$ is called a *fuzzy generalized topology* [3] on X. The number $T(A)$ is called the *degree of generalized openness* of A.

A mapping $T^*: I^X \rightarrow I$ is called a *fuzzy generalized cotopology* if the following three conditions are satisfied:

(GO1) $T^*(1_X)=1$.

(GO2) $T^*(\wedge A_i) \geq \wedge T^*(A_i)$ for all $i \in J$, $A_i \in I^X$.

Then $T^*(A)$ is called the *degree of generalized closedness* of A.

Theorem 1.1 ([3]). (1) If T is a fuzzy generalized topology on X, then the mapping $T^*: I^X \rightarrow I$ defined by $T^*(A)=T(A^c)$, is a fuzzy generalized cotopology on X.

(2) If T^* is a fuzzy generalized cotopology on a non-empty set X, then the mapping $T: I^X \rightarrow I$ defined by $T(A)=T^*(A^c)$, is a fuzzy generalized topology on X.

Let (X, T) be a FGTS and $A \in I^X$. Then

(1) The *r-closure* of A [4], denoted by $gCl_r(A)$, is defined by

$$gCl_r(A)=\cap\{K \in I^X : T^*(K) \geq r, A \subseteq K\},$$

where $T^*(K)=T(K^c)$.

(2) The *r-interior* of A [4], denoted by $gInt_r(A)$, is defined by

$$gInt_r(A)=\cup\{K \in I^X : T(K) \geq r, K \subseteq A\}.$$

We will call A a *fuzzy r-generalized open set* [4] if $T(A) \geq r$, A a *fuzzy r-generalized closed set* if $T^*(A) \geq r$.

Theorem 1.2 ([4]). Let (X, T) be a FGTS and A, $B \in I^X$. Then

(1) $gInt_r(0_X)=0_X$ and $gCl_r(1_X)=1_X$.

(2) $gInt_r(A) \subseteq A \subseteq gCl_r(A)$.

(3) $gInt_r(gInt_r(A))=gInt_r(A)$ and $gCl_r(gCl_r(A))=gCl_r(A)$

(4) $A \subseteq B \Rightarrow gInt_r(A) \subseteq gInt_r(B)$, $gCl_r(A) \subseteq gCl_r(B)$.

(5) $(gCl_r(A))^c=gInt_r(A^c)$ and $(gInt_r(A))^c=gCl_r(A^c)$.

(6) A is fuzzy r-generalized open iff $A=gInt_r(A)$.

(7) A is fuzzy r-generalized closed iff $A=gCl_r(A)$.

2. Main Results

Definition 2.1([4]). Let $f:(X, T_1) \rightarrow (Y, T_2)$ be a mapping on FGTS's. Then f is said to be *fuzzy r-generalized continuous* if for every $A \in I^Y$, we have

$$T_2(A) \geq r \Rightarrow T_1(f^{-1}(A)) \geq r.$$

Theorem 2.2([4]). Let $f: X \rightarrow Y$ be a mapping between FGTS's (X, T_1) and (Y, T_2). Then the following are equivalent:

(1) f is fuzzy r-generalized continuous.

(2) For every fuzzy r-generalized open set A in Y, $f^{-1}(A)$ is fuzzy r-generalized open in X.

(3) $T_2^*(B) \geq r \Rightarrow T_1^*(f^{-1}(B)) \geq r$ for $B \in I^Y$.

(4) For every fuzzy r-generalized closed set A in Y, $f^{-1}(A)$ is fuzzy r-generalized closed in X.

(5) $f(gCl_r(A)) \subseteq gCl_r(f(A)$ for $A \in I^X$.

(6) $gCl_r(f^{-1}(B)) \subseteq f^{-1}(gCl_r(B))$ for $B \in I^Y$.

(7) $f^{-1}(gInt_r(B)) \subseteq gInt_r(f^{-1}(B))$ for $B \in I^Y$.

Theorem 2.3. Let $f: X \rightarrow Y$ be a mapping between FGTS's (X, T_1) and (Y, T_2). Then f is fuzzy r-generalized continuous if and only if for fuzzy point x_α in X and each fuzzy r-generalized open set V containing $f(x_\alpha)$, there is a fuzzy r-generalized open set U containing x_α such that $f(U) \subseteq V$.

