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Fuzzy β-(r, s)-Open Sets in Smooth Bitopological Spaces

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Abstract

We introduce and investigate the concepts of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open sets, $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-closed sets and fuzzy pairwise β-(r, s)-continuous mappings in smooth bitopological spaces.

Key Words : fuzzy β-(r, s)-open sets, fuzzy β-(r, s)-closed sets, fuzzy pairwise β-(r, s)-continuous mappings

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [12] in his classical paper. Using the concept of fuzzy sets, Chang [2] was the first to introduce the concept of a fuzzy topology on a set X by axiomatizing a collection T of fuzzy subsets of X, where he referred to each member of T as an open set. In his definition of fuzzy topology, fuzziness in the concept of openness of a fuzzy subset was absent. These spaces and its generalizations are later studied by several authors, one of which, developed by Šostak [11], used the idea of degree of openness. This type of generalization of fuzzy topological spaces was later rephrased by Chattopadhyay, Hazra, and Samanta [3], and by Ramadan [9]. Kandil [4] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. Lee [5] introduced the concept of smooth bitopological spaces as a generalization of smooth topological spaces and Kandil's fuzzy bitopological spaces.

In this paper, we introduce the concepts of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open sets and fuzzy pairwise β-(r, s)-continuous mappings in smooth bitopological spaces and then we investigate some of their characteristic properties.

2. Preliminaries

Let I be the closed unit interval $[0, 1]$ of the real line and let I_0 be the half open interval $(0, 1]$ of the real line. For a set X, I^X denotes the collection of all mapping from X to I. A member μ of I^X is called a fuzzy set of X. By $\tilde{0}$ and $\tilde{1}$ we denote constant mappings on X with value 0 and 1, respectively. For any $\mu \in I^X$, μ^c denotes the complement $\tilde{1} - \mu$. All other notations are the standard notations of fuzzy set theory.

A *Chang's fuzzy topology* on X [2] is a family T of fuzzy sets in X which satisfies the following properties:

(1) $\tilde{0}, \tilde{1} \in T$.

(2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.

(3) If $\mu_k \in T$ for all k, then $\bigvee \mu_k \in T$.

The pair (X, T) be called a *Chang's fuzzy topological space*. Members of T are called T-fuzzy open sets of X and their complements T-fuzzy closed sets of X.

A system (X, T_1, T_2) consisting of a set X with two Chang's fuzzy topologies T_1 and T_2 on X is called a *Kandil's fuzzy bitopological space*.

A *smooth topology* on X is a mapping $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.

(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$.

(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *smooth topological space*. For $r \in I_0$, we call μ a $\mathcal{T}$-fuzzy r-open set of X if $\mathcal{T}(\mu) \geq r$ and μ a $\mathcal{T}$-fuzzy r-closed set of X if $\mathcal{T}(\mu^c) \geq r$.

A system $(X, \mathcal{T}_1, \mathcal{T}_2)$ consisting of a set X with two smooth topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X is called a *smooth bitopological space*. Throughout this paper the indices i, j take values in $\{1, 2\}$ and $i \neq j$.

Let $(X, \mathcal{T})$ be a smooth topological space. Then it is easy to see that for each $r \in I_0$, an r-cut

$$\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$$

is a Chang's fuzzy topology on X.

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Let (X, T) be a Chang's fuzzy topological space and $r \in I_0$. Then the mapping $T^r : I^X \to I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise} \end{cases}$$

becomes a smooth topology.

Hence, we obtain that if $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a smooth bitopological space and $r, s \in I_0$, then $(X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$ is a Kandil's fuzzy bitopological space. Also, if (X, T_1, T_2) is a Kandil's fuzzy bitopological space and $r, s \in I_0$, then $(X, (T_1)^r, (T_2)^s)$ is a smooth bitopological space.

