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Fuzzy r-minimal Continuous Functions Between Fuzzy Minimal Spaces and Fuzzy Topological Spaces

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Abstract

In this paper, we introduce the concepts of fuzzy r-minimal continuous function and fuzzy r-minimal open function between a fuzzy r-minimal space and a fuzzy topological space. We also investigate characterizations and properties for such functions.

Key Words : r-minimal structure, fuzzy r-minimal continuous, fuzzy r-minimal open function

1. Introduction

The concept of fuzzy set was introduced by Zadeh [10]. Chang [2] defined fuzzy topological spaces using fuzzy sets. In [3, 8], Chattopadhyay, Hazra and Samanta introduced a smooth topological space which is a generalization of fuzzy topological space.

In [9], Yoo et al. introduced the concept of fuzzy r-minimal space which is an extension of the smooth topological space. The concept of fuzzy r-M continuity was also introduced and investigated in [9]. In this paper, we introduce the concepts of fuzzy r-minimal continuous function and fuzzy r-minimal open function on fuzzy r-minimal spaces and investigate characterizations for such functions.

2. Preliminaries

Let I be the unit interval $[0,1]$ of the real line. A member A of I^X is called a fuzzy set of X. By $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$ we denote constant maps on X with value 0 and 1, respectively. For any $A \in I^X$, A^c denotes the complement $\tilde{\mathbf{1}} - \mathbf{A}$. All other notations are standard notations of fuzzy set theory.

An *fuzzy point* x_α in X is a fuzzy set x_α defined as follows

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

A fuzzy point x_α is said to belong to a fuzzy set A in X, denoted by $x_\alpha \in A$, if $\alpha \leq A(x)$ for $x \in X$. A fuzzy set A in X is the union of all fuzzy points which belong to A.

Let $f : X \to Y$ be a function and $A \in I^X$ and $B \in I^Y$. Then $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise }, \end{cases}$$

for $y \in Y$ and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x)), x \in X$.

A *fuzzy topology* (or *smooth topology*) [3, 5] on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:
(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.
(2) $\mathcal{T}(A_1 \wedge A_2) \geq \mathcal{T}(A_1) \wedge \mathcal{T}(A_2)$ for $A_1, A_2 \in I^X$.
(3) $\mathcal{T}(\bigvee A_i) \geq \bigwedge \mathcal{T}(A_i)$ for $A_i \in I^X$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*. And $A \in I^X$ is said to be *fuzzy r-open* (resp., *fuzzy r-closed*) if $\mathcal{T}(A) \geq r$ (resp., $\mathcal{T}(A^c) \geq r$).

The *r-closure* of A, denoted by $cl(A,r)$, is defined as $cl(A,r) = \cap\{B \in I^X : A \subseteq B$ and B is fuzzy r-closed$\}$.

The *r-interior* of A, denoted by $int(A,r)$, is defined as $int(A,r) = \cup\{B \in I^X : B \subseteq A$ and B is fuzzy r-open$\}$.

Definition 2.1 ([9]). Let X be a nonempty set and $r \in (0,1] = I_0$. A fuzzy family $\mathcal{M} : I^X \to I$ on X is said to have a *fuzzy r-minimal structure* if the family

$$\mathcal{M}_r = \{A \in I^X \mid \mathcal{M}(A) \geq r\}$$

contains $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$.

Then the $(X, \mathcal{M})$ is called a *fuzzy r-minimal space* (simply r-FMS) if $\mathcal{M}$ has a fuzzy r-minimal structure. Every member of $\mathcal{M}_r$ is called a *fuzzy r-minimal open* set. A fuzzy set A is called a *fuzzy r-minimal closed* set if the complement of A (simply, A^c) is a fuzzy r-minimal open set.

Let $(X, \mathcal{M})$ be an r-FMS and $r \in I_0$. The fuzzy r-minimal closure and the fuzzy r-minimal interior of A [9],

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denoted by $mC(A,r)$ and $mI(A,r)$, respectively, are defined as

$$mC(A,r) = \cap\{B \in I^X : B^c \in \mathcal{M}_r \text{ and } A \subseteq B\},$$

$$mI(A,r) = \cup\{B \in I^X : B \in \mathcal{M}_r \text{ and } B \subseteq A\}.$$

Theorem 2.2 ([9]). Let $(X, \mathcal{M})$ be an r-FMS and A, B in I^X.

