Interval-Valued Fuzzy Almost M-Continuous Mappings On Interval-Valued Fuzzy Topological Spaces

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Abstract

We introduce the concept of IVF almost M-continuity and investigate characterizations for such mappings on the interval-valued fuzzy topological spaces. We study the relationships between IVF almost M-continuous mappings and IVF compactness.

Key Words : IVF minimal structure, IVF M-continuous, IVF weakly M-continuous, IVF almost M-continuous

1. Introduction

Zadeh [9] introduced the concept of fuzzy set and investigated basic properties. Gorzalczany [1] introduced the concept of interval-valued fuzzy set which is a generalization of fuzzy sets. In [4], the author introduced and studied an IVF minimal structure as a generalization of interval-valued fuzzy topology introduced by Mondal and Samanta [8]. The author and Kim [7] introduced the concepts of interval-valued fuzzy M-continuous mappings defined between an IVF minimal space and an IVF topological space. And we studied some characterizations and basic properties of such mappings. In this paper, we introduce the concept of IVF almost M-continuity and investigate characterizations for IVF almost M-continuous mappings on the interval-valued fuzzy topological spaces. We study the relationships between IVF almost M-continuous mappings and IVF compactness.

2. Preliminaries

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$ and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points, respectively. Especially, we denote $\mathbf{0} = [0,0], \mathbf{1} = [1,1]$, and $\mathbf{a} = [a,a]$ for $a \in (0,1)$. We also note that

(1) For all $M, N \in D[0,1]$,

$$M = N \Leftrightarrow M^L = N^L, M^U = N^U.$$

(2) For all $M, N \in D[0,1]$,

$$M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U.$$

For every $M \in D[0,1]$, the complement of M, denoted by M^c, is defined by $M^c = \mathbf{1} - M = [1 - M^U, 1 - M^L]$.

Let X be a nonempty set. A mapping $A : X \rightarrow D[0,1]$ is called *an interval-valued fuzzy set* (simply, IVF set) in X. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^L$ and $A(x)^U$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$. In particular, for any $c \in [a,b]$, the IVF set whose value is $\mathbf{c} = [c,c]$ for all $x \in X$ is denoted by simply $\widetilde{c}$. For a point $p \in X$ and for $[a,b] \in D[0,1]$ with $b > 0$, the IVF set which takes the value $[a,b]$ at p and $\mathbf{0}$ elsewhere in X is called *an interval-valued fuzzy point* (simply, IVF point) and is denoted by $[a,b]_p$. In particular, if $b = a$, then it is also denoted by a_p. Denoted by $IVF(X)$ the set of all IVF sets in X.

For every $A, B \in IVF(X)$, we define

$$A = B \Leftrightarrow (\forall x \in X)([A(x)]^L = [B(x)]^L \text{ and } [A(x)]^U = [B(x)]^U),$$

$$A \subseteq B \Leftrightarrow (\forall x \in X)([A(x)]^L \subseteq [B(x)]^L \text{ and } [A(x)]^U \subseteq [B(x)]^U).$$

The complement A^c of A is defined by, for all $x \in X$,

$$[A^c(x)]^L = 1 - [A(x)]^U \text{ and } [A^c(x)]^U = 1 - [A(x)]^L.$$

For a family of IVF sets $\{A_i : i \in J\}$ where J is an index set, the union $G = \cup_{i \in J} A_i$ and $F = \cap_{i \in J} A_i$ are defined by

$$(\forall x \in X)\ ([G(x)]^L = \sup_{i \in J}[A_i(x)]^L, [G(x)]^U = \sup_{i \in J}[A_i(x)]^U),$$
$$(\forall x \in X)\ ([F(x)]^L = \inf_{i \in J}[A_i(x)]^L, [F(x)]^U = \inf_{i \in J}[A_i(x)]^U),$$ respectively.

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Let $f : X \to Y$ be a mapping and let A be an IVF set in X. Then the image of A under f, denoted by $f(A)$, is defined by

$$[f(A)(y)]^L = \begin{cases} \sup_{f(x)=y}[A(x)]^L, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise }, \end{cases}$$

$$[f(A)(y)]^U = \begin{cases} \sup_{f(x)=y}[A(x)]^U, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise }, \end{cases}$$

for all $y \in Y$.

