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The Intuitionistic Fuzzy Normal Subgroup

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Abstract

In this paper we continue the study of intuitionistic fuzzy groups by introducing the notion of intuitionistic fuzzy normal subgroup based on intuitionistic fuzzy space as a generalization of fuzzy normal subgroup.

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1. Introduction

The theory of intuitionistic fuzzy set is expected to play an important role in modern mathematics in general as it represents a generalization of fuzzy set. The notion of intuitionistic fuzzy set was first defined by Atanassov [3] as a generalization of Zadeh's [10] fuzzy set. After the concept of intuitionistic fuzzy set was introduced, several papers have been published by mathematicians to extend the classical mathematical concepts and fuzzy mathematical concepts to the case of intuitionistic fuzzy sets. The difficulty in such generalizations lies in how to pick out the rational generalization from the large number of available approaches.

The study of fuzzy groups was first started with the introduction of the concept of fuzzy subgroups by Rosenfeld [9]. In 1979 Anthony and Sherwood [2] redefined fuzzy subgroups using the concept of triangular norm. In his remarkable paper Dib [4] introduced a new approach to define fuzzy groups using his definition of fuzzy space which serves as the universal set in classical group theory. Dib remarked the absence of the fuzzy universal set and discussed some problems in Rosenfeld's approach.

The notion of fuzzy normal subgroup was first initiated by Abdul Razak [1] to continue the theory of fuzzy groups obtained by Dib. In 1998 Dib and Hassan introduced and discussed the fuzzy normal group in a similar manner to Razak. In this paper we continue the study of intuitionistic fuzzy groups by generalizing the notion of the fuzzy normal subgroup to the intuitionistic fuzzy normal subgroup case.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions required in the sequel.

Let $L = I \times I$, where $I = [0,1]$. Define a partial order on L , in terms of the partial order on I , as follows: For every $(r_1, r_2), (s_1, s_2) \in L$

1. $(r_1, r_2) \le (s_1, s_2)$ iff $r_1 < s_1, r_2 < s_2$ (or $r_1 = s_1$ and $r_2 = s_2$) whenever $s_1 \ne 0 \ne s_2$.
2. $(0,0) = (s_1, s_2)$ whenever $s_1 = 0 = s_2$.

Thus the cartesian product $L = I \times I$ is a distributive, not complemented lattice. The operation of infimum and supremum in L are given respectively by

$(r_1, r_2) \wedge (s_1, s_2) = (r_1 \wedge s_1, r_2 \wedge s_2)$ and

$(r_1, r_2) \vee (s_1, s_2) = (r_1 \vee s_1, r_2 \vee s_2)$.

Definition 2.1 *(Atanassov [3]).* Let X be a nonempty fixed set. An intuitionistic fuzzy set A is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership and the degree of nonmembership respectively of each element $x \in X$ to the set A , and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in X$.

Remark 2.2 The intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in X will be denoted by $A = \{\langle x, \underline{A}(x), \overline{A}(x)\rangle : x \in X\}$ or simply $A = \left(x, \underline{A}(x), \overline{A}(x)\right)$ where $\underline{A}(x) = \mu_A(x)$ and $\overline{A}(x) = \nu_A(x)$.

The support of the intuitionistic fuzzy set $A = \{\langle x, \underline{A}(x), \overline{A}(x)\rangle : x \in X\}$ in X is the subset $A_\circ$ of X defined by:

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$A_\circ = \{x \in X : \underline{A}(x) \neq 0 \text{ and } \overline{A}(x) \neq 1\}$.

Definition 2.3 Let X be a nonempty set. An intuitionistic fuzzy space (simply IFS) denoted by (X,I,I) is the set of all ordered triples (x,I,I), where $(x,I,I) = \{(x,r,s) : r,s \in I$ with $r+s \leq 1$ and $x \in X\}$. The ordered triplet (x,I,I) is called an intuitionistic fuzzy element of the intuitionistic fuzzy space (X,I,I) and the condition $r,s \in I$ with $r+s \leq 1$ will be referred to as the ``intuitionistic condition".

Therefore an intuitionistic fuzzy space is an (ordinary) set with ordered triples. In each triplet the first component indicates the (ordinary) element while the second and the third components indicate its set of possible membership and nonmembership values respectively.