Definition 2.1. [5] Let $(X, \mathcal{T})$ be a smooth topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *$\mathcal{T}$-fuzzy r-closure* is defined by

$$\mathcal{T}\text{-Cl}(\mu, r) = \bigwedge\{\rho \in I^X \mid \mu \le \rho, \mathcal{T}(\rho^c) \ge r\}$$

and the *$\mathcal{T}$-fuzzy r-interior* is defined by

$$\mathcal{T}\text{-Int}(\mu, r) = \bigvee\{\rho \in I^X \mid \mu \ge \rho, \mathcal{T}(\rho) \ge r\}.$$

Lemma 2.2. [5] Let μ be a fuzzy set of a smooth topological space $(X, \mathcal{T})$ and let $r \in I_0$. Then we have:

(1) $\mathcal{T}\text{-Cl}(\mu, r)^c = \mathcal{T}\text{-Int}(\mu^c, r)$.

(2) $\mathcal{T}\text{-Int}(\mu, r)^c = \mathcal{T}\text{-Cl}(\mu^c, r)$.

Definition 2.3. [5, 6] Let μ be a fuzzy set of a smooth bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $r, s \in I_0$. Then μ is said to be

(1) a *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-semiopen* set if
$\mu \le \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mu, r), s)$,

(2) a *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-semiclosed* set if
$\mathcal{T}_j\text{-Int}(\mathcal{T}_i\text{-Cl}(\mu, r), s) \le \mu$,

(3) a *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen* set if
$\mu \le \mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu, s), r)$,

(4) a *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preclosed* set if
$\mathcal{T}_i\text{-Cl}(\mathcal{T}_j\text{-Int}(\mu, s), r) \le \mu$.

Definition 2.4. [5, 6] Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ be a mapping from a smooth bitopological space X to a smooth bitopological space Y and $r, s \in I_0$. Then f is called

(1) a *fuzzy pairwise (r, s)-continuous* mapping if the induced mapping $f : (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1)$ is fuzzy r-continuous and the induced mapping $f : (X, \mathcal{T}_2) \to (Y, \mathcal{T}_2)$ is fuzzy s-continuous,

(2) a *fuzzy pairwise (r, s)-semicontinuous* mapping if $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-semiopen set of X for each $\mathcal{U}_1$-fuzzy r-open set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-semiopen set of X for each $\mathcal{U}_2$-fuzzy s-open set ν of Y,

(3) a *fuzzy pairwise (r, s)-precontinuous* mapping if $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preopen set of X for each $\mathcal{U}_1$-fuzzy r-open set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preopen set of X for each $\mathcal{U}_2$-fuzzy s-open set ν of Y.

3. $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open sets

Definition 3.1. Let μ be a fuzzy set of a smooth bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $r, s \in I_0$. Then μ is said to be

(1) a *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open* set if
$\mu \le \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu, s), r), s)$,

(2) a *$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-closed* set if
$\mathcal{T}_j\text{-Int}(\mathcal{T}_i\text{-Cl}(\mathcal{T}_j\text{-Int}(\mu, s), r), s) \le \mu$.

Theorem 3.2. Let μ be a fuzzy set of a smooth bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $r, s \in I_0$. Then the following statements are equivalent:

(1) μ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open set.

(2) μ^c is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-closed set.

Proof. It follows from Lemma 2.2. □

Remark 3.3. It is clear that every $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-semiopen set is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open set and every $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen set is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r, s)-open set. However, the following example show that all of the converses need not be true.

Example 3.4. Let $X = \{x, y\}$ and $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ and μ_6 be fuzzy sets of X defined as

$$\mu_1(x) = 0.4, \quad \mu_1(y) = 0.7;$$
$$\mu_2(x) = 0.1, \quad \mu_2(y) = 0.2;$$
$$\mu_3(x) = 0.8, \quad \mu_3(y) = 0.5;$$
$$\mu_4(x) = 0.8, \quad \mu_4(y) = 0.1;$$
$$\mu_5(x) = 0.7, \quad \mu_5(y) = 0.6;$$

and

$$\mu_6(x) = 0.5, \quad \mu_6(y) = 0.2.$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $(\mathcal{T}_1, \mathcal{T}_2)$ is a smooth bitopology on X. The fuzzy set μ_3 is $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-$(\frac{1}{2}, \frac{1}{3})$-open which is not $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiopen and μ_4 is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-$(\frac{1}{3}, \frac{1}{2})$-open set which is not a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy $(\frac{1}{3}, \frac{1}{2})$-semiopen set. Also, μ_5 is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-$(\frac{1}{2}, \frac{1}{3})$-open set which is not a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy $(\frac{1}{2}, \frac{1}{3})$-preopen set and μ_6 is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-$(\frac{1}{3}, \frac{1}{2})$-open set which is not a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy $(\frac{1}{3}, \frac{1}{2})$-preopen set.

