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STABILITY COMPUTATION VIA GRÖBNER BASIS

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Abstract. In this article, we discuss a Gröbner basis algorithm related to the stability of algebraic varieties in the sense of Geometric Invariant Theory. We implement the algorithm with Macaulay 2 and use it to prove the stability of certain curves that play an important role in the log minimal model program for the moduli space of curves.

1. Introduction and preliminaries

In this article, we discuss a Gröbner basis algorithm related to the stability of algebraic varieties in the sense of Geometric Invariant Theory. We implement the algorithm with Macaulay 2, and give some applications to the moduli theory of curves.

Given an algebraic group G acting on a projective variety X linearized by a line bundle L, the stability of a point $x \in X$ can be determined by examining its stability with respect to one-parameter subgroups $\rho : \mathbb{G}_m \to G$. For each ρ, we let $\rho(\alpha).x$ specialize to a point $x^\star \in X$ and look at the character with which G acts on the fibre $L_{x^\star}$: If the character is negative (resp. positive, nonnegative), then x and $x^\star$ are stable (resp. unstable, semistable) with respect to ρ. The negative of this character is called the *Hilbert-Mumford index* $\mu^L(x,\rho)$ of x with respect to ρ. Assuming L is very ample, X is a closed subvariety of $\mathbb{P}^N := \mathbb{P}(\Gamma(L))$ and the Hilbert-Mumford index of x with respect to ρ admits the following simple description:

$$\mu^L(x,\rho) = -\min\{wt_\rho(x_i) \,|\, x_i(x) \neq 0\},$$

where x_i's are homogeneous coordinates of $\mathbb{P}^N$ that diagonalize the action of ρ.

While computing the Hilbert-Mumford index of a given point in a projective space is simple and does not require an algorithm, this becomes quite a daunting task if the 'point' is itself complicated, sitting inside a large projective space. Our object of study in this paper is the prime example: In many moduli

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problems, algebraic geometers use the *Hilbert scheme* that parametrizes subschemes, and describing its points is not suitable for manual computation even for relatively simple subvarieties of a projective space of reasonable size: for instance, describing Hilbert points of genus two, degree six curves in $\mathbb{P}^4$ using degree two generators would require 1365 variables!

The main algorithm in this paper uses Gröbner bases to effectively compute the Hilbert-Mumford index of the Hilbert point of a variety. The algorithm is implemented with Macaulay 2 in §2.2: Interested readers are invited to copy and paste the code `mumfordIndex` and use it to verify our computations or to carry out other stability computations. The Macaulay 2 script is available at `http://www.science.marshall.edu/hyeond`.

As an application, we use the algorithms to prove the stability (with respect to a ρ) of certain curves of genus two with cusps, which play an important role in the geometry of the moduli space of tri-canonical curves [8]. We also prove the instability of the bicanonical *elliptic bridges* (Definition 5), which was used in working out the GIT of bi-canonical curves [6, Proposition 10].

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2. Hilbert-Mumford index of Hilbert points

Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a projective variety with Hilbert polynomial P and *Hilb*, the component containing X of the Hilbert scheme parametrizing the subschemes of $\mathbb{P}^N$ that have Hilbert polynomial P. We let $[X]$ denote the Hilbert point of X in *Hilb*.

For $m \gg 0$, we have a projective embedding

$$\phi_m : Hilb \hookrightarrow \mathrm{Gr}(P(m), Sym^m V^*) \hookrightarrow \mathbb{P} := \mathbb{P}\left(\bigwedge^{P(m)} Sym^m V^*\right). \tag{2.1}$$

The *mth Hilbert point* $[X]_m$ of X is the image $\phi_m([X])$ in $\mathbb{P}$.

Let $x_0, \ldots, x_N$ be homogeneous coordinates for $\mathbb{P}^N$. Let B_m denote the monomial basis $\{x^a = \prod_{i=0}^N x_i^{a_i} \mid \sum a_i = m\}$ of $Sym^m V^*$. The exterior products

$$x^{a(1)} \wedge x^{a(2)} \wedge \cdots \wedge x^{a(P(m))}, \quad x^{a(i)} \in B_m$$

form a basis W_m for $\bigwedge^{P(m)} Sym^m V^*$.

Let $\rho'' : \mathbb{G}_m \to SL_{N+1}(k)$ be a one-parameter subgroup. If $x_0, \dots, x_N$ diagonalize the action of ρ'', that is,

$$\rho''(t) \cdot x_i = t^{r''_i} x_i, \quad r''_0 \geq \cdots \geq r''_N, \quad \sum r''_i = 0,$$

then the bases B_m and W_m diagonalizes the action of ρ'' on $Sym^m V^*$ and $\bigwedge^{P(m)} Sym^m V^*$: For $x^a := \prod_{i=0}^N x_i^{a_i}$, we have

$$\rho''(t) \cdot x^a = t^{wt_{\rho''}(x^a)} x^a, \quad wt_{\rho''}(x^a) = \sum_{i=0}^N r''_i a_i.$$

For $M = x^{a(1)} \wedge x^{a(2)} \wedge \cdots \wedge x^{a(P(m))}$,

$$\rho''(t) \cdot M = t^{wt_{\rho''}(M)} M, \quad wt_{\rho''}(M) = \sum_{j=1}^{P(m)} wt_{\rho''}(x^{a(j)}).$$

By definition, the Hilbert-Mumford index is

$$\mu([X]_m, \rho'') = \max\{-wt_{\rho''}(M) \mid M \neq 0 \text{ on } [X]_m\}. \tag{2.2}$$

Let $\rho : \mathbb{G}_m \to \mathrm{GL}_{N+1}(k)$ be the associated one-parameter subgroup with weights $r_i = r''_i - r''_N$ such that $r_0 \geq r_1 \geq \cdots \geq r_N = 0$ and $r''_i = r_i - \frac{1}{N+1} \sum r_j$. In practice, we frequently start with a 1-PS ρ of $\mathrm{GL}_{N+1}(k)$ with weight r_i and compute the Hilbert-Mumford index with respect to the 1-PS ρ' of $SL_{N+1}(k)$ with integral weights $(N+1)r_i - \sum r_j$. Note that the weights $r_i - \frac{1}{N+1} \sum r_j$ may not be integral but $\mu([X]_m, \rho'')$ still makes sense and since $\rho' = (N+1)\rho''$, the (semi)stability with respect to ρ' is equivalent to the (semi)stability with respect to ρ''.