Definition 2.4 Let $U_\circ$ be a given subset of X. An intuitionistic fuzzy subspace U of the IFS (X,I,I) is the collection of all ordered triples $(x,\underline{u}_x,\overline{u}_x)$, where $x \in U_\circ$ and $\underline{u}_x,\overline{u}_x$ are subsets of I such that $\underline{u}_x$ contains at least one element beside the zero element and $\overline{u}_x$ contains at least one element beside the unit. If $x \notin U_\circ$, then $\underline{u}_x = 0$ and $\overline{u}_x = 1$. The ordered triple $(x,\underline{u}_x,\overline{u}_x)$ will be called an intuitionistic fuzzy element of the intuitionistic fuzzy subspace U. The empty fuzzy subspace denoted by ϕ is defined to be $\phi = \{(x,I,I) : x \in \phi\}$.

Remark 2.5 For the sake of simplicity throughout this paper by saying an intuitionistic fuzzy space X we mean the intuitionistic fuzzy space (X,I,I).

Let $U = \{(x,\underline{u}_x,\overline{u}_x) : x \in U_\circ\}$ and $V = \{(x,\underline{v}_x,\overline{v}_x) : x \in V_\circ\}$ be two intuitionistic fuzzy subspaces of the intuitionistic fuzzy space X. The union, intersection and difference between the intuitionistic fuzzy subspaces U and V are given respectively as follows:

$$U \cup V = \{(x,\underline{u}_x \cup \underline{v}_x,\overline{u}_x \cap \overline{v}_x) : x \in U_\circ \cup V_\circ\},$$

$$U \cap V = \{(x,\underline{u}_x \cap \underline{v}_x,\overline{u}_x \cup \overline{v}_x) : x \in U_\circ \cap V_\circ\},$$

$$U - V = \{(x,\{0\} \cup (\underline{u}_x - \underline{v}_x),\{1\} \cup (\overline{u}_x - \overline{v}_x)) : x \in U_\circ - V_\circ\}.$$

Definition 2.6 Given any two nonempty intuitionistic fuzzy spaces X and Y. The intuitionistic fuzzy cartesian product denoted by $(X,I,I) \boxtimes (Y,I,I)$ is defined as:

$$(X,I,I) \boxtimes (Y,I,I) = \{((x,y),(r_1,r_2),(s_1,s_2)) : x \in X, y \in Y \text{ and } (r_1,r_2),(s_1,s_2) \in I \times I\}$$

Definition 2.7 An intuitionistic fuzzy relation ρ from an IFS X to an IFS Y is a subset of the intuitionistic fuzzy cartesian product $(X,I,I) \boxtimes (Y,I,I)$. An intuitionistic fuzzy relation from an IFS X into itself is called an intuitionistic fuzzy relation in the IFS X.

Definition 2.8 An intuitionistic fuzzy function between two intuitionistic fuzzy spaces X and Y is an intuitionistic fuzzy relation $\boldsymbol{F}$ from X to Y satisfying the following conditions:

1. For every $x \in X$ with $r,s \in I$, there exists a unique element $y \in Y$ with $w,z \in I$; such that $((x,y),(r,w),(s,z)) \in A$ for some $A \in \boldsymbol{F}$.
2. If $((x,y),(r_1,w_1),(s_1,z_1)) \in A \in \boldsymbol{F}$, and $((x,y'),(r_2,w_2),(s_2,z_2)) \in B \in \boldsymbol{F}$ then $y = y'$
3. If $((x,y),(r_1,w_1),(s_1,z_1)) \in A \in \boldsymbol{F}$, and $((x,y),(r_2,w_2),(s_2,z_2)) \in B \in \boldsymbol{F}$ then $(r_1 > r_2)$ implies $(w_1 > w_2)$ and $(s_1 > s_2)$ implies $(z_1 > z_2)$.
4. If $((x,y),(r,w),(s,z)) \in A \in \boldsymbol{F}$, then $r=0$ implies $w=0$, $s=1$ implies $z=0$, and $r=1$ implies $w=1$, $s=0$ implies $z=1$.

Thus conditions (1) and (2) imply that there exists a unique (ordinary) function from X to Y, namely $F : X \to Y$ and that for every $x \in X$ there exists unique (ordinary) functions from I to I, namely $\underline{f}_x,\overline{f}_x : I \to I$. On the other hand conditions (3) and (4) are respectively equivalent to the following conditions:

(i) $\underline{f}_x$ is nondecreasing on I and $\overline{f}_x$ is nonincreasing on I.