(1) $mI(A,r) \subseteq A$ and if A is a fuzzy r-minimal open set, then $mI(A,r) = A$.

(2) $A \subseteq mC(A,r)$ and if A is a fuzzy r-minimal closed set, then $mC(A,r) = A$.

(3) If $A \subseteq B$, then $mI(A,r) \subseteq mI(B,r)$ and $mC(A,r) \subseteq mC(B,r)$.

(4) $mI(A,r) \cap mI(B,r) \supseteq mI(A \cap B,r)$ and $mC(A,r) \cup mC(B,r) \subseteq mC(A \cup B,r)$.

(5) $mI(mI(A,r),r) = mI(A,r)$ and $mC(mC(A,r),r) = mC(A,r)$.

(6) $\tilde{\mathbf{1}} - mC(A,r) = mI(\tilde{\mathbf{1}} - A,r)$ and $\tilde{\mathbf{1}} - mI(A,r) = mC(\tilde{\mathbf{1}} - A,r)$.

Definition 2.3 ([9]). Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be two r-FMS's. Then a function $f : X \to Y$ is said to be

(1) *fuzzy r-M continuous* if for every fuzzy r-minimal open set A in Y, $f^{-1}(A)$ is fuzzy r-minimal open in X,

(2) *fuzzy r-M open* if for every fuzzy r-minimal open set G in X, $f(G)$ is fuzzy r-minimal open in Y.

3. Fuzzy r-minimal continuous function and fuzzy r-minimal open functions

Definition 3.1. Let $(X, \mathcal{M}_X)$ be an r-FMS and (Y, σ) a fuzzy topological space. Then $f : X \to Y$ is said to be *fuzzy r-minimal continuous* if for every fuzzy r-open set A in Y, $f^{-1}(A)$ is fuzzy r-minimal open in X.

Theorem 3.2. Let $f : X \to Y$ be a function between an r-FMS $(X, \mathcal{M}_X)$ and a fuzzy topological space (Y, σ). Then we have the following:

(1) f is fuzzy r-minimal continuous.

(2) $f^{-1}(B)$ is a fuzzy r-minimal closed set for each fuzzy r-closed set B in Y.

(3) $f(mC(A,r)) \subseteq cl(f(A),r)$ for $A \in I^X$.

(4) $mC(f^{-1}(B),r) \subseteq f^{-1}(cl(B,r))$ for $B \in I^Y$.

(5) $f^{-1}(int(B,r)) \subseteq mI(f^{-1}(B),r)$ for $B \in I^Y$.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

Proof. (1) $\Leftrightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) For $A \in I^X$,

$$\begin{aligned} &f^{-1}(cl(f(A),r)) \\ &= f^{-1}(\cap\{F \in I^Y : f(A) \subseteq F, F \text{ is fuzzy } r\text{-closed}\}) \\ &= \cap\{f^{-1}(F) : A \subseteq f^{-1}(F), F \text{ is fuzzy } r\text{-closed}\} \\ &\supseteq \cap\{K \in I^X : A \subseteq K, K \text{ is fuzzy } r\text{-minimal closed}\} \\ &= mC(A,r). \end{aligned}$$

Hence $f(mC(A,r)) \subseteq cl(f(A),r)$.

(3) $\Leftrightarrow$ (4) Fort $B \in I^Y$, from (3), it follows

$$f(mC(f^{-1}(B),r)) \subseteq cl(f(f^{-1}(B)),r) \subseteq cl(B,r).$$

Hence (4) is obtained and similarly, we get (4) $\Rightarrow$ (3).