Given $M \in W_m$, the ρ-weight and the ρ'-weight of M are related by

$$wt_{\rho'}(M) = (N+1) wt_\rho(M) - r \cdot m \cdot P(m), \tag{2.3}$$

where $r = \sum_{i=0}^N r_i$. Combining (2.2) and (2.3), we obtain

$$\mu([X]_m, \rho') = (N+1) \cdot \max\{-wt_\rho(M) \mid M \neq 0 \text{ on } [X]_m\} + r \cdot m \cdot P(m). \tag{2.4}$$

For notational convenience, we define $\mu([X]_m, \rho) := \mu([X]_m, \rho')$.

2.1. A Gröbner basis algorithm for computing the Hilbert-Mumford index

This algorithm seems to have been known to certain experts (see [1] and [2]). Indeed, Lemma 3.3 and Corollary 3.4 of [2] deal with the generic case of our Proposition 1. We write the details here in a form convenient for our application.

Given a one-parameter subgroup ρ of $\mathrm{GL}_{N+1}(k)$ such that $\rho(\alpha) \cdot x_i = \alpha^{r_i} x_i$, $r_0 \geq r_1 \geq \cdots \geq r_N = 0$, introduce the following ρ-weighted graded lexicographic order, denoted simply by '$\prec$'. This is a total order on the set of monomials $\{x^a\}$ defined by declaring that $x^a \prec x^b$ if

(1) $\deg x^a < \deg x^b$ or
(2) $\deg x^a = \deg x^b$ and $wt_\rho(x^a) < wt_\rho(x^b)$ or
(3) $\deg x^a = \deg x^b$ and $wt_\rho(x^a) = wt_\rho(x^b)$ and $a_j < b_j$, where

$$j = \min\{i \mid a_i \neq b_i\}.$$

Given $f \in S := k[x_0, \dots, x_N]$, we let $in_\prec(f)$ denote the term of f with maximal order. For an ideal I of S, we let

$$in_\prec(I) := \langle in_\prec(f) \mid f \in I \rangle.$$

Let I_X be the homogeneous ideal of a subscheme $X \subset \mathbb{P}^N$. Note that the monomials $\{x^{a(1)}, \dots, x^{a(P(m))}\}$ that are not in $in_\prec(I_X)$ form a basis of $(S/I_X)_m$ and $(S/in_\prec(I_X))_m$.

Proposition 1. *The Hilbert-Mumford index of the mth Hilbert point of X with respect to the associated one-parameter subgroup ρ' of $SL_{N+1}(k)$ is:*

$$\mu([X]_m, \rho') = -(N+1) \sum_{i=1}^{P(m)} wt_\rho(x^{a(i)}) + m \cdot P(m) \cdot \sum_{j=0}^{N} r_j, \tag{2.5}$$

where $\{x^{a(1)}, \dots, x^{a(P(m))}\}$ are degree m monomials not in $in_\prec(I_X)$

Proof. Let $Z = \{x^{b(1)}, \dots, x^{b(P(m))}\}$ be another $P(m)$-element subset of B_m that gives rise to a basis for $(S/I_X)_m$. Note that Z being a basis is equivalent to that $x^{b(1)} \wedge \cdots \wedge x^{b(P(m))}$ is nonzero on the Hilbert point $[X]_m$. Consider the normal form $\sum_{j=1}^{P(m)} c_{ij} x^{a(j)}$ of $x^{b(i)}$ determined uniquely by

$$x^{b(i)} \equiv \sum_{j=1}^{P(m)} c_{ij} x^{a(j)} \pmod{I_X}, \quad c_{ij} \in k.$$

Since both $\{x^{a(1)}, \dots, x^{a(P(m))}\}$ and Z are bases for the quotient space $(S/I_X)_m$, the matrix (c_{ij}) is invertible. This allows us to reorder $x^{b(i)}$'s as follows: Determine τ_1 by the condition that the τ_1th row contains the pivot element of the matrix (c_{ij}), where *pivot* is simply the first nonzero entry of the first column such that the corresponding minor (in the cofactor expansion along the first column) is not zero. Then we inductively define τ_j's: Given new $\{b(\tau_1), \dots, b(\tau_{j-1})\}$, we define τ_j by the condition that the τ_jth row contains the pivot of the $(P(m)-j+1) \times (P(m)-j+1)$ matrix obtained from (c_{ij}) by deleting the rows and columns that contain the first $j-1$ pivots. Our choice of τ_i's insures that after the reordering, we have a one to one correspondence $x^{a(i)} \mapsto x^{b(\tau_i)}$ between $\{x^{a(1)}, \dots, x^{a(P(m))}\}$ and Z such that

$$x^{b(\tau_i)} \equiv \sum_{k=1}^{P(m)} c'_{ik} x^{a(k)} \pmod{I_X},$$

where $c'_{ik} := c_{\tau_i k}$ and $c'_{ii} \neq 0$. It follows that

$$wt_\rho\left(in_\prec\left(g_i\right)\right) = wt_\rho(x^{b(\tau_i)}) \geq wt_\rho(x^{a(i)}), \quad g_i := x^{b(\tau_i)} - \sum_{k=1}^{P(m)} c'_{ik} x^{a(k)} \in I_X.$$

Hence

$$\sum_{i=1}^{P(m)} wt_\rho(x^{b(\tau_i)}) \geq \sum_{i=1}^{P(m)} wt_\rho(x^{a(i)})$$

and the assertion follows from (2.4). □

The proposition translates into the following stability statements:

Corollary 1. $[X]_m$ *is stable* (*resp. semistable*) *with respect to* ρ *if and only if*

$$\sum wt_\rho(x^{a(i)}) < (resp. \leq) \frac{mP(m)}{N+1} \sum r_i.$$

In terms of the corresponding one-parameter subgroup ρ' of $SL_{N+1}(k)$,

Corollary 2. $[X]_m$ *is stable* (*resp. semistable*) *with respect to* ρ' *if and only if*

$$\sum_{i=1}^{P(m)} wt_{\rho'}(x^{a(i)}) < (resp. \leq) 0.$$

The upshot of the formula (2.5) is that the monomials $x^{a(1)}, \dots, x^{a(P(m))}$ can be systematically computed by using Gröbner basis and can easily be implemented with a computer algebra system.