Let B be an IVF set in Y. Then the inverse image of B under f, denoted by $f^{-1}(B)$, is defined by

$([f^{-1}(B)(x)]^L = [B(f(x))]^L,\ [f^{-1}(B)(x)]^U = [B(f(x))]^U)$ for all $x \in X$.

Definition 2.1 ([8]). A family τ of IVF sets in X is called an *interval-valued fuzzy topology* on X if it satisfies:

(1) $\mathbf{0}, \mathbf{1} \in \tau$.

(2) $A, B \in \tau \Rightarrow A \cap B \in \tau$.

(3) For $i \in J$, $A_i \in \tau \Rightarrow \cup_{i\in J} A_i \in \tau$.

Every member of τ is called an IVF open set. An IVF set A is called an IVF closed set if the complement of A is an IVF open set. And (X, τ) is called an *interval-valued fuzzy topological space*.

In an IVF topological space (X, τ), for an IVF set A in X, the IVF closure and the IVF interior of A [8], denoted by $cl(A)$ and $int(A)$, respectively, are defined as

$cl(A) = \cap\{B \in IVF(X) : B^c \in \tau \text{ and } A \subseteq B\}$,

$int(A) = \cup\{B \in IVF(X) : B \in \tau \text{ and } B \subseteq A\}$.

An IVF set A is said to be

(1) *IVF semiopen* [2] if there is an IVF -open set B in X such that $B \subseteq A \subseteq cl(B)$,

(2) *IVF preopen* [2] if $A \subseteq int(cl(A))$,

(3) *IVF regular open* (resp., *IVF regular closed*) [5] if $A = int(cl(A))$ (resp., $A = cl(int(A))$),

(4) *IVF β-open* [5] if $A \subseteq cl(int(cl(A)))$.

A family $\mathcal{M}$ of interval-valued fuzzy sets in X is called an *interval-valued fuzzy minimal structure* [4] on X if

$$\mathbf{0}, \mathbf{1} \in \mathcal{M}.$$

In this case, $(X, \mathcal{M})$ is called an *interval-valued fuzzy minimal space* (simply, *IVF minimal space*). Every member of $\mathcal{M}$ is called an IVF m-open set. An IVF set A is called an IVF m-closed set if the complement of A (simply, A^c) is an IVF m-open set.

Let $(X, \mathcal{M})$ be an IVF minimal space and A in IVF(X). The IVF minimal-closure of A [4], denoted by $mCl(A)$, is defined as

$mCl(A) = \cap\{B \in IVF(X) : B^c \in \mathcal{M} \text{ and } A \subseteq B\}$;

the IVF minimal-interior of A [4], denoted by $mInt(A)$, is defined as

$mInt(A) = \cup\{B \in IVF(X) : B \in \mathcal{M} \text{ and } B \subseteq A\}$.

Theorem 2.2 ([4]). Let $(X, \mathcal{M})$ be an IVF minimal space and A, B in IVF(X).

(1) $mInt(A) \subseteq A$ and if A is an IVF m-open set, then $mInt(A) = A$.

(2) $A \subseteq mCl(A)$ and if A is an IVF m-closed set, then $mCl(A) = A$.

(3) If $A \subseteq B$, then $mInt(A) \subseteq mInt(B)$ and $mCl(A) \subseteq mCl(B)$.

(4) $mInt(A) \cap mInt(B) \supseteq mInt(A \cap B)$ and $mCl(A) \cup mCl(B) \subseteq mCl(A \cup B)$.

(5) $mInt(mInt(A)) = mInt(A)$ and $mCl(mCl(A)) = mCl(A)$.

(6) $\mathbf{1} - mCl(A) = mInt(\mathbf{1} - A)$ and $\mathbf{1} - mInt(A) = mCl(\mathbf{1} - A)$.