Proof. Suppose f is fuzzy r-generalized continuous. For each fuzzy point x_α in X and each fuzzy r-generalized open set V containing $f(x_\alpha)$, since f is fuzzy r-generalized continuous, from Theorem 2.2 (2), $f^{-1}(V)$ is a fuzzy r-generalized open set containing x_α. Set $U = f^{-1}(V)$. Then the fuzzy r-generalized open set U satisfies $f(U) \subseteq V$.

For the converse, let V be any fuzzy r-generalized open set in Y. For each fuzzy point $x_\alpha \in f^{-1}(V)$, by hypothesis, there exists a fuzzy r-generalized open set U containing x_α such that $f(U) \subseteq V$. So $x_\alpha \in U \subseteq f^{-1}(V)$ and $x_\alpha \in gInt_r(f^{-1}(V))$. This implies $f^{-1}(V) \subseteq gInt_r(f^{-1}(V))$ and from Theorem 1.2 (6), $f^{-1}(V)$ is fuzzy r-generalized open. Hence from Theorem 2.2(2), f is fuzzy r-generalized continuous.

Definition 2.4. Let $f: X \rightarrow Y$ be a mapping between FGTS's (X, T_1) and (Y, T_2). Then f is said to be *fuzzy r-generalized almost continuous* if for fuzzy point x_α in X and each fuzzy r-generalized open set V containing $f(x_\alpha)$, there is a fuzzy r-generalized open set U containing x_α such that

$$f(U) \subseteq gInt_r(gCl_r(V)).$$

Every fuzzy r-generalized continuous mapping f is clearly fuzzy r-generalized almost continuous but the converse is not always true.

Example 2.5. Let $X=I$, let A, B and C be fuzzy sets defined as follows

$$A(x)=\frac{1}{2}(x+1),\ x\in I;$$

$$B(x)=-\frac{1}{2}(x-2),\ x\in I;$$

and

$$C(x)=\begin{cases}-\frac{1}{2}(x-2), & \text{if } 0\le x\le \frac{1}{2},\\ \frac{1}{2}(x+1), & \text{if } \frac{1}{2}\le x\le 1.\end{cases}$$

Consider two fuzzy families T_1 and T_2 defined as the following:

$$T_1(\mu)=\begin{cases}\frac{2}{3}, & \text{if } \mu=\tilde{0},\\ \frac{1}{2}, & \text{if } \mu=C,\\ 0, & otherwise;\end{cases}$$

and

$$T_2(\mu)=\begin{cases}\frac{1}{2}, & \text{if } \mu=\tilde{0},\\ \frac{2}{3}, & \text{if } \mu=A,B,A\cup B,C,\\ 0, & otherwise.\end{cases}$$

Note $gInt_r(gCl_r(\tilde{0}))=\tilde{0}$ and $gInt_r(gCl_r(A))=gInt_r(gCl_r(B))=gInt_r(gCl_r(C))=C$. Obviously the identity mapping $f:(X,T_1)\to(X,T_2)$ is fuzzy $\frac{1}{3}$-generalized almost continuous but not fuzzy $\frac{1}{3}$-generalized continuous.

Theorem 2.6. Let $f:X\to Y$ be a mapping between FGTS's (X,T_1) and (Y,T_2). Then the following statements are equivalent:

(1) f is fuzzy r-generalized almost continuous.

(2) $f^{-1}(B)\subseteq gInt_r(f^{-1}(gInt_r(gCl_r(B))))$ for each fuzzy r-generalized open set B in Y.

(3) $gCl_r(f^{-1}(gCl_r(gInt_r(F))))\subseteq f^{-1}(F)$ for each fuzzy r-generalized closed set F in Y.

(4) $gCl_r(f^{-1}(gCl_r(gInt_r(gCl_r(B)))))\subseteq f^{-1}(gCl_r(B))$ for each $B\in I^Y$.

(5) $f^{-1}(gInt_r(B))\subseteq gInt_r(f^{-1}(gInt_r(gCl_r(gInt_r(B)))))$ for each $B\in I^Y$.

Proof. (1) $\Rightarrow$ (2) Let B be a fuzzy r-generalized open set in Y. From the definition of fuzzy r-generalized almost continuity, there exists a fuzzy r-generalized open set U of x_α such that $f(U)\subseteq gInt_r(gCl_r(B))$ for each $x_\alpha\in f^{-1}(B)$. It implies $x_\alpha\in gInt_r(f^{-1}(gInt_r(gCl_r(B))))$. Hence we have $f^{-1}(B)\subseteq gInt_r(f^{-1}(gInt_r(gCl_r(B))))$.