Moreover, considering the functoriality of the Hilbert-Mumford index and the tautological ring of the Hilbert scheme reveals that one only needs to compute the Hilbert-Mumford index for mth Hilbert points for finitely many m to obtain the Hilbert-Mumford indices for all m. The results in the remainder of the section are taken from [5].

Proposition 2. *Let* X, ρ, $\{x^{a(1)}, \dots, x^{a(P(m))}\}$ *be as before. The* filtered Hilbert function $P_{X,\rho}$ *on* $\mathbb{Z}$ *defined by*

$$P_{X,\rho}(m) = \sum_{i=1}^{P(m)} wt_{\rho'}(x^{a(i)})$$

is a polynomial in m *for* $m \gg 0$.

Proof. For $m \gg 0$, we have the Grothendieck embedding (2.1) such that $\phi_m^* \mathcal{O}(+1) = \det \pi_* \mathcal{O}_{\mathcal{X}}(m)$, where $\pi : \mathcal{X} \to Hilb$ is the universal variety. Let n be the dimension of X. There are Cartier divisors ([10]) $L_0, \dots, L_{n+1}$ on $Hilb$ such that

$$\det \pi_* \mathcal{O}_{\mathcal{X}}(m) = \sum_{i=0}^{n+1} \binom{m}{i} L_i$$

and it follows from the functoriality of the Hilbert-Mumford index that

$$\mu^{\phi_m^* \mathcal{O}(+1)}([X]_m, \rho) = \sum_{i=0}^{n+1} \binom{m}{i} \mu^{L_i}([X], \rho)$$

which is a polynomial in m. □

When this is put into practice to actually compute $\mu([X]_m, \rho)$ for all m, one needs to somehow determine M for which $P_{X,\rho}(m)$ is a polynomial for all $m \geq M$. An obviously necessary condition is that the mth Hilbert point of X be defined, which leads us to the Castelnuovo-Mumford regularity:

Proposition 3. *If X and $\lim_{t\to 0} \rho(t) \cdot X$ are M-regular, then we have*

$$P_{X,\rho}(m) = \sum_{i=0}^{n+1} \binom{m}{i} \mu^{L_i}([X], \rho)$$

for $m \geq M$.

A very useful corollary of this is:

Corollary 3. *Let $C \subset \mathbb{P}(V)$ be a projective variety, $\rho : \mathbb{G}_m \to SL(V)$ a one-parameter subgroup, and $C^\star$, the variety to which $\rho(t).C$ specializes. Suppose that C and $C^\star$ satisfy*

(1) *C (resp. $C^\star$) is connected of pure dimension one*;
(2) *$V^* \to \Gamma(\mathcal{O}_C(1))$ (resp. $\Gamma(\mathcal{O}_{C^\star}(1))$) is an isomorphism*;
(3) *$\mathcal{O}_C$ (resp. $\mathcal{O}_{C^\star}$) is 2-regular.*

Then for each $m \geq 2$ we have

(2.6)

$$\mu([C]_m, \rho) = (m-1)\left(\left(\frac{1}{2}\mu([C]_3, \rho) - \mu([C]_2, \rho)\right) m + 3\mu([C]_2, \rho) - \mu([C]_3, \rho)\right).$$

Proof. (1) and (3) together imply that $\mu([C]_m, \rho)$ is a polynomial in m for $m \geq 2$. (2) implies that $\det \pi_* \mathcal{O}_{\mathcal{X}}(1) = L_0 + L_1$ is trivial, and the formula (2.6) follows immediately. □

The conditions in the above corollary are satisfied by a large class of curves, including the *c-semistable* curves, i.e., reduced complete connected curves C such that

- C has nodes, ordinary cusps and tacnodes as singularities;
- the dualizing sheaf ω_C is ample;
- C does not have a genus-one connected subcurve that meets the rest of the curve in one point not counting multiplicity.

Corollary 4. *Let C be a bicanonical c-semistable curve, i.e., a c-semistable curve embedded by the bicanonical system $\left|\omega_C^{\otimes 2}\right|$:*

$$C \hookrightarrow \mathbb{P}\Gamma(C, \omega_C^{\otimes 2}) \simeq \mathbb{P}(V).$$

Let $C^\star$ denote the curve to which $\rho(t).C$ specializes. If $C^\star$ is also a bicanonical c-semistable curve, then for all $m \geq 2$, C is

(1) *m-Hilbert stable if and only if $\mu([C]_3, \rho) \geq 2\mu([C]_2, \rho) > 0$;*
(2) *m-Hilbert strictly semistable if and only if $\mu([C]_3, \rho) = \mu([C]_2, \rho) = 0$;*
(3) *m-Hilbert unstable if and only if $\mu([C]_3, \rho) \leq 2\mu([C]_2, \rho) < 0$.*

2.2. Macaulay 2 implementation

Here we give a Macaulay 2 [4] implementation of the algorithm according to Proposition 1. The code has

Input: A homogeneous ideal `I` of a graded ring S and a weight vector `w`.

Output: A sequence consisting of

(1) The regularity reg(`I`) of `I`;
(2) Values of the filtered Hilbert function $P_{X,\rho}(m)$ for $m < reg$(`I`), where X is the projective variety defined by `I` and ρ is the 1-PS whose weight vector is `w`;
(3) The polynomial which coincides with $P_{X,\rho}(m)$ for $m \geq reg$(`I`).

Function:

```
mumfordIndex = (I,w) -> (
S = ring I;
r = dim Proj(S/I);
regI = regularity resolution I;
MUm = (I,w,m) -> (
  S = ring I;
  N = numgens S;
  K = coefficientRing S;
  Sw = K[gens S, Weights => w, MonomialOrder => GLex];
  W = map(Sw, S, vars Sw);
  I = W(I);
  P = hilbertPolynomial I;
  inI = ideal leadTerm I;
  Sbar = Sw/inI;
  F = map(Sbar, Sw, vars Sbar);
  Bm = basis(m, Sw);
  -- Computes a basis of the degree m piece of Sw.
  Bmbar = basis(m, Sbar);
  -- Computes a basis of the degree m piece of Sbar.
  Bm = flatten entries Bm;
  PSm = #Bm;
  monomialWeight = (f) ->
   (expf = flatten exponents f;
    sum(expf, w, times));
  e = apply(0..(PSm-1),i->(if F(Bm_i)===F(0) then 0 else 1));
```

```
    TOTALWT = sum for i from 0 to PSm-1 list
    product{monomialWeight(Bm_i), e_i};
    -- Computes the total weight.
    mu = sum{product{N,-TOTALWT}, product{m, P(m), sum w}}
    -- Computes the Hilbert-Mumford index.
    );
  b = transpose matrix(QQ,table(1,r+2,(i,j)->MUm(I,w,regI + j)));
  A = matrix(QQ,table(r+2,r+2, (i,j) -> (regI + i)^j));
  c = A^(-1)*b;
  QQ[m];
  fHilbFun = sum(r+2, i->c_(i,0)*m^i);
  -- Computes the filtered Hilbert polynomial.
  if regI > 2 then
   val = apply(i = 2..(regI-1), i->MUm(I,w,i))
    else val = ();
  print(regI, val, fHilbFun)
  )
```