Theorem 3.5. (1) Any union of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-open sets is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-open set.

(2) Any intersection of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-closed sets is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-closed set.

Proof. (1) Let $\{\mu_k\}$ be a collection of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-open sets. Then for each k, $\mu_k \le \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu_k, s), r), s)$. So

$$\begin{aligned} \bigvee \mu_k &\le \bigvee \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu_k, s), r), s) \\ &\le \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\bigvee \mu_k, s), r), s). \end{aligned}$$

Thus $\bigvee \mu_k$ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-open set.

(2) It follows from (1) using Theorem 3.2. □

Theorem 3.6. Let μ be a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-open and $(\mathcal{T}_j, \mathcal{T}_i)$-fuzzy (s,r)-semiclosed set. Then μ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r,s)-semiopen set.

Proof. Let μ be a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-open and $(\mathcal{T}_j, \mathcal{T}_i)$-fuzzy (s,r)-semiclosed set. Since μ is $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-open, $\mu \le \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu, s), r), s)$. Also since μ is $(\mathcal{T}_j, \mathcal{T}_i)$-fuzzy (s,r)-semiclosed, $\mu \ge \mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu, s), r)$. Thus

$$\begin{aligned} \mu &\le \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu, s), r), s) \\ &= \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu, s), r), r), s) \\ &\le \mathcal{T}_j\text{-Cl}(\mathcal{T}_i\text{-Int}(\mu, r), s). \end{aligned}$$

Hence μ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r,s)-semiopen set. □

Theorem 3.7. Let μ be a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-closed and $(\mathcal{T}_j, \mathcal{T}_i)$-fuzzy (s,r)-semiopen set. Then μ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r,s)-semiclosed set.

Proof. Let μ be a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-closed and $(\mathcal{T}_j, \mathcal{T}_i)$-fuzzy (s,r)-semiopen set. Since μ is $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy β-(r,s)-closed, $\mu \ge \mathcal{T}_j\text{-Int}(\mathcal{T}_i\text{-Cl}(\mathcal{T}_j\text{-Int}(\mu, s), r), s)$. Also since μ is $(\mathcal{T}_j, \mathcal{T}_i)$-fuzzy (s,r)-semiopen, $\mu \le \mathcal{T}_i\text{-Cl}(\mathcal{T}_j\text{-Int}(\mu, s), r)$. Thus

$$\begin{aligned} \mu &\ge \mathcal{T}_j\text{-Int}(\mathcal{T}_i\text{-Cl}(\mathcal{T}_j\text{-Int}(\mu, s), r), s) \\ &= \mathcal{T}_j\text{-Int}(\mathcal{T}_i\text{-Cl}(\mathcal{T}_i\text{-Cl}(\mathcal{T}_j\text{-Int}(\mu, s), r), r), s) \\ &\ge \mathcal{T}_j\text{-Int}(\mathcal{T}_i\text{-Cl}(\mu, r), s). \end{aligned}$$

Hence μ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r,s)-semiclosed set. □

4. Fuzzy pairwise β-continuous mappings

Definition 4.1. Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ be a mapping from a smooth bitopological space X to a smooth bitopological space Y and $r, s \in I_0$. Then f is called

(1) a *fuzzy pairwise* β-(r,s)-*continuous* mapping if $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-(r,s)-open set of X for each $\mathcal{U}_1$-fuzzy r-open set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-(s,r)-open set of X for each $\mathcal{U}_2$-fuzzy s-open set ν of Y,