(ii) $\underline{f}_x(0) = 0 = \overline{f}_x(1)$ and $\underline{f}_x(1) = 1 = \overline{f}_x(0)$.

That is, an intuitionistic fuzzy function between two intuitionistic fuzzy spaces X and Y is a function $\boldsymbol{F}$ from X to Y characterized by the ordered triple

$$\left(F(x),\{\underline{f}_x\}_{x\in X},\{\overline{f}_x\}_{x\in X}\right),$$

where $F(x)$ is a function from X to Y and $\{\underline{f}_x\}_{x\in X},\{\overline{f}_x\}_{x\in X}$ are family of functions from I to I satisfying the conditions (i) and (ii) such that the image of any intuitionistic fuzzy subset A of the IFS X under $\boldsymbol{F}$ is the intuitionistic fuzzy subset $\boldsymbol{F}(A)$ of the IFS Y defined by

$$F(A)y = \begin{cases} \left(\bigvee_{x\in F^{-1}(y)} \underline{f}_x(\mu_A(x)), \bigwedge_{x\in F^{-1}(y)} \overline{f}_x(\nu_A(x))\right) & if\ F^{-1}(y) \neq \phi \\ (0,1) & if\ F^{-1}(y) = \phi \end{cases}$$

We will call the functions $\underline{f}_x,\overline{f}_x$ the *comembership functions* and the *cononmembership functions* respectively. The intuitionistic fuzzy function F will be denoted by

$$F = (F, \underline{f}_x, \overline{f}_x).$$

An intuitionistic fuzzy binary operation F on an IFS (X,I,I) is an intuitionistic fuzzy function $F : X \times X \to X$ with comembership functions $\underline{f}_{xy}$ and cononmembership functions $\overline{f}_{xy}$ satisfying:

1. $\underline{f}_{xy}(r,s) \neq 0$ iff $r \neq 0$ and $s \neq 0$, and $\overline{f}_{xy}(w,z) \neq 1$ iff $w \neq 1$ and $z \neq 1$.
2. $\underline{f}_{xy}, \overline{f}_{xy}$ are onto. That is, $\underline{f}_{xy}(I \times I) = I$ and $\overline{f}_{xy}(I \times I) = I$.

Thus for any two intuitionistic fuzzy elements $(x,I,I),(y,I,I)$ of the IFS X and any intuitionistic fuzzy binary operation $F = \left(F, \underline{f}_{xy}, \overline{f}_{xy}\right)$ defined on an IFS X, the action of the intuitionistic fuzzy binary operation ***F*** $= \left(F, \underline{f}_{xy}, \overline{f}_{xy}\right)$ over the IFS X is given by

$$(x,I,I)F(y,I,I) = F((x,I,I),(y,I,I))$$

$$= (F(x,y), \underline{f}_{xy}(I \times I), \overline{f}_{xy}(I \times I))$$

$$= (F(x,y), I, I).$$

An intuitionistic fuzzy binary operation is said to be *uniform* if the associated comembership and cononmembership functions are identical. That is, if $\underline{f}_{xy} = \underline{f}, \overline{f}_{xy} = \overline{f}$ for all $x,y \in X$. A left semiuniform (right semiuniform) fuzzy binary operation is an intuitionistic fuzzy binary operation having identical comembership functions (cononmembership functions).

An intuitionistic fuzzy groupoid, denoted by $((X,I,I),F)$, is an IFS (X,I,I) together with an intuitionistic fuzzy binary operation F defined over it. A uniform (left semiuniform, right semiuniform) intuitionistic fuzzy groupoid is an intuitionistic fuzzy groupoid with uniform (left semiuniform, right semiuniform) intuitionistic fuzzy binary operation. The ordered pair $(U;F)$ is said to be an intuitionistic fuzzy subgroupoid of the intuitionistic fuzzy groupoid $((X,I,I),F)$ iff U is an intuitionistic fuzzy subspace of the intuitionistic fuzzy space X and U is closed under the intuitionistic fuzzy binary operation ***F***. An intuitionistic fuzzy semigroup is an intuitionistic fuzzy groupoid where ***F*** is associative. That is, for any choice of $(x,I,I),(y,I,I),(z,I,I) \in ((G,I,I),F)$:

$$\left((x,I,I)F(y,I,I)\right)F(z,I,I) = (x,I,I)F\left((y,I,I)F(z,I,I)\right)$$

.