Remark 1. The subprogram `MUm` computes $\mu([X]_m, \rho)$ for a given m. After running `MUm`, the initial ideal $in_{\prec}(I)$ and the monomial basis $\{x^{a(1)}, \dots, x^{a(P(m))}\}$ for $(S/I)_m$ can be retrieved with the commands `inI` and `Bmbar`, respectively.

2.3. State polytopes

In [2], Bayer and Morrison considered the weight polytope of the mth Hilbert point $[I]_m := \bigwedge^{P(m)} Sym^m V^*/I_m$: For a fixed maximal torus $H \subset SL_{N+1}(k)$, the weight polytope is simply the convex hull of the characters of H that appear in the weight decomposition. This is called the (mth) *state polytope* of I and is denoted by $State_m(I)$. The main theorem of [2] says that the vertices of $State_m(I)$ are precisely

$$\sum_{x^a \in (in_{\prec} I)_m} a, \quad \prec \text{ a monomial order.}$$

Let ρ be a 1-PS of $SL_{N+1}(k)$ with weight vector w. It follows from the definition of $State_m(I)$ that

$$\mu([I]_m, \rho) = \max\{-w.v \mid v \text{ a vertex of } State_m(I)\}$$

and Proposition 1 says that the maximum is achieved precisely at the vertex associated to $in_{\prec_w}(I)$, where $\prec_w$ is the w-weighted lexicographic total order on the monomials.

State polytopes have received deserved attention after the fundamental work [1] and [2]. Especially of our interest is [9] which proves that the Chow polytope can be realized as a suitable limit of the state polytopes. The Chow polytope $Chow(I)$ of an ideal I is the weight polytope (with respect to a maximal torus) of the Chow form $Ch(I)$. The precise statement is

$$\lim_{m\to\infty} \frac{(n+1)!}{m^{n+1}} State_m(I) = Chow(I),$$

where n is the dimension of the projective variety defined by I. Let $w \in \mathbb{R}^{N+1}$ and consider the linear functional $L_w(x) = -w.x$. Since L_w achieves its maximum on $State_m(I)$ at the vertex associated to $in_{\prec_w}(I)_m$, its maximum on $Chow(I)$ is achieved at

$$\lim_{m\to\infty} \frac{(n+1)!}{m^{n+1}} \sum_{x^a \in in_{\prec_w}(I)_m} a.$$

It follows that:

Corollary 5 ([11], Corollary 3.5)**.** *For any 1-PS ρ of $SL_{N+1}(k)$, we have*

$$\lim_{m\to\infty} \frac{(n+1)!}{m^{n+1}} \mu([I]_m, \rho) = \mu(Ch(I), \rho).$$

In particular, if a projective variety is asymptotically Hilbert semistable, then it is Chow semistable.

3. Applications

In this section, we shall give two concrete applications of the algorithm developed in the previous section. These examples played important roles in our work on moduli problems of ν-canonical curves for $\nu = 2, 3$ [7, 8, 6, 5].

3.1. The rational bicuspidal curve of genus two

When constructing a moduli space, one hopes to avoid objects with infinite automorphisms as the moduli space often fails to be separated at such points. Fortunately, such objects are often destablized by a one-parameter subgroups of the automorphism group. However, if the object is not destabilized by one of these subgroups, then it has a rather good chance of being semistable. In the moduli problem of tri-canonical curves of genus two [8], the rational curve C_0 with two cusps and no other singularities turns out to be at the focal point of the whole problem: It is the only *pseudo-stable curve* ([12]) with infinite automorphisms to which other cuspidal pseudo-stable curves specialize under the action of $\mathrm{Aut}(C_0)$ (Figure 1).

In this section, we test C_0 against the one-parameter subgroups ρ coming from $\mathrm{Aut}(C_0)$ and show that it is Hilbert strictly semistable with respect to ρ. Then we prove in §3.2 that $[C_0]_m$ is the flat limit of the families $\{\rho(\alpha) \cdot [C']_m\}$ where C' is any other pseudo-stable cuspidal curve. This implies that all cuspidal curves are strictly semistable with respect to ρ, and that all such curves are semistable if one of them is. The results in this section appeared without computational details in [8] where we proved that these curves are semistable using a standard degeneration argument.

We first find a normalization map for C_0 from the classical projective geometric construction of a cusp. Let $\nu_6(\mathbb{P}^1)$ denote the rational sextic curve

$$\begin{array}{rcl} \nu_6 : \mathbb{P}^1 & \to & \mathbb{P}^6 \\ [s,t] & \mapsto & \left[s^6, s^5t, s^4t^2, s^3t^3, s^2t^4, st^5, t^6\right]. \end{array}$$

To create a cusp at $\nu_6(0) = [0,\dots,0,1]$, we project $\nu_6(\mathbb{P}^1)$ from $[0,\dots,0,1,0]$ on the tangent line $T_{\nu_6(0)}\nu_6(\mathbb{P}^1) = \{X_0 = \cdots = X_4 = 0\}$. The image under this projection is

$$\pi_{[0,\dots,0,1,0]} \circ \nu_6(\mathbb{P}^1) = \{[s^6, s^5t, s^4t^2, s^3t^3, s^2t^4, t^6] \mid [s,t] \in \mathbb{P}^1\}.$$

We successively project C' from $[0,1,0,\dots,0] \in T_{[1,0,\dots,0]}C' = \{X_2 = \cdots = X_5 = 0\}$ and get

$$C_0 = \{[s^6, s^4t^2, s^3t^3, s^2t^4, t^6] \mid [s,t] \in \mathbb{P}^1\}$$

which has ordinary cusps at $[0,\dots,0,1]$ and $[1,0,\dots,0]$. From this, it is clear that C_0 admits automorphisms coming from automorphisms $[s,t] \mapsto [\alpha s, t]$, $\alpha \in \mathbb{G}_m$, of $\mathbb{P}^1$. Such an automorphism corresponds to the one-parameter subgroup ρ of $\mathrm{GL}_5(k)$ with weights $(6,4,3,2,0)$. We shall prove that $[C_0]_m$ is semistable with respect to ρ, for all m, via an explicit computation of Hilbert-Mumford index $\mu([C_0]_m, \rho)$. Although this can be done by simply plugging the ideal of C_0 and `w` $= \{6,4,3,2,0\}$ in `mumfordIndex` (§2.2), we shall first carry out the algorithm step by step and present the computations in a traditional manner as if we did them by hand.