(4) $\Leftrightarrow$ (5) From Theorem 2.2, it is obvious. □

Example 3.3. Let $X = I$ and let us consider two fuzzy sets A, B defined as

$$A(x) = \frac{1}{2}x, \ \ x \in X;$$

$$B(x) = -\frac{1}{2}(x-1), \ \ x \in X.$$

Consider a fuzzy family

$$\mathcal{M}(U) = \begin{cases} \frac{1}{2}, & \text{if } U = \tilde{\mathbf{0}}, \tilde{\mathbf{1}}, \\ \frac{2}{3}, & \text{if } U = A, B, \\ 0, & \text{otherwise,} \end{cases}$$

and a fuzzy topology

$$\sigma(U) = \begin{cases} 1, & \text{if } U = \tilde{\mathbf{0}}, \tilde{\mathbf{1}}, \mathbf{A}, \mathbf{B}, \\ \frac{1}{3}, & \text{if } U = A \cap B, A \cup B, \\ 0, & \text{otherwise.} \end{cases}$$

Then the identity function $f : (X, \mathcal{M}) \to (X, \sigma)$ satisfies the part (3) in Theorem 3.2 but f is not fuzzy $\frac{1}{3}$-minimal continuous.

Let X be a nonempty set and $\mathcal{M} : I^X \to I$ a fuzzy family on X. The fuzzy family $\mathcal{M}$ is said to have the property $(\mathcal{U})$ [9] if for $A_i \in \mathcal{M}$ $(i \in J)$,

$$\mathcal{M}(\cup A_i) \geq \wedge \mathcal{M}(A_i).$$

Theorem 3.4. Let $f : X \to Y$ be a function between an r-FMS $(X, \mathcal{M}_X)$ and a fuzzy topological space (Y, σ). If $\mathcal{M}_X$ has the property $(\mathcal{U})$, then f is fuzzy r-minimal continuous iff for a fuzzy point x_α in X and each fuzzy r-open set V containing $f(x_\alpha)$, there is a fuzzy r-minimal open set U containing x_α such that $f(U) \subseteq V$.

Proof. Let f be fuzzy r-minimal continuous, then for fuzzy point x_α in X and each fuzzy r-open set V containing $f(x_\alpha)$, from Theorem 3.2 (5), $x_\alpha \in f^{-1}(V) = f^{-1}(int(V,r)) \subseteq mI(f^{-1}(V),r)$. So there exists a fuzzy r-minimal open set U containing x_α such that $x_\alpha \in U \subseteq f^{-1}(V)$. Hence the result is obtained.

For the converse, let a fuzzy set A be fuzzy r-open in Y. Then by hypothesis, we have $\cup U = f^{-1}(A)$ for each fuzzy r-minimal open set U in X. Hence by the property $(\mathcal{U})$, $f^{-1}(A)$ is fuzzy r-minimal open. □

Corollary 3.5. Let $f : X \to Y$ be a function between an r-FMS $(X, \mathcal{M}_X)$ and a fuzzy topological space (Y, τ). If $\mathcal{M}_X$ has property $(\mathcal{U})$, then the following are equivalent:

(1) f is fuzzy r-minimal continuous.

(2) For fuzzy point x_α in X and each fuzzy r-open set V containing $f(x_\alpha)$, there is a fuzzy r-minimal open set U containing x_α such that $f(U) \subseteq V$.

(3) $f^{-1}(B)$ is fuzzy r-minimal closed, for each fuzzy r-closed set B in Y.

(4) $f(mC(A, r)) \subseteq cl(f(A), r)$ for $A \in I^X$.

(5) $mC(f^{-1}(B), r) \subseteq f^{-1}(cl(B, r))$ for $B \in I^Y$.

(6) $f^{-1}(int(B, r)) \subseteq mI(f^{-1}(B), r)$ for $B \in I^Y$.

Proof. It follows from Theorem 3.2 and Theorem 3.4. □

Definition 3.6. Let (X, σ) be a fuzzy topological space and $(Y, \mathcal{M}_Y)$ an r-FMS. Then $f : (X, \sigma) \to (Y, \mathcal{M}_Y)$ is said to be *fuzzy r-minimal open* if for every fuzzy r-open set A in X, $f(A)$ is fuzzy r-minimal open in Y.

Theorem 3.7. Let $f : X \to Y$ be a function on a fuzzy topological space (X, σ) and an r-FMS $(Y, \mathcal{M}_Y)$.

(1) f is fuzzy r-minimal open.

(2) $f(int(A, r)) \subseteq mI(f(A), r)$ for $A \in I^X$.

(3) $int(f^{-1}(B), r) \subseteq f^{-1}(mI(B), r)$ for $B \in I^Y$.

Then (1) ⇒ (2) ⇔ (3).