Let $(X, \mathcal{M}_X)$ be an IVF minimal space and let (Y, τ) be an IVF topological space. Then $f : X \to Y$ is said to be

(1) *interval-valued fuzzy M-continuous* [7] (simply, IVF M-continuous) if for every $A \in \tau$, $f^{-1}(A)$ is in $\mathcal{M}_X$;

(2) *IVF weakly M-continuous* [6] if for every IVF point M_x and each IVF open set V of $f(M_x)$, there exists IVF m-open set U of M_x such that $f(U) \subseteq cl(V)$.

Let $(X, \mathcal{M})$ be an IVF minimal space and $\mathcal{A} = \{A_i : i \in J\}$. $\mathcal{A}$ is called an *interval-valued fuzzy cover* (simply, IVF cover) if $\cup\{A_i : i \in J\} = \mathbf{1}$. And it is called an *IVF m-open cover* [4] if each A_i is an IVF m-open set. An IVF set A in X is said to be *IVF m-compact* if every IVF m-open cover $\mathcal{A} = \{A_i : i \in J\}$ of B has a finite IVF subcover.

3. IVF almost M-continuous mappings

Definition 3.1. Let $f : X \to Y$ be a mapping between an IVF minimal space $(X, \mathcal{M})$ and an IVF topological space (Y, τ). Then f is said to be *IVF almost M-continuous* if for every IVF point M_x and each IVF open set V containing $f(M_x)$, there exists IVF m-open set U containing M_x such that $f(U) \subseteq int(cl(V))$.

Remark 3.2. Let $f : X \to Y$ be a mapping between an IVF minimal space $(X, \mathcal{M})$ and an IVF topological space (Y, τ). If the IVF minimal structure $\mathcal{M}$ is an IVF topology in X, the IVF almost M-continuous (resp., IVF M-continuous, IVF weakly M-continuous) is IVF almost continuous [3] (resp., IVF continuous [7], IVF weakly continuous [5]). And

IVF continuous $\Rightarrow$ IVF almost continuous
$\Rightarrow$ IVF weakly continuous

Thus obviously the following implications are obtained

IVF M-continuous $\Rightarrow$ IVF almost M-continuous
$\Rightarrow$ IVF weakly M-continuous

Theorem 3.3. Let $f : X \to Y$ be a mapping between an IVF minimal space $(X, \mathcal{M})$ and an IVF topological space (Y, τ). Then the following statements are equivalent:

(1) f is IVF almost M-continuous.

(2) $f^{-1}(B) \subseteq mInt(f^{-1}(int(cl(B))))$ for each IVF open set B of Y.

(3) $mCl(f^{-1}(cl(int(F)))) \subseteq f^{-1}(F)$ for each IVF closed set F in Y.

(4) $mCl(f^{-1}(cl(int(cl(B))))) \subseteq f^{-1}(cl(B))$ for each $B \in IVF(Y)$.

(5) $f^{-1}(int(B)) \subseteq mInt(f^{-1}(int(cl(int(B)))))$ for each $B \in IVF(Y)$.

(6) $f^{-1}(V) = mInt(f^{-1}(V))$ for an IVF regular open set V in Y.

(7) $f^{-1}(F) = mCl(f^{-1}(F))$ for an IVF regular closed set F in Y.

Proof. (1) ⇒ (2) Let B be an IVF open set in Y. Then for each $M_x \in f^{-1}(B)$, there exists an IVF m-open set U of M_x such that $f(U) \subseteq int(cl(B))$. This implies $M_x \in mInt(f^{-1}(int(cl(B))))$. Thus $f^{-1}(B) \subseteq mInt(f^{-1}(int(cl(B))))$.

(2) ⇒ (1) Let M_x be an IVF point in X and V an IVF open set containing $f(M_x)$. Then by (2), $M_x \in f^{-1}(V) \subseteq mInt(f^{-1}(int(cl(V))))$, and so there exists an IVF m-open set U containing M_x such that $U \subseteq f^{-1}(int(cl(V)))$. This implies $f(U) \subseteq f(f^{-1}(int(cl(V)))) \subseteq int(cl(V))$. Hence f is IVF almost M-continuous.