- We first find the ideal I_{C_0} of C_0 from the parametrization map:

$$I_{C_0} = \langle -x_1x_4 + x_3^2, -x_0x_4 + x_1x_3, -x_0x_4 + x_2^2, -x_0x_3 + x_1^2 \rangle.$$

- We compute a Gröbner basis for I_{C_0} with respect to the ρ-weighted GLex:

$$x_1x_4 - x_3^2,\ x_0x_4 - x_2^2,\ x_1x_3 - x_2^2,\ x_0x_3 - x_1^2,\ x_2^2x_4 - x_3^3,\ x_0x_2^2 - x_1^3.$$

- The leading terms of the Gröbner basis elements are:

$$x_1x_4, x_0x_4, x_1x_3, x_0x_3, x_2^2x_4, x_0x_2^2.$$

These generate the initial ideal $in_{\prec}(I_{C_0})$.

- The degree 2 monomials not in the initial ideal are:

$$x_0^2,\ x_0x_1,\ x_0x_2,\ x_1^2,\ x_1x_2,\ x_2^2,\ x_2x_3,\ x_2x_4,\ x_3^2,\ x_3x_4,\ x_4^2. \tag{3.1}$$

These monomials have total weight 66. On the other hand, we have

$$\frac{P(2)\cdot 2}{5}\sum_{i=0}^{4} r_i = \frac{11\cdot 2}{5}\cdot 15 = 66.$$

Therefore, by Proposition 1, the 2nd Hilbert point of the tri-canonical image of C_0 is at best strictly semistable with respect to ρ. Similarly, we find that the degree three monomials not contained in the initial ideal $in_{\prec}(I_{C_0})$ are:

$$x_0^3,\ x_0^2x_1,\ x_0^2x_2,\ x_0x_1^2,\ x_0x_1x_2,\ x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,$$
$$x_2^2x_3,\ x_2x_3^2,\ x_2x_3x_4,\ x_2x_4^2,\ x_3^3,\ x_3^2x_4,\ x_3x_4^2,\ x_4^3$$

of which total weight is 153. On the other hand, we have

$$\frac{P(3)\cdot 3}{5}\cdot\sum_{i=0}^{4} r_i = \frac{17\cdot 3}{5}\cdot 15 = 153.$$

Therefore, by Proposition 1, the 3rd Hilbert point of the tri-canonical image of C_0 is strictly semistable with respect to ρ. Now it follows from Corollary 4 that C_0 is m-Hilbert strictly semistable for all $m \geq 2$.

Remark 2 ($\mu([C_0]_m, \rho)$ as computed by Macaulay 2). First, compute the ideal of C_0:

```
i12 : P1 = QQ[s,t];
i13 : P4 = QQ[x_0..x_4];
i14 : f = map(P1,P4,{s^6,s^4*t^2,s^3*t^3,s^2*t^4,t^6})
                  6   4 2   3 3   2 4   6
o14 = map(P1,P4,{s , s t , s t , s t , t })
o14 : RingMap P1 <--- P4
i15 : C0 = kernel f
              2                        2          2
o15 = ideal (x  - x x , x x  - x x , x  - x x , x  - x x )
              3    1 4   1 3    0 4   2    0 4   1    0 3
o15 : Ideal of P4
```

Run `mumfordIndex` to compute the filtered Hilbert function $P_{C_0/\rho}(m)$:

```
i7 : mumfordIndex(C0, {6,4,3,2,0})
(2, (), 0)
```

Reading the output sequence, the regularity of C_0 is 2 and the filtered Hilbert function $P_{C_0/\rho}(m)$ agrees with the zero polynomial for all $m \geq 2$ (hence the empty sequence `()` in the second entry). Thus C_0 is m-Hilbert strictly semistable for all $m \geq 2$.

We can run the subprogram `MUm` to find $\mu([C_0]_m, \rho)$ for $m = 2, 3$ and the monomials not in the initial ideal:

```
i8 : MUm(C0, {6,4,3,2,0}, 2)
o8 = 0
i11 : inI
                                        2       2
o11 = ideal (x x , x x , x x , x x , x x , x x )
              1 4   0 4   1 3   0 3   2 4   0 2
o11 : Ideal of Sw
```

The degree two monomials not in the initial ideal are

```
i12 : Bmbar
o12 = | x_0^2 x_0x_1 x_0x_2 x_1^2 x_1x_2 x_2^2 x_2x_3 x_2x_4
        x_3^2 x_3x_4 x_4^2 |
                 1          11
o12 : Matrix Sbar  <--- Sbar
```

whose weights sum up to

```
i13 : TOTALWT
o13 = 66
```

Similarly, we compute $\mu([C_0]_3, \rho)$ by

```
i14 : MUm(C0, {6,4,3,2,0}, 3)
o14 = 0
```

The degree 3 monomials not in the initial ideal are

```
i15 : Bmbar
o15 = | x_0^3 x_0^2x_1 x_0^2x_2 x_0x_1^2 x_0x_1x_2 x_1^3
        x_1^2x_2 x_1x_2^2 x_2^3 x_2^2x_3 x_2x_3^2 x_2x_3x_4
        x_2x_4^2 x_3^3 x_3^2x_4 x_3x_4^2 x_4^3 |
                 1          17
o15 : Matrix Sbar  <--- Sbar
```

and their weights sum up to

```
i16 : TOTALWT
o16 = 153
```

3.2. Degeneration of cuspidal curves

There are three types of genus two pseudo-stable curves with a cusp:

(a) C_0, the rational curve with two cusps;
(b) C_0', the rational curve with a cusp and a node;
(c) C_1, a curve with one cusp and no other singularities, normalized by a smooth elliptic curve.