An intuitionistic fuzzy monoid is an intuitionistic fuzzy semigroup that admits an identity. That is, there exists an intuitionistic fuzzy element $(e,I,I) \in (G,I,I)$ such that for all (x,I,I) in $((G,I,I),F)$:

$$(e,I,I)F(x,I,I) = (x,I,I)F(e,I,I) = (x,I,I).$$

An intuitionistic fuzzy group is an intuitionistic fuzzy monoid in which each intuitionistic fuzzy element has an inverse. That is, for every intuitionistic fuzzy element (x,I,I) in $((G,I,I),F)$ there exists an intuitionistic fuzzy element (x^{-1},I,I) in $((G,I,I),F)$ such that:

$$(x,I,I)F(x^{-1},I,I) = (x^{-1},I,I)F(x,I,I) = (e,I,I).$$

An intuitionistic fuzzy group $((G,I,I),F)$ is called an abelian (commutative) intuitionistic fuzzy group if and only if for all $(x,I,I),(y,I,I) \in ((G,I,I),F)$

$$(x,I,I)F(y,I,I) = (y,I,I)F(x,I,I).$$

If S is an intuitionistic fuzzy subspace of the IFS (G,I,I), then the ordered pair $(S;F)$ is called an intuitionistic fuzzy subgroup of the IFG $((G,I,I),F)$, denoted by $(S;F) \leq ((G,I,I),F)$, if $(S;F)$ defines an IFG under the intuitionistic fuzzy binary operation F.

3. Intuitionistic fuzzy normal subgroup

In this section we will introduce the notion of the associated intuitionistic fuzzy subgroup and then we define the intuitionistic fuzzy normal subgroup.

Let $((G,I,I),F)$ be an intuitionistic fuzzy group having the intuitionistic fuzzy subgroup $(U;F)$. Similar to the fuzzy case and on the contrary to the ordinary case, intuitionistic fuzzy elements $(x, \underline{u}_x, \overline{u}_x)$ of the intuitionistic fuzzy subgroup $(U;F)$ are *not necessary* associative with intuitionistic fuzzy elements (x,I,I) of the intuitionistic fuzzy group $((G,I,I),F)$. That is

$$\alpha F(\beta F\gamma) \neq (\alpha F\beta)F\gamma,$$

where α, β and γ are some intuitionistic fuzzy elements of U or (G,I,I) such that one or two of these intuitionistic fuzzy elements belong to U.

Example 3.1 Let $X = \{-1, 1, -i, i\}$. Define the intuitionistic fuzzy binary operation $F = \left(F, \underline{f}_{xy}, \overline{f}_{xy}\right)$ on (X,I,I) such that $F : X \times X \to X$ is the ordinary multiplication of complex numbers and the comembership and cononmembership functions have the following forms:

$$\underline{f}_{11}(r,s) = \begin{cases} \dfrac{rs}{\alpha} & \text{if } rs \leq \alpha^2 \\ 1 + \dfrac{rs-1}{1+\alpha} & \text{if } rs > \alpha^2, \end{cases}$$

$$\overline{f}_{11}(r,s) = \begin{cases} 1-\dfrac{rs}{\alpha} & if\ rs \le \alpha^2 \\ \dfrac{1-rs}{1+\alpha} & if\ rs > \alpha^2 \end{cases}$$

$$\underline{f}_{-11}(r,s) = \underline{f}_{1-1}(r,s) = \begin{cases} \dfrac{rs}{\alpha} & if\ rs \le \alpha\beta \\ 1+\dfrac{1-\beta}{1-\alpha\beta}(rs-1) & if\ rs > \alpha\beta \end{cases},$$

$$\overline{f}_{-11}(r,s) = \overline{f}_{1-1}(r,s) = \begin{cases} 1-\dfrac{rs}{\alpha} & if\ rs \le \alpha\beta \\ \dfrac{1-\beta}{1-\alpha\beta}(1-rs) & if\ rs > \alpha\beta \end{cases},$$

and the other comembership and cononmembership functions are defined by the product rs, where α,β are given fixed real numbers satisfying $0<\beta<\alpha<1$.