(2) a *fuzzy pairwise* β-(r,s)-*open* mapping if $f(\rho)$ is a $(\mathcal{U}_1, \mathcal{U}_2)$-fuzzy β-(r,s)-open set of Y for each $\mathcal{T}_1$-fuzzy r-open set ρ of X and $f(\lambda)$ is a $(\mathcal{U}_2, \mathcal{U}_1)$-fuzzy β-(s,r)-open set of Y for each $\mathcal{T}_2$-fuzzy s-open set λ of X,

(3) a *fuzzy pairwise* β-(r,s)-*closed* mapping if $f(\rho)$ is a $(\mathcal{U}_1, \mathcal{U}_2)$-fuzzy β-(r,s)-closed set of Y for each $\mathcal{T}_1$-fuzzy r-closed set ρ of X and $f(\lambda)$ is a $(\mathcal{U}_2, \mathcal{U}_1)$-fuzzy β-(s,r)-closed set of Y for each $\mathcal{T}_2$-fuzzy s-closed set λ of X.

Theorem 4.2. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$, $(Y, \mathcal{U}_1, \mathcal{U}_2)$ and $(Z, \mathcal{V}_1, \mathcal{V}_2)$ be smooth bitopological spaces and let $f : X \to Y$ and $g : Y \to Z$ be mappings and $r, s \in I_0$. Then the following statements are true.

(1) If f is fuzzy pairwise β-(r,s)-continuous and g is fuzzy pairwise (r,s)-continuous then $g \circ f$ is fuzzy pairwise β-(r,s)-continuous.

(2) If f is fuzzy pairwise (r,s)-open and g is fuzzy pairwise β-(r,s)-open then $g \circ f$ is fuzzy pairwise β-(r,s)-open.

(3) If f is fuzzy pairwise (r,s)-closed and g is fuzzy pairwise β-(r,s)-closed then $g \circ f$ is fuzzy pairwise β-(r,s)-closed.

Proof. Straightforward. □

Theorem 4.3. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(Y, \mathcal{U}_1, \mathcal{U}_2)$ be smooth bitopological spaces and let $r, s \in I_0$. Then $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ is a fuzzy pairwise β-(r,s)-continuous mapping if and only if $f : (X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s) \to (Y, (\mathcal{U}_1)_r, (\mathcal{U}_2)_s)$ is a fuzzy pairwise β-continuous mapping.

Proof. Let $\mu \in (\mathcal{U}_1)_r$ and $\nu \in (\mathcal{U}_2)_s$. Then $\mathcal{U}_1(\mu) \geq r$ and $\mathcal{U}_2(\nu) \geq s$ and hence μ is a $\mathcal{U}_1$-fuzzy r-open set and ν is a $\mathcal{U}_2$-fuzzy s-open set of Y. Since $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ is a fuzzy pairwise β-(r,s)-continuous mapping, $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-(r,s)-open set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-(s,r)-open set of $(X, \mathcal{T}_1, \mathcal{T}_2)$. So $f^{-1}(\mu)$ is $((\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$-fuzzy β-open and $f^{-1}(\nu)$ is $((\mathcal{T}_2)_s, (\mathcal{T}_1)_r)$-fuzzy β-open of $(X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$. Thus $f : (X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s) \to (Y, (\mathcal{U}_1)_r, (\mathcal{U}_2)_s)$ is a fuzzy pairwise β-continuous mapping.

Conversely, let μ be any $\mathcal{U}_1$-fuzzy r-open set and ν any $\mathcal{U}_2$-fuzzy s-open set of $(Y, \mathcal{U}_1, \mathcal{U}_2)$. Then $\mathcal{U}_1(\mu) \geq r$ and $\mathcal{U}_2(\nu) \geq s$. So $\mu \in (\mathcal{U}_1)_r$ and $\nu \in (\mathcal{U}_2)_s$. Since $f : (X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s) \to (Y, (\mathcal{U}_1)_r, (\mathcal{U}_2)_s)$ is a fuzzy pairwise β-continuous mapping, $f^{-1}(\mu)$ is a $((\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$-fuzzy β-open set and $f^{-1}(\nu)$ is a $((\mathcal{T}_2)_s, (\mathcal{T}_1)_r)$-fuzzy β-open set of $(X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$. So $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-(r,s)-open set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-(s,r)-open set of $(X, \mathcal{T}_1, \mathcal{T}_2)$. Thus $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ is a fuzzy pairwise β-(r,s)-continuous mapping. □