In this section, we shall prove C_0' and C_1 specialize to C_0 under the ρ-action. Since C_0' is in the closure of the locus of C_1 in the Hilbert scheme, we only need to prove it for C_1.

3.2.1. *Flat limit of* $\rho(\alpha).[C_1]$. Computation of flat limits is rather well known (cf. [2]). We quickly recapitulate the algorithm here: Given $S = k[x_0, \ldots, x_N]$ and a one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL}_{N+1}(k)$ defined by $\rho(\alpha).x_i = \alpha^{wt_\rho(x_i)} x_i$, we define the graded ρ-weight order $\prec_\rho$ as follows: let x^a and x^b be monomials. Then $x^a \prec_\rho x^b$ if

(1) $\deg x^a < \deg x^b$ or
(2) $\deg x^a = \deg x^b$ and $wt_\rho(x^a) < wt_\rho(x^b)$.

Note that $\prec_\rho$ is a partial order and $\prec$ in §2.1 is a total order that refines $\prec_\rho$.

Given $g = \sum c_a x^a \in S$, and a homogeneous ideal I of S, we define

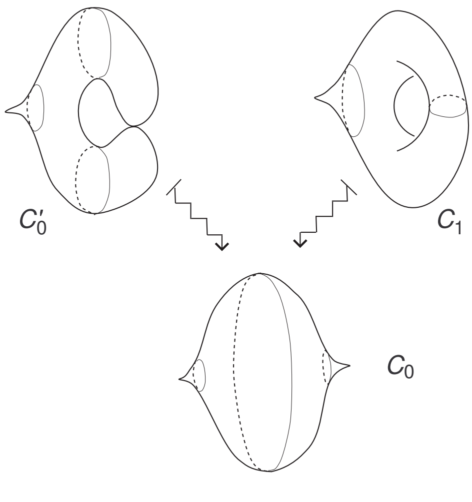

FIGURE 1. C_1 and C_0' degenerate to C_0 along the action of ρ

(a) $in_{\prec_\rho}(g)$ is the sum of the terms (of g) of maximal order[1];
(b) $in_{\prec_\rho}(I) := \langle in_{\prec_\rho}(f) \,|\, f \in I\rangle$;
(c) $\widetilde{g}(x_0, \ldots, x_N, \alpha) := \alpha^b g(\alpha^{-r_0}x_0, \ldots, \alpha^{-r_N}x_N)$, $b = \max\{wt_\rho(x^a) \,|\, c_a \neq 0\}$;
(d) $\widetilde{I} := \langle \widetilde{g}, g \in I\rangle \subset S[\alpha]$.

Note that for a fixed $\alpha \neq 0$, $I_\alpha = \langle \widetilde{g} \,|\, g \in I\rangle$ is the ideal defining $\rho(\alpha).C$, where $C \subset \mathbb{P}^N$ is the projective variety defined by I. The problem at hand is to compute the ideal of the limit variety of the family $\rho(\alpha).C$ as α tends to zero. First, we have:

Theorem 3 ([3], p. 343). *For any ideal $I \subset S$, the $k[\alpha]$-algebra $S[\alpha]/\widetilde{I}$ is free as $k[\alpha]$-algebra. Furthermore, we have*

$$S[\alpha]/\widetilde{I} \otimes_{k[\alpha]} k[\alpha, \alpha^{-1}] \simeq (S/I)[\alpha, \alpha^{-1}],$$
$$S[\alpha]/\widetilde{I} \otimes_{k[\alpha]} k[\alpha]/(\alpha) \simeq S/in_{\prec_\rho}(I).$$

This means precisely that $\widetilde{I}$ is the homogeneous ideal of the flat projective closure in $\mathbb{P}^N$ of the family $\rho(\alpha).C$, and that the flat limit is given by the initial ideal $in_{\prec_\rho}(I)$. These ideals can be readily computed by using Gröbner basis:

Proposition 4 ([3], p. 369). *Let $\{g_1, \ldots, g_t\}$ be a Gröbner basis for I with respect to $\prec$ (§2.1). Then*

(1) *$\widetilde{g}_1, \ldots, \widetilde{g}_t$ generate $\widetilde{I}$;*
(2) *$in_{\prec_\rho}(g_1), \ldots, in_{\prec_\rho}(g_t)$ generate $in_{\prec_\rho}(I)$.*

[1]Bayer and Mumford take the minimal weight term to be the initial term. Here we are using the dual action of ρ on the ideal, hence the reversal of the signs of the weights.

Remark 4. This algorithm can be easily implemented with Macaulay 2. The following function `flatLimit` takes an ideal `I` and a weight vector `w`, and computes the projective closure `tI` and the flat limit of the one-parameter family $\rho(\alpha).I$, $\alpha \in \mathbb{C}^*$, where ρ is the one-parameter subgroup with the prescribed weight vector `w`. The Gröbner basis computation occurs in `saturate(I,a)`.

```
flatLimit = (I,w) -> (
R = ring I;
N = numgens R - 1;
K = coefficientRing R;
Ra = K[gens R, a];
wmax = max w;
f = map(Ra, R, gens ideal apply(0..N, j -> a^(wmax-w_j)*(gens Ra)_j));
-- the new weights wmax-w_j corresponds to those
of the inverse of lambda
I = f(I);
tI = saturate(I,a);
substitute(tI, {a=>0})
)
```

Using this algorithm, we shall prove that $\lim_{\alpha\to 0}\rho(\alpha).C_1 = C_0$.
(A) We first compute the ideal of C_1. Let $\nu : C_1^\nu \to C_1$ be the normalization of C_1 and $p \in C_1^\nu$ be the closed point over the cusp q of C_1. The dualizing sheaf ω_{C_1} can be expressed

$$\omega_{C_1}(U) = \left\{ \zeta \in \omega_{C_1^\nu}(\nu^{-1}U) \;\middle|\; \sum_{y\in\nu^{-1}(x)} \mathrm{Res}_y(\nu^* f\cdot\zeta) = 0 \text{ for all } x \in U \text{ and } f \in \mathcal{O}_{C_1,x} \right\}$$

to an open set $U \subset C_1$. It follows that $\nu^*\omega_{C_1} = \omega_{C_1^\nu}(2p) = \mathcal{O}_{C_1^\nu}(2p)$ and $\nu^*\mathcal{O}_{C_1}(1) \simeq \mathcal{O}_{C_1^\nu}(6p)$. This means that the tri-canonical image of C_1 is given by a $\mathfrak{g}^4_6$ of C_1^ν. In other words, C_1 is the image of a suitable projection

$$\mathbb{P}(\Gamma(C_1^\nu, \mathcal{O}_{C_1^\nu}(6p))) \dashrightarrow \mathbb{P}^4$$

following the embedding $\eta : C_1^\nu \hookrightarrow \mathbb{P}(\Gamma(C_1^\nu, \mathcal{O}_{C_1^\nu}(6p))) = \mathbb{P}^5$ given by $|\mathcal{O}_{C_1^\nu}(6p)|$. The projection is from a point on the tangent line $T_p(C_1^\nu)$, creating the cusp q.