(2) ⇒ (3) Let F be any IVF closed set of Y. Then from (2) and Theorem 2.2, it follows

$$\begin{aligned} f^{-1}(\mathbf{1}-F) &\subseteq mInt(f^{-1}(int(cl(\mathbf{1}-F)))) \\ &= mInt(f^{-1}(\mathbf{1}-cl(int(F)))) \\ &= mInt(\mathbf{1}-f^{-1}(cl(int(F)))) \\ &= \mathbf{1}-mCl(f^{-1}(cl(int(F)))). \end{aligned}$$

Hence $mCl(f^{-1}(cl(int(F)))) \subseteq f^{-1}(F)$.

(3) ⇒ (4) It is obvious.

(4) ⇒ (5) It is obvious from (4) and Theorem 2.2

(5) ⇒ (6) Let V be any IVF regular open set of Y. Then by (5), $f^{-1}(V) = f^{-1}(int(V)) \subseteq mInt(f^{-1}(int(cl(int(V))))) = mInt(f^{-1}(V)$ It implies $f^{-1}(V) = mInt(f^{-1}(V)$.

(6) ⇔ (7) Obvious.

(6) ⇒ (1) Let V be an IVF open set containing $f(M_x)$. Since $int(cl(V))$ is IVF regular open, by (6), $f^{-1}(int(cl(V))) = mInt(f^{-1}(int(cl(V))))$ and so there is an IVF m-open set containing M_x such that $U \subseteq f^{-1}(int(cl(V)))$. Then this implies that f is an IVF almost M-continuous mapping. □

Let $\mathcal{M}$ be an IVF minimal structure on X. Then $\mathcal{M}$ said to have property $(\mathcal{B})$ [4] if the union of any family of IVF sets belong to $\mathcal{M}$ belongs to $\mathcal{M}$.

Lemma 3.4 ([4]). Let $\mathcal{M}$ be an IVF minimal structure on X. Then the following are equivalent.

(1) $\mathcal{M}$ has the property $(\mathcal{B})$.

(2) If $mInt(B) = B$, then $B \in \mathcal{M}$.

(3) If $mCl(F) = F$, then $X - F \in \mathcal{M}$.

Theorem 3.5. Let $f : X \to Y$ be a mapping between an IVF minimal space $(X, \mathcal{M})$ and an IVF topological space (Y, τ). If $\mathcal{M}$ has the property $(\mathcal{B})$, then the following statements are equivalent:

(1) f is IVF almost M-continuous.

(2) For each IVF regular open set V in Y, $f^{-1}(V)$ is IVF m-open.

(3) For each IVF regular closed set F in Y, $f^{-1}(F)$ is IVF m-closed.

Proof. It follows from Theorem 3.3 and Lemma 3.4. □

Theorem 3.6. Let $f : X \to Y$ be a mapping between an IVF minimal space $(X, \mathcal{M})$ and an IVF topological space (Y, τ). Then the following statements are equivalent:

(1) f is IVF almost M-continuous.

(2) $mCl(f^{-1}(G)) \subseteq f^{-1}(cl(G))$ for each IVF β-open set G in Y.

(3) $mCl(f^{-1}(G)) \subseteq f^{-1}(cl(G))$ for each IVF semiopen set G in Y.

Proof. (1) ⇒ (2) Let G be an IVF β-open set. Then $G \subseteq cl(int(cl(G)))$ and $cl(G)$ is IVF regular closed. Hence from Theorem 3.3 (7), it follows

$$mCl(f^{-1}(G)) \subseteq mCl(f^{-1}(cl(G))) = f^{-1}(cl(G)).$$

(2) ⇒ (3) Since every IVF semiopen set is IVF β-open, it is obvious.

(3) ⇒ (1) Let F be an IVF regular closed set. Then F is IVF semiopen, and so from (3), we have

$$mCl(f^{-1}(F)) \subseteq f^{-1}(cl(F)) = f^{-1}(F).$$

Hence, from Theorem 3.3 (7), f is an IVF almost M-continuous mapping. □

Theorem 3.7. Let $f : X \to Y$ be a mapping between an IVF minimal space $(X, \mathcal{M})$ and an IVF topological space (Y, τ). Then f is IVF almost M-continuous if and only if $mCl(f^{-1}(cl(int(cl(G))))) \subseteq f^{-1}(cl(G))$ for each IVF preopen set G in Y.