It is easy to check that $((X,I,I),F)$ is an intuitionistic fuzzy group. Also the intuitionistic fuzzy subspace $U=\{(-1,[0,\beta],[\beta,1]),(1,[0,\alpha],[\alpha,1])\}$ together with the intuitionistic fuzzy binary operation F define an intuitionistic fuzzy subgroup of $((X,I,I),F)$. Now

$$((1,I,I)F(1,[0,\alpha],[\alpha,1]))F(-1,[0,\beta],[\beta,1]) =$$

$$\left(-1,\left[0,1-\frac{1-\beta}{1-\alpha\beta}\left(1-\frac{2\alpha\beta}{1+\alpha}\right)\right],\left[\frac{1-\beta}{1-\alpha\beta}\left(1-\frac{\beta-\alpha\beta}{1+\alpha}\right),1\right]\right).$$

On the other hand

$$(1,I,I)F((1,[0,\alpha],[\alpha,1])F(-1,[0,\beta],[\beta,1])) =$$

$$\left(-1,\left[0,1-\frac{(1-\beta)^2}{1-\alpha\beta}\right],\left[1-\frac{(1-\beta)^2}{1-\alpha\beta},1\right]\right).$$

That is, intuitionistic fuzzy elements of U are not associative with intuitionistic fuzzy elements of (X,I,I).

Definition 3.1 An intuitionistic fuzzy subgroup $(U;F)$ of an intuitionistic fuzzy group $((G,I,I),F)$ is said to be associative in $((G,I,I),F)$ if the intuitionistic fuzzy elements of U are associative with intuitionistic fuzzy elements of $((G,I,I),F)$, i.e.

$$\alpha F(\beta F\gamma) = (\alpha F\beta)F\gamma$$

for arbitrary choices of intuitionistic fuzzy elements α,β and γ of U and (G,I,I).

Example 3.2 Let $X=\{-1,1,-i,i\}$. Define the intuitionistic fuzzy binary operation $F=\left(F,\underline{f}_{xy},\overline{f}_{xy}\right)$ on (X,I,I) such that $F: X\times X\to X$ is the ordinary multiplication of complex numbers and the comembership and cononmembership functions are given respectively for all $x,y\in X$ by $\underline{f}_{xy}(r,s)=r\wedge s$ and $\overline{f}_{xy}(r,s)=r\wedge s.$

Let $U=\{(-1,[0,\frac{1}{2}],[\frac{1}{2},1]),(1,[0,\frac{1}{2}],[\frac{1}{2},1])\}$ be an intuitionistic fuzzy subspace of the IFS (X,I,I). It is easy to see that the U defines an associative intuitionistic fuzzy subgroup of $((X,I,I),F)$.

Following the above definitions and examples we have the following interesting result regarding associative intuitionistic fuzzy subgroups.

Theorem 3.1 Let $((G,I,I),F)$ be an intuitionistic fuzzy group and $F=\left(F,\underline{f},\overline{f}\right)$ be a uniform intuitionistic fuzzy binary operation such that $\underline{f}(r,1)=\underline{f}(1,r)=r$ and $\overline{f}(r,1)=\overline{f}(1,r)=r$. Then every intuitionistic fuzzy subgroup of $((G,I,I),F)$ is an associative intuitionistic fuzzy subgroup in $((G,I,I),F)$.

Proof. The proof is straightforward using the properties of $\underline{f}$ and $\overline{f}$.

Corollary 3.1 *Let $((G,I,I),F)$ be an intuitionistic fuzzy group and $F=\left(F,\underline{f},\overline{f}\right)$ be a uniform intuitionistic fuzzy binary operation. If $\underline{f}$ and $\overline{f}$ are t-norm functions then every intuitionistic fuzzy subgroup of the intuitionistic fuzzy group $((G,I,I),F)$ is an associative intuitionistic fuzzy subgroup.*

Before introducing the intuitionistic fuzzy normal subgroup we will define now the notion of left (right) coset of intuitionistic fuzzy subgroup.