Consider the normal form $E := \{x_0^2x_2 = x_1(x_1 - x_2)(x_1 - \ell x_2)\} \subset \mathbb{P}^2$ of C_1^ν given by $|\mathcal{O}_{C_1^\nu}(3p)|$, where $p = [1,0,0]$. Then $\eta(C_1^\nu) \subset \mathbb{P}^5$ is the image of E under the second Veronese embedding

$$(\star) \qquad \begin{array}{ccc} v_2 : \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ [x_0,x_1,x_2] & \mapsto & \left[x_0^2, x_0x_1, x_1^2, x_0x_2, x_1x_2, x_2^2\right]. \end{array}$$

The tangent line T to E at p is $\{x_2 = 0\}$. If $y_0,\dots,y_5$ are the homogeneous coordinates of $\mathbb{P}^5$ in $(\star)$, the second Veronese image of T is given by the ideal $\langle y_3, y_4, y_5, y_1^2 - y_0y_2\rangle$. Hence the tangent line to $v_2(T)$ at $p = \eta([1,0,0]) = [1,0,\dots,0]$ is $\{y_0 = y_3 = y_4 = y_5 = 0\}$ and y_1 is a local parameter of $v_2(T)$ at p. Since $v_2(T)$ and $\eta(C_1^\nu)$ agree to order one, it follows that the tangent line to $\eta(C_1^\nu)$ at $p = \eta([1,0,0]) = [1,0,\ldots,0]$ is $\{y_0 = y_3 = y_4 = y_5 = 0\}$ and y_1 is a local parameter of $\mathcal{O}_{\eta(C_1^\nu),p}$. Therefore, the projection

$$\begin{array}{rcc} pr : \mathbb{P}^5 & \dashrightarrow & \mathbb{P}^4 \\ [y_0,\ldots,y_5] & \mapsto & [y_0, y_2, y_3, y_4, y_5] \end{array}$$

kills the tangent direction and replaces p with a cusp.

The ideal $I_{v_2(E)}$ of $\eta(C_1^\nu) = v_2(E) \subset \mathbb{P}^5$ is generated by:

$$\begin{aligned} &y_1^2 - y_0y_2,\ y_4^2 - y_2y_5,\ y_3y_4 - y_1y_5,\ y_3^2 - y_0y_5,\ y_2y_3 - y_1y_4,\ y_1y_3 - y_0y_4, \\ &y_2y_4\ell - y_4y_5\ell - y_2^2 + y_0y_4 + y_0y_5 + y_2y_5,\ y_2y_5\ell - y_4y_5\ell - y_2y_4 + y_0y_5 + y_2y_5, \\ &y_1y_4\ell - y_1y_5\ell - y_1y_2 + y_0y_3 + y_1y_4. \end{aligned}$$

The ideal of $pr \circ \eta(C_1^\nu) \subset \mathbb{P}^4$ is the kernel of the homomorphism

$$\begin{array}{ll} k[z_0,\ldots,z_4] & \to k[y_0,\ldots,y_5]/I_{v_2(E)} \\ (z_0,\ldots,z_4) & \mapsto (y_0, y_2, y_3, y_4, y_5). \end{array}$$

It is generated by:

$$\begin{gathered} z_3^2 - z_1z_4,\ z_2^2 - z_0z_4,\ z_1z_4\ell - z_3z_4\ell - z_1z_3 + z_0z_4 + z_1z_4, \\ z_1z_3\ell - z_3z_4\ell - z_1^2 + z_0z_3 + z_0z_4 + z_1z_4. \end{gathered}$$

(B) Second, we compute the Gröbner basis of I_{C_1} with respect to the total weight order:

$$\begin{gathered} z_1z_4 - z_3^2,\ z_1z_3 - z_2^2 - z_3^2\ell - z_3^2 + z_3z_4\ell,\ z_0z_4 - z_2^2, \\ z_0z_3 - z_1^2 + z_2^2\ell + z_2^2 + z_3^2\ell^2 + z_3^2\ell + z_3^2 - z_3z_4\ell^2 - z_3z_4\ell, \\ z_2^2z_4 - z_3^3 + z_3^2z_4\ell + z_3^2z_4 - z_3z_4^2\ell, \\ z_0z_2^2 - z_1^3 + 2z_1z_2^2\ell + 2z_1z_2^2 + z_2^2z_3\ell^2 + z_2^2z_3 \\ +z_3^3\ell^3 + z_3^3\ell^2 + z_3^3\ell + z_3^3 - z_3^2z_4\ell^3 - z_3^2z_4\ell^2 - z_3^2z_4\ell. \end{gathered} \tag{3.2}$$

(C) From (B) we obtain a Gröbner basis for $\widetilde{I}_{C_1}$ with terms without α underlined:

$$\begin{gathered} z_3z_4\ell\alpha^4 - z_3^2\ell\alpha^2 - z_3^2\alpha^2\underline{-z_2^2 + z_1z_3},\ \underline{-z_2^2 + z_0z_4},\ \underline{-z_3^2 + z_1z_4}, \\ -z_3z_4^2\ell\alpha^4 + z_3^2z_4\ell\alpha^2 + z_3^2z_4\alpha^2\underline{-z_3^3 + z_2^2z_4}, \\ -z_3z_4\ell^2\alpha^6 - z_3z_4\ell\alpha^6 + z_3^2\ell^2\alpha^4 + z_3^2\ell\alpha^4 + z_3^2\alpha^4 + z_2^2\ell\alpha^2 + z_2^2\alpha^2\underline{-z_1^2 + z_0z_3}, \\ -z_3^2z_4\ell^3\alpha^8 - z_3^2z_4\ell^2\alpha^8 + z_3^3\ell^3\alpha^6 - z_3^2z_4\ell\alpha^8 + z_3^3\ell^2\alpha^6 + z_3^3\ell\alpha^6 \\ +z_2^2z_3\ell^2\alpha^4 + z_3^3\alpha^6 + z_2^2z_3\alpha^4 + 2z_1z_2^2\ell\alpha^2 + 2z_1z_2^2\alpha^2\underline{-z_1^3 + z_0z_2^2}. \end{gathered}$$