Proof. (1) ⇒ (2) For $A \in I^X$,

$$\begin{aligned} & f(int(A), r) \\ & = f(\cup\{B \in I^X : B \subseteq A, B \text{ is fuzzy } r\text{-open}\}) \\ & = \cup\{f(B) \in I^Y : f(B) \subseteq f(A), \\ & \qquad f(B) \text{ is fuzzy } r\text{-minimal open}\} \\ & \subseteq \cup\{U \in I^Y : U \subseteq f(A), U \text{ is fuzzy } r\text{-minimal open}\} \\ & = mI(f(A), r). \end{aligned}$$

Hence $f(int(A), r) \subseteq mI(f(A), r)$.

(2) ⇔ (3) For $B \in I^Y$, from (3) it follows that

$$f(int(f^{-1}(B), r)) \subseteq mI(f(f^{-1}(B)), r) \subseteq mI(B, r).$$

This implies (3). Similarly, we get the implication (3) ⇒ (2). □

Remark 3.8. In Example 3.3, consider the identity function $f : (X, \sigma) \to (X, \mathcal{M}_X)$. Then f satisfies the statement (2) in Theorem 3.7, but it is not fuzzy r-minimal open.

Lemma 3.9. Let $f : X \to Y$ be a function on a fuzzy topological space (X, σ) and an r-FMS $(Y, \mathcal{M}_Y)$. If f is fuzzy r-minimal open, then $f(A) = mI(f(A), r)$ for every fuzzy r-open set A in X.

Proof. It follows from Theorem 2.2 (1). □

Theorem 3.10. ([9]) Let $(X, \mathcal{M})$ be an r-FMS with the property $(\mathcal{U})$. Then

(1) $mI(A, r) = A$ if and only if $A \in \mathcal{M}_r$ for $A \in I^X$.

(2) $mC(A, r) = A$ if and only if $A^c \in \mathcal{M}_r$ for $A \in I^X$.

From Lemma 3.9 and Theorem 3.10, obviously the next corollary is obtained:

Corollary 3.11. Let $f : (X, \sigma) \to (Y, \mathcal{M}_Y)$ be a function on a fuzzy topological space (X, σ) and an r-FMS $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ has property $(\mathcal{U})$, then the following are equivalent:

(1) f is fuzzy r-minimal open.

(2) $f(int(A, r)) \subseteq mI(f(A), r)$ for $A \in I^X$.

(3) $int(f^{-1}(B), r) \subseteq f^{-1}(mI(B, r))$ for $B \in I^Y$.

Definition 3.12. Let (X, σ) be a fuzzy topological space and $(Y, \mathcal{M}_Y)$ be an r-FMS. Then $f : (X, \sigma) \to (Y, \mathcal{M}_Y)$ is said to be *fuzzy r-minimal closed* if for every fuzzy r-minimal closed set A in X, $f(A)$ is fuzzy r-minimal closed in Y.

Theorem 3.13. Let $f : (X, \sigma) \to (Y, \mathcal{M}_Y)$ be a function on a fuzzy topological space (X, σ) and an r-FMS $(Y, \mathcal{M}_Y)$.

(1) f is fuzzy r-minimal closed.

(2) $mC(f(A), r) \subseteq f(cl(A, r))$ for $A \in I^X$.

(3) $f^{-1}(mC(B, r)) \subseteq cl(f^{-1}(B), r)$ for $B \in I^Y$.

Then (1) ⇒ (2) ⇔ (3).

Proof. It is similar to the proof of Theorem 3.7. □

Lemma 3.14. Let $f : (X, \sigma) \to (Y, \mathcal{M}_Y)$ be a function on a fuzzy topological space (X, σ) and an r-FMS $(Y, \mathcal{M}_Y)$. If f is fuzzy r-minimal closed, then $f(A) = mC(f(A), r)$ for every fuzzy r-minimal closed set A in X.

Proof. It is obvious. □

Corollary 3.15. Let $f : X \to Y$ be a function on a fuzzy topological space (X, σ) and an r-FMS $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ has property $(\mathcal{U})$, then the following are equivalent:

(1) f is fuzzy r-minimal closed.

(2) $mC(f(A), r) \subseteq f(cl(A), r)$ for $A \in I^X$.

(3) $f^{-1}(mC(B, r)) \subseteq cl(f^{-1}(B), r)$ for $B \in I^Y$.

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