Theorem 4.4. Let (X, T_1, T_2) and (Y, U_1, U_2) be Kandil's fuzzy bitopological spaces and let $r, s \in I_0$. Then $f : (X, T_1, T_2) \to (Y, U_1, U_2)$ is a fuzzy pairwise β-continuous mapping if and only if $f : (X, (T_1)^r, (T_2)^s) \to (Y, (U_1)^r, (U_2)^s)$ is a fuzzy pairwise β-(r,s)-continuous mapping.

Proof. Let μ be a $(U_1)^r$-fuzzy r-open set and ν a $(U_2)^s$-fuzzy s-open set of $(Y, (U_1)^r, (U_2)^s)$. Then $(U_1)^r(\mu) \geq r$ and $(U_2)^s(\nu) \geq s$, and hence $\mu \in U_1$ and $\nu \in U_2$. Since $f : (X, T_1, T_2) \to (Y, U_1, U_2)$ is a fuzzy pairwise β-continuous mapping, $f^{-1}(\mu)$ is a (T_1, T_2)-fuzzy β-open set and $f^{-1}(\nu)$ is a (T_2, T_1)-fuzzy β-open set of (X, T_1, T_2). So $f^{-1}(\mu)$ is a $((T_1)^r, (T_2)^s)$-fuzzy β-(r,s)-open set and $f^{-1}(\nu)$ is a $((T_2)^s, (T_1)^r)$-fuzzy β-(s,r)-open set of $(X, (T_1)^r, (T_2)^s)$. Thus $f : (X, (T_1)^r, (T_2)^s) \to (Y, (U_1)^r, (U_2)^s)$ is a fuzzy pairwise β-(r,s)-continuous mapping.

Conversely, let $\mu \in U_1$ and $\nu \in U_2$. Then $(U_1)^r(\mu) \geq r$ and $(U_2)^s(\nu) \geq s$, and hence μ is a $(U_1)^r$-fuzzy r-open set and ν is a $(U_2)^s$-fuzzy s-open set of Y. Since $f : (X, (T_1)^r, (T_2)^s) \to (Y, (U_1)^r, (U_2)^s)$ is a fuzzy pairwise β-(r,s)-continuous mapping, $f^{-1}(\mu)$ is a $((T_1)^r, (T_2)^s)$-fuzzy β-(r,s)-open set and $f^{-1}(\nu)$ is a $((T_2)^s, (T_1)^r)$-fuzzy β-(s,r)-open set of $(X, (T_1)^r, (T_2)^s)$. So $f^{-1}(\mu)$ is a (T_1, T_2)-fuzzy β-open set and $f^{-1}(\nu)$ is a (T_2, T_1)-fuzzy β-open set of (X, T_1, T_2). Thus $f : (X, T_1, T_2) \to (Y, U_1, U_2)$ is a fuzzy pairwise β-continuous mapping. □

Theorem 4.5. Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ be a mapping and $r, s \in I_0$. Then the following statements are equivalent:

(1) f is a fuzzy pairwise β-(r,s)-continuous mapping.

(2) $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-(r,s)-closed set of X for each $\mathcal{U}_1$-fuzzy r-closed set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-(s,r)-closed set of X for each $\mathcal{U}_2$-fuzzy s-closed set ν of Y.