Proof. Suppose f is IVF almost M-continuous. Let G be an IVF preopen set in Y. Then we have $cl(G) = cl(int(cl(G)))$, so $cl(G)$ is IVF regular open. From Theorem 3.7 (7), we have $f^{-1}(cl(G)) = mCl(f^{-1}(cl(G))) = mCl(f^{-1}(cl(int(cl(G)))))$.

Thus it implies

$$mCl(f^{-1}(cl(int(cl(G))))) \subseteq f^{-1}(cl(G)).$$

For the converse, let A be an IVF regular closed set in Y. Then $int(A)$ is IVF preopen. From hypothesis, it follows

$$\begin{aligned} f^{-1}(A) &= f^{-1}(cl(int(A))) \\ &\supseteq mCl(f^{-1}(cl(int(cl(int(A)))))) \\ &= mCl(f^{-1}(cl(int(A)))) \\ &= mCl(f^{-1}(A)). \end{aligned}$$

This implies $f^{-1}(A) = mCl(f^{-1}(A))$, and hence f is IVF almost M-continuous. □

Theorem 3.8. Let $f : X \to Y$ be a mapping between an IVF minimal space $(X, \mathcal{M})$ and an IVF topological space (Y, τ). Then f is IVF almost M-continuous if and only if $f^{-1}(G) \subseteq mInt(f^{-1}(int(cl(G))))$ for each IVF preopen set G in Y.

Proof. Suppose f is IVF almost M-continuous and let G be an IVF preopen set in Y. Then $int(cl(G))$ is IVF regular open. From Theorem 3.3, it follows

$$f^{-1}(G) \subseteq f^{-1}(int(cl(G))) = mInt(f^{-1}(int(cl(G)))).$$

For the converse, let U be IVF regular open. Then U is obviously IVF preopen. By hypothesis, $f^{-1}(U) \subseteq mInt(f^{-1}(int(cl(U)))) = mInt(f^{-1}(U))$. This implies $f^{-1}(U) = mInt(f^{-1}(U))$ and by Theorem 3.7, f is IVF almost M-continuous. □

Let (X, τ) be an IVFTS. An IVF set A in X is said to be *nearly IVF compact* [4] if for every IVF open cover $\mathcal{A} = \{A_i \in IVF(X) : i \in J\}$ of A, there exists $J_0 = \{1, 2, \cdots, n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} int(cl(A_i))$.

Theorem 3.9. Let $f : X \to Y$ be a mapping between an IVF minimal space $(X, \mathcal{M})$ and an IVF topological space (Y, τ). If f is an IVF almost M-continuous surjection and X is IVF m-compact, then Y is nearly IVF compact.

Proof. Let $\mathcal{C} = \{B_i \in IVF(Y) : i \in J\}$ be an IVF open cover of Y. Then by Theorem 3.3, for each $i \in J$, $f^{-1}(B_i) \subseteq mInt(f^{-1}(int(cl(B_i))))$.

Set $\mathcal{G} = \{U(x_{i_\alpha}) \in \mathcal{M} : x_{i_\alpha} \in U(x_{i_\alpha}) \in f^{-1}(int(cl(B_i)))$ for all $x_{i_\alpha} \in f^{-1}(B_i), i \in J\}$. Then $\mathcal{G}$ is an IVF m-open cover of X, and so by IVF m-compactness, finally, there exists $J_0 = \{1, 2, \cdots n\} \subseteq J$ such that $X \subseteq \cup_{i \in J_0} mInt(f^{-1}(int(cl(B_i))))$. So

$$\begin{aligned} f(X) &\subseteq f(\cup_{i \in J_0} mInt(f^{-1}(int(cl(B_i))))) \\ &\subseteq \cup_{i \in J_0} f(f^{-1}(int(cl(B_i)))) \\ &= \cup_{i \in J_0} int(cl(B_i)). \end{aligned}$$

Since f is surjective, consequently Y is nearly IVF compact. □

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