(2) $\Rightarrow$ (1) Let x_α be any fuzzy point in X and V a fuzzy r-generalized open set containing $f(x_\alpha)$. Then since $x_\alpha\in f^{-1}(V)\subseteq gInt_r(f^{-1}(gInt_r(gCl_r(V))))$, there exists a fuzzy r-generalized open set U containing x_α such that $x_\alpha\in U\subseteq f^{-1}(gInt_r(gCl_r(V)))$. This implies $f(x_\alpha)\in f(U)\subseteq f(f^{-1}(gInt_r(gCl_r(V))))\subseteq gInt_r(gCl_r(V))$. Hence f is fuzzy r-generalized almost continuous.

(2) $\Rightarrow$ (3) Let F be any fuzzy r-generalized closed set of Y. Then it follows

$$\begin{aligned}f^{-1}(\tilde{1}-F)&\subseteq gInt_r(f^{-1}(gInt_r(gCl_r(\tilde{1}-F))))\\ &=gInt_r(f^{-1}(\tilde{1}-gCl_r(gInt_r(F))))\\ &=gInt_r(\tilde{1}-f^{-1}(gCl_r(gInt_r(F))))\\ &=\tilde{1}-gCl_r(f^{-1}(gCl_r(gInt_r(F)))).\end{aligned}$$

Hence we have $gCl_r(f^{-1}(gCl_r(gInt_r(F))))\subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (5) For $B\in I^Y$, from hypothesis, it follows

$$\begin{aligned}f^{-1}(gInt_r(B))&=\tilde{1}-(f^{-1}(gCl_r(\tilde{1}-B)))\\ &\subseteq \tilde{1}-gCl_r(f^{-1}(gCl_r(gInt_r(gCl_r(\tilde{1}-B)))))\\ &=gInt_r(f^{-1}(gInt_r(gCl_r(gInt_r(B))))).\end{aligned}$$

Hence

$$f^{-1}(gInt_r(B))\subseteq gInt_r(f^{-1}(gInt_r(gCl_r(gInt_r(B))))).$$

(5)$\Rightarrow$ (1) For each fuzzy point x_α in X and each fuzzy r-generalized open set V containing $f(x_\alpha)$, by (5), we have $x_\alpha\in f^{-1}(V)=f^{-1}(gInt_r(V))\subseteq gInt_r(f^{-1}(gInt_r(gCl_r(V))))$. So there is a fuzzy r-generalized open set U of x_α such that $x_\alpha\in U\subseteq f^{-1}(gInt_r(gCl_r(V)))$. This implies $f(U)\subseteq gInt_r(gCl_r(V))$. Thus f is fuzzy r-generalized almost continuous.

Definition 2.7. Let (X,T) be a FGTS and $A\in I^X$. Then A is called a *fuzzy r-generalized regular open set* if $A=gInt_r(gCl_r(A))$.

Theorem 2.8. Let (X,T) be a FGTS and $A,B\in I^X$. Then

(1) Every fuzzy r-generalized regular open set is fuzzy r-generalized open.

(2) If A and B are fuzzy r-generalized regular open set, so also is $A\cap B$.

Proof. (1) Let A be fuzzy r-generalized regular open. Then

$gInt_r(A)=gInt_r(gInt_r(gCl_r(A)))=gInt_r(gCl_r(A))=A$.

From Theorem 1.2, A is fuzzy r-generalized open.

(2) Obvious.

In general, every fuzzy r-generalized open set is not fuzzy r-generalized regular open and the union of two fuzzy r-generalized regular open sets is not fuzzy r-generalized regular open as shown in the next example.

Example 2.9. Let X = I, let A, B, C and D be fuzzy sets defined as follows

$$A(x)=\frac{1}{2}(x+1),\ x\in I;$$

$$B(x)=-\frac{1}{2}(x-2),\ x\in I;$$

$$C(x)=\begin{cases}0, & \text{if } 0\le x\le \frac{1}{4},\\ \frac{2}{3}(x-\frac{1}{4}), & \text{if } \frac{1}{4}\le x\le 1;\end{cases}$$

and

$$D(x)=\begin{cases}-\frac{2}{3}(x-\frac{3}{4}), & \text{if } 0\le x\le \frac{3}{4},\\ 0, & \text{if } \frac{3}{4}\le x\le 1.\end{cases}$$

Consider a fuzzy family T defined as the following:

$$T(\mu)=\begin{cases}\frac{2}{3}, & \text{if } \mu=\tilde{0},\\ \frac{1}{2}, & \text{if } \mu=A,B,A\cup B,C,D,C\cup D,(C\cap D)^c,\\ 0, & otherwise.\end{cases}$$

Let $r=\frac{1}{2}$. Then for fuzzy r-generalized regular open sets A, B, we know that

$$gInt_r(gCl_r(A\cup B))=(C\cap D)^c\neq A\cup B,$$

and so $A\cup B$ is not fuzzy r-generalized regular open. On the other hand, since $A\cup B$ is a fuzzy r-generalized regular open set, we can say that every fuzzy r-generalized open set is not generally fuzzy r-generalized regular open.