Definition 3.2 If $U=\{(x,u_x): x\in U_\circ\}$ is an intuitionistic fuzzy subgroup of the intuitionistic fuzzy group $((G,I,I),F)$ then for every intuitionistic fuzzy element (x,I,I) of (G,I,I), the intuitionistic fuzzy subspace defined by

$$(x,I,I)U = (x,I,I)FU = \{(xFz,\underline{f}_{xz}(I,u_z),\overline{f}_{xz}(I,u_z))\},$$

is called a left coset of the intuitionistic fuzzy subgroup $(U;F)$. A right coset of the intuitionistic fuzzy subgroup $(U;F)$ is defined by the intuitionistic fuzzy subspace

$$U(x,I,I) = UF(x,I,I) = \{(zFx, \underline{f}_{zx}(u_z, I), \overline{f}_{zx}(u_z, I))\}.$$

Theorem 3.2 *For any associative intuitionistic fuzzy subgroup* $(U;F)$ *of the intuitionistic fuzzy group* $((G,I,I),F)$ *the following hold*

1. $(x,I,I)U = (h,\underline{I}_h,\overline{I}_h)U$ for every intuitionistic fuzzy element $(h,\underline{I}_h,\overline{I}_h) \in (x,I,I)U$ where $\underline{I}_h,\overline{I}_h$ denote the possible membership and nonmbership values of h respectively.
2. There is a one-to-one correspondence between any two left (right) cosets of the intuitionistic fuzzy subgroup $(U;F)$.
3. There is a one-to-one correspondence between the family of right cosets and the family of left cosets of the intuitionistic fuzzy subgroup $(U;F)$.
4. Any two right cosets (left cosets) of the intuitionistic fuzzy subgroup $(U;F)$ are identical or disjoint intuitionistic fuzzy subspaces.

Proof. (i) Let $(h,\underline{I}_h,\overline{I}_h)$ be any intuitionistic fuzzy element in $(x,I,I)U$ then $(h,\underline{I}_h,\overline{I}_h) = (x,I,I)(y,\underline{u}_y,\overline{u}_y)$ for some $y \in U_\circ$. If $(z,\underline{u}_z,\overline{u}_z)$ is an arbitrary element of U then

$$(x,I,I)(z,\underline{u}_z,\overline{u}_z) = (x,I,I)\left((y,\underline{u}_y,\overline{u}_y)(y^{-1},\underline{u}_{y^{-1}},\overline{u}_{y^{-1}})\right)(z,\underline{u}_z,\overline{u}_z)$$

$$= \left((x,I,I)(y,\underline{u}_y,\overline{u}_y)\right)\left((y^{-1},\underline{u}_{y^{-1}},\overline{u}_{y^{-1}})(z,\underline{u}_z,\overline{u}_z)\right) \in (h,\underline{I}_h,\overline{I}_h)U.$$

(ii) Let $(x,I,I)U$ and $(y,I,I)U$ be any two left cosets of the intuitionistic fuzzy group U, then $(x,I,I)(z,\underline{u}_z,\overline{u}_z) \leftrightarrow (y,I,I)(z,\underline{u}_z,\overline{u}_z)$ is the required one-to-one correspondence between $(x,I,I)U$ and $(y,I,I)U$. For the right cosets we use the same arrangement.

(iii) Let $\{(x,I,I)U : x \in G\}$ and $\{U(x,I,I) : x \in G\}$ denotes the family of left and right cosets respectively of the intuitionistic fuzzy group U, then the required one-to-one correspondence is define by

$$(x,I,I)U \leftrightarrow U(x,I,I).$$

(iv) Let $(x,I,I)U$ and $(y,I,I)U$ be any two intersecting left cosets of the intuitionistic fuzzy subgroup U, then there exists $\alpha,\beta \in U_\circ$ such that

$$(x,I,I)(\alpha,\underline{u}_\alpha,\overline{u}_\alpha) = (y,I,I)(\beta,\underline{u}_\beta,\overline{u}_\beta).$$

Choose any intuitionistic fuzzy element $(x,I,I)(z,\underline{u}_z,\overline{u}_z) \in (x,I,I)U$ then

$$(x,I,I)(z,\underline{u}_z,\overline{u}_z) = (x,I,I)\left((\alpha,\underline{u}_\alpha,\overline{u}_\alpha)(\alpha^{-1},\underline{u}_{\alpha^{-1}},\overline{u}_{\alpha^{-1}})\right)(z,\underline{u}_z,\overline{u}_z)$$

$$= \left((x,I,I)(\alpha,\underline{u}_\alpha,\overline{u}_\alpha)\right)\left((\alpha^{-1},\underline{u}_{\alpha^{-1}},\overline{u}_{\alpha^{-1}})(z,\underline{u}_z,\overline{u}_z)\right)$$

$$= \left((y,I,I)(\beta,\underline{u}_\beta,\overline{u}_\beta)\right)\left((\alpha^{-1},\underline{u}_{\alpha^{-1}},\overline{u}_{\alpha^{-1}})(z,\underline{u}_z,\overline{u}_z)\right)$$

$$= (y,I,I)\left((\beta,\underline{u}_\beta,\overline{u}_\beta)(\alpha^{-1},\underline{u}_{\alpha^{-1}},\overline{u}_{\alpha^{-1}})\right)(z,\underline{u}_z,\overline{u}_z) \in (y,I,I)U.$$