(3) For each fuzzy set μ of Y,

$$f^{-1}(\mathcal{U}_1\text{-Cl}(\mu, r)) \geq \mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu), s), r), s)$$

and

$$f^{-1}(\mathcal{U}_2\text{-Cl}(\mu, s)) \geq \mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\mu), r), s), r).$$

(4) For each fuzzy set ρ of X,

$$\mathcal{U}_1\text{-Cl}(f(\rho), r) \geq f(\mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(\rho, s), r), s))$$

and

$$\mathcal{U}_2\text{-Cl}(f(\rho), s) \geq f(\mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(\rho, r), s), r)).$$

Proof. (1) ⇔ (2) It follows from Theorem 3.2.

(2) ⇒ (3) Let μ be any fuzzy set of Y. Then $\mathcal{U}_1\text{-Cl}(\mu, r)$ is a $\mathcal{U}_1$-fuzzy r-closed set and $\mathcal{U}_2\text{-Cl}(\mu, s)$ is a $\mathcal{U}_2$-fuzzy s-closed set of Y. By (2), $f^{-1}(\mathcal{U}_1\text{-Cl}(\mu, r))$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy β-(r,s)-closed set and $f^{-1}(\mathcal{U}_2\text{-Cl}(\mu, s))$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy β-(s,r)-closed set of X. Thus

$$\begin{aligned} f^{-1}(\mathcal{U}_1\text{-Cl}(\mu, r)) &\geq \mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mathcal{U}_1\text{-Cl}(\mu, r)), s), r), s) \\ &\geq \mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu), s), r), s) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\mathcal{U}_2\text{-Cl}(\mu, s)) &\geq \mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\mathcal{U}_2\text{-Cl}(\mu, s)), r), s), r) \\ &\geq \mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\mu), r), s), r). \end{aligned}$$

(3) ⇒ (4) Let ρ be any fuzzy set of X. Then $f(\rho)$ is a fuzzy set of Y. By (3),

$$\begin{aligned} f^{-1}(\mathcal{U}_1\text{-Cl}(f(\rho), r)) &\geq \mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}f(\rho), s), r), s) \\ &\geq \mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(\rho, s), r), s) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\mathcal{U}_2\text{-Cl}(f(\rho), s)) &\geq \mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}f(\rho), r), s), r) \\ &\geq \mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(\rho, r), s), r). \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{U}_1\text{-Cl}(f(\rho), r) &\geq ff^{-1}(\mathcal{U}_1\text{-Cl}(f(\rho), r)) \\ &\geq f(\mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(\rho, s), r), s)) \end{aligned}$$

and

$$\begin{aligned}\mathcal{U}_2\text{-Cl}(f(\rho),s) &\geq ff^{-1}(\mathcal{U}_2\text{-Cl}(f(\rho),s))\\ &\geq f(\mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(\rho,r),s),r)).\end{aligned}$$

(4) $\Rightarrow$ (2) Let μ be any $\mathcal{U}_1$-fuzzy r-closed set and ν any $\mathcal{U}_2$-fuzzy s-closed set of Y. Then $f^{-1}(\mu)$ and $f^{-1}(\nu)$ are fuzzy sets of X. By (4),

$$\begin{aligned}\mu &= \mathcal{U}_1\text{-Cl}(\mu,r)\\ &\geq \mathcal{U}_1\text{-Cl}(ff^{-1}(\mu),r)\\ &\geq f(\mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu),s),r),s))\end{aligned}$$

and

$$\begin{aligned}\nu &= \mathcal{U}_2\text{-Cl}(\nu,s)\\ &\geq \mathcal{U}_2\text{-Cl}(ff^{-1}(\nu),s)\\ &\geq f(\mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\nu),r),s),r)).\end{aligned}$$

Thus

$$\begin{aligned}f^{-1}(\mu) &\geq f^{-1}f(\mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu),s),r),s))\\ &\geq \mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu),s),r),s)\end{aligned}$$

and

$$\begin{aligned}f^{-1}(\nu) &\geq f^{-1}f(\mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\nu),r),s),r))\\ &\geq \mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\nu),r),s),r).\end{aligned}$$

Therefore $f^{-1}(\mu)$ is a $(\mathcal{T}_1,\mathcal{T}_2)$-fuzzy β-(r,s)-closed set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2,\mathcal{T}_1)$-fuzzy β-(s,r)-closed set of X. □

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