Theorem 2.10. Let $f: X\to Y$ be a mapping between FGTS's (X, T_1) and (Y, T_2). Then f is fuzzy r-generalized almost continuous if and only if $gCl_r(f^{-1}(V))\subseteq f^{-1}(gCl_r(V))$ for a fuzzy r-generalized regular open set V in Y.

Proof. Suppose f is fuzzy r-generalized almost continuous. Let V be any fuzzy r-generalized regular open set of Y. Then since $(\tilde{1}-V)$ is fuzzy r-generalized regular closed, it follows

$$\begin{aligned}\tilde{1}-f^{-1}(gCl_r(V))&=f^{-1}(gInt_r(\tilde{1}-V))\\ &\subseteq gInt_r(f^{-1}(gInt_r(gCl_r(gInt_r(\tilde{1}-V)))))\\ &=gInt_r(f^{-1}(gInt_r(\tilde{1}-V)))\\ &=gInt_r(\tilde{1}-(f^{-1}(gCl_r(V))))\\ &=\tilde{1}-gCl_r(f^{-1}(gCl_r(V)))\subseteq \tilde{1}-gCl_r(f^{-1}(V)).\end{aligned}$$

Hence we have $gCl_r(f^{-1}(V))\subseteq f^{-1}(gCl_r(V))$.

For the converse, let F be any fuzzy r-generalized closed set in Y. Since $gInt_r(F)$ is a fuzzy r-generalized regular open set, from hypothesis and $gCl_r(gInt_r(F))\subseteq gInt_r(F)\subseteq F$, it follows

$gCl_r(f^{-1}(gCl_r(gInt_r(F))))\subseteq gCl_r(f^{-1}(gInt_r(F)))\subseteq f^{-1}(gCl_r(gInt_r(F)))\subseteq f^{-1}(F)$.

By Theorem 2.6 (3), f is fuzzy r-generalized almost continuous.

Theorem 2.11. Let $f: X\to Y$ be a mapping between FGTS's (X, T_1) and (Y, T_2). Then the following statements are equivalent:

(1) f is fuzzy r-generalized almost continuous.

(2) $f^{-1}(V)$ is fuzzy r-generalized open for a fuzzy r-generalized regular open set V in Y.

(3) $f^{-1}(F)$ is fuzzy r-generalized closed for a fuzzy r-generalized regular closed set F in Y.

Proof. (1) $\Rightarrow$ (2) Let V be a fuzzy r-generalized regular open set in Y. For each $x_\alpha\in f^{-1}(V)$, from the fuzzy r-generalized almost continuity of f, there a fuzzy r-generalized open set U in X such that $f(U)\subseteq gInt_r(gCl_r(V))$. Since V is a fuzzy r-generalized regular open, $x_\alpha\in U\subseteq f^{-1}(V)$ and so $f^{-1}(V)$ is fuzzy r-generalized open set.

(2) $\Rightarrow$ (1) Let V be a fuzzy r-generalized open set containing $f(x_\alpha)$. Since $gInt_r(gCl_r(V))$ is fuzzy r-generalized regular open, by (2), $f^{-1}(gInt_r(gCl_r(V)))$ is a fuzzy r-generalized open set containing x_α. Set U = $f^{-1}(gInt_r(gCl_r(V)))$. Then U is a fuzzy r-generalized open set satisfying $f(U)\subseteq gInt_r(gCl_r(V))$. Thus f is a fuzzy r-generalized almost continuous mapping.

(2) $\Leftrightarrow$ (3) Obvious.

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저 자 소 개

민원근 (Won Keun Min)

1988년 - 현재 : 강원대학교 수학과 교수

관심분야 : 퍼지 위상, 퍼지 이론, 일반 위상
Phone : 033-250-8419
Fax : 033-252-7289
E-mail : wkmin@kangwon.ac.kr