That is $(x,I,I)U \subset (y,I,I)U$. Similarly we can show that $(y,I,I)U \subset (x,I,I)U$ and also we can show the same result for the right cosets of U which proves (iv).

Definition 3.3 An intuitionistic fuzzy subgroup U of the intuitionistic fuzzy group $((G,I,I),F)$ is called an intuitionistic fuzzy normal subgroup if

1. U is associative in $((G,I,I),F)$,
2. $(x,I,I)U = (x,I,I)U$, for all $x \in G$.

Example 3.3 (1) Let $((X,I,I),F)$ be as defined in Example 3.2, then one can easily check that the intuitionistic fuzzy subspace $U = \{(-1,[0,\frac{1}{2}],[\frac{1}{2},1]),(1,[0,\frac{1}{2}],[\frac{1}{2},1])\}$ together with the intuitionistic fuzzy binary operation F define an intuitionistic fuzzy normal subgroup of $((X,I,I),F)$.

(2) Let $X = S_3$ be the set of all permutations on $\{1,2,3\}$. Define the intuitionistic fuzzy binary operation $F = \left(F,\underline{f},\overline{f}\right)$ over the intuitionistic fuzzy space (X,I,I) where F is the ordinary composition of permutations and $\underline{f}_{xy}(r,s) = r \wedge s$, $\overline{f}_{xy}(r,s) = r \vee s$ for all $x,y \in X$. Consider the intuitionistic fuzzy subspace $U = \{(\varepsilon,[0,\frac{1}{2}],[\frac{1}{2},1]),(\alpha,[0,\frac{1}{2}],[\frac{1}{2},1])\}$ where ε denotes the identity permutation and $\alpha = (12)$ .One can easily investigate that $(U;F)$ is an associative intuitionistic fuzzy subgroup which is not normal.

The next theorem gives a necessary and sufficient condition for intuitionistic fuzzy normal subgroups.

Theorem 3.3 *An intuitionistic fuzzy subgroup* $U = \{(z,\underline{u}_z,\overline{u}_z) : z \in U_\circ\}$ *of the intuitionistic fuzzy group* $((G,I,I),F)$ *is an intuitionistic fuzzy normal subgroup iff*

1. $(U_\circ,F)$ is an ordinary normal subgroup of the ordinary group (G,F).
2. $\underline{f}_{xz}(I,\underline{u}_z) = \underline{f}_{z'x}(\underline{u}_{z'},I), \overline{f}_{xz}(I,\overline{u}_z) = \overline{f}_{z'x}(\overline{u}_{z'},I) : xFz = z'Fx$ where $x \in X$ and $z,z' \in U_\circ$.

Proof. Assume $U = \{(z,\underline{u}_z,\overline{u}_z) : z \in U_\circ\}$ is a intuitionistic fuzzy normal subgroup $((G,I,I),F)$, from [8] we have $(U_\circ,F)$ is an ordinary (normal) subgroup of the ordinary group (G,F). Using the normality of U we have $(x,I,I)U = U(x,I,I)$ for all $x \in G$. That is,

$$\{(xFz, \underline{f}_{xz}(I, \underline{u}_z), \overline{f}_{xz}(I, \overline{u}_z)) : z \in U_\circ\} =$$
$$\{(zFx, \underline{f}_{zx}(\underline{u}_z.I), \overline{f}_{zx}(\overline{u}_z, I)) : z \in U_\circ\}.$$

Therefore for every $z \in U_\circ$ there exists $z' \in U_\circ$ such that $xFz = z'Fx$. In other words $xFU_\circ = U_\circ Fx$. Hence $U_\circ$ is an ordinary normal subgroup of the ordinary group (G, F) which proves (i). (ii) follow directly from the definition.

The other part of the proof is direct.

Based on the correspondence theorem [8] and using the definition of fuzzy normal subgroup [5] we can obtain the following corollary

Corollary 3.1 *An intuitionistic fuzzy subgroup* $(U;F)$ *of the intuitionistic fuzzy group* $((G,I,I),F)$ *where* $F = \left(F, \underline{f}, \overline{f}\right)$ *is an intuitionistic fuzzy normal subgroup iff* $(U;\underline{F})$ *and* $(U;\overline{F})$ *are both fuzzy normal subgroups of the fuzzy groups* $((G,I),\underline{F})$ *and* $((G,I),\overline{F})$ *respectively, where* $\underline{F} = (F, \underline{f})$ *and* $\overline{F} = (F, 1-\overline{f})$.

Theorem 3.4 Every intuitionistic fuzzy normal subgroup U of $((G,I,I),F)$ defines an equivalence relation $\mathcal{R}$ on the intuitionistic fuzzy space (G,I,I) given by

$$(x,I,I)\mathcal{R}(y,I,I) \Leftrightarrow (x,I,I)U = U(y,I,I).$$

The equivalence relation $\mathcal{R}$ on the intuitionistic fuzzy space (G,I,I) induces an equivalence relation on G by the correspondence $(x,I,I) \leftrightarrow x$. That is,

$$(x,I,I)\mathcal{R}(y,I,I) \Leftrightarrow x\mathcal{R}y,$$

which is equivalent to $xU_\circ = U_\circ y$.

Let $U = \{(z, \underline{u}_z, \overline{u}_z) : z \in U_\circ\}$ be a intuitionistic fuzzy normal subgroup of $((G,I,I),F)$ and let z_1, z_2, z_3, z_3 be elements of $U_\circ$. For all $x, y \in G$ and using the normality of U we have

$$\left((x,I,I)F(z_1,\underline{u}_{z_1},\overline{u}_{z_1})\right)F\left((y,I,I)F(z_2,\underline{u}_{z_2},\overline{u}_{z_2})\right)$$
$$= (x,I,I)F\left((z_1,\underline{u}_{z_1},\overline{u}_{z_1})F(y,I,I)\right)F(z_2,\underline{u}_{z_2},\overline{u}_{z_2})$$
$$= (x,I,I)F\left((y,I,I)F(z_3,\underline{u}_{z_3},\overline{u}_{z_3})\right)F(z_2,\underline{u}_{z_2},\overline{u}_{z_2})$$
$$= \left((x,I,I)F(y,I,I)\right)F\left((z_3,\underline{u}_{z_3},\overline{u}_{z_3})F(z_2,\underline{u}_{z_2},\overline{u}_{z_2})\right)$$
$$= (xFy,I,I)F(z_4,\underline{u}_{z_4},\overline{u}_{z_4}).$$

Therefore it follows that

$$\left((x,I,I)U\right)F\left((y,I,I)U\right) = (xFy,I,I)U, \ldots \quad (\mathbf{P}).$$

That is the intuitionistic fuzzy binary operation F over the intuitionistic fuzzy group $((G,I,I),F)$ induces a binary operation defined by $\mathbf{P}$ on the family of cosets over the intuitionistic fuzzy subgroup U. This family of cosets together with the induced binary operation $\mathbf{P}$ defines an ordinary group called the factor group of $((G,I,I),F)$ module U and denoted by $((G,I,I),F)/U$.

Theorem 3.5 If $U = \{(z, \underline{u}_z, \overline{u}_z) : z \in U_\circ\}$ is a normal intuitionistic fuzzy subgroup of the intuitionistic fuzzy group $((G,I,I),F)$ then the factor group $((G,I,I),F)/U$ is isomorphic to the factor group $(G,F)/U_\circ$ by the correspondence

$$(x,I,I)U \leftrightarrow xU_\circ.$$

4. Conclusion

In this paper, we continue the study initiated in [8] about intuitionistic fuzzy groups. In the absence of the intuitionistic fuzzy universal set, formulation of the intrinsic definition for an intuitionistic fuzzy subgroup is not evident. In this paper we define the notion of intuitionistic fuzzy normal subgroup as a genralization of fuzzy normal subgroup defined in [1, 5] using the notion intuitionistic fuzzy group based on intuitionistic fuzzy space. The use of intuitionistic fuzzy space as a universal set corrects the deviation in the definition of intuitionistic fuzzy subgroups. This concept can be considered as a new formulation of the classical theory of intuitionistic fuzzy groups.

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