(D) Substituting $\alpha = 0$, we obtain the ideal of the flat limit:

$$\langle z_3^2 - z_1z_4, z_1z_3 - z_0z_4, z_2^2 - z_0z_4, z_1^2 - z_0z_3\rangle.$$

This is precisely the ideal of the tri-canonical model of C_0, regardless of ℓ.

3.3. Hilbert unstable curves - Instability of elliptic bridges

Definition 5. An *elliptic tail* (resp. *elliptic bridge*) is a connected subcurve of arithmetic genus one meeting the rest of the curve in one node (resp. two nodes).

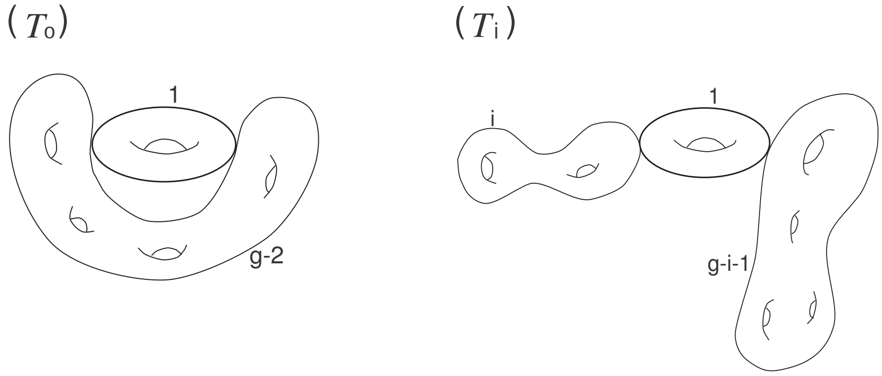

FIGURE 2. Generic elliptic bridges

In this section, we shall prove that a bicanonically embedded elliptic bridge is Hilbert unstable. Readers looking for context as to why this particular stability problem is important are invited to take a look at [5] and [6].

Let C be a generic elliptic bridge of genus g consisting of a genus $g-2$ curve D meeting in two nodes q and r with a genus one subcurve E.

Proposition 5. *Let C_0 be the curve in Figure 3 consisting of D and two conics C_1 and C_2, where D is embedded by $|\omega_D^{\otimes 2}(2q+2r)|$ and C_1 and C_2 meet D in nodes q and r respectively and meet each other in a tacnode. Then there is a one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{SL}_{N+1}$ such that*

(1) *$\rho(t).C$ specializes to a bicanonical c-semistable curve C_0;*
(2) *$P_{C_0,\rho}(m) = -3(g-1)(m-1)$. In particular, C_0 is Hilbert unstable.*

Since $P_{C,\rho}(m) = P_{C_0,\rho}(m)$, it follows that C is also Hilbert unstable.

Proof. Note that $\omega_C^{\otimes 2}|_D = \omega_D^{\otimes 2}(2q+2r)$ and $\omega_C^{\otimes 2}|_E = \mathcal{O}_E(2q+2r)$, which imply that D and E are embedded in linear subspaces of $\mathbb{P}^{3g-4}$ of dimensions $3g-6$ and 3, respectively. Hence we can choose coordinates such that

$$x_{N-1} = x_N = 0 \quad \text{on } D,$$
$$x_0 = \cdots = x_{N-4} = 0 \quad \text{on } E.$$

We can extract equations for E embedded by $|2q+2r|$ by argument similar to extracting the normal form of elliptic curve embedded in $\mathbb{P}^2$ by $|3p_0|$.

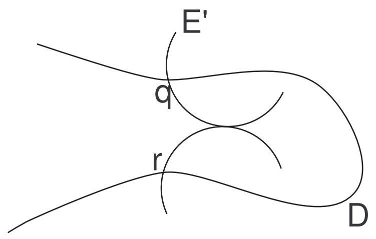

FIGURE 3. Flat limit of $\rho(t).C$

Let $\{1_q, x\}$ and $\{1_r, y\}$ be bases for $\Gamma(2q)$ and $\Gamma(2r)$, respectively. We may assume that $2q \not\equiv 2r$: we first prove that an elliptic bridge with $2q \not\equiv 2r$ is unstable, and can deduce that an elliptic bridge with $2q \equiv 2r$ is also unstable since the unstable locus is closed. Under this assumption, we can choose x and y such that $x \in \Gamma(2q - r)$ and $y \in \Gamma(2r - q)$ and hence the vanishing order at q and r (on E) are as follows:

	1_q	x	1_r	y
ord_q	2	0	0	1
ord_r	0	1	2	0

(3.3)

	$x \cdot 1_r$	$y \cdot 1_q$	xy	$1_q \cdot 1_r$
ord_q	0	3	1	2
ord_r	3	0	1	2

Let $x_{N-3} = x \cdot 1_r$, $x_{N-2} = y \cdot 1_q$, $x_{N-1} = xy$ and $x_N = 1_q \cdot 1_r$. One sees immediately that the image of E under $|2q + 2r|$ lies on the Segre surface

$$\{f_1 := x_{N-3}x_{N-2} - x_{N-1}x_N = 0\}.$$

Also, since $\dim \Gamma(4q + 4r) = 8$, there is a nontrivial linear relation between the 9 elements

1	x	y	xy	x^2	y^2	x^2y	xy^2	x^2y^2
x_N^2	$x_{N-3}x_N$	$x_{N-2}x_N$	$x_{N-3}x_{N-2}$	x_{N-3}^2	x_{N-2}^2	$x_{N-3}x_{N-1}$	$x_{N-2}x_{N-1}$	x_{N-1}^2

Let f_2 denote a linear relation:

$$\begin{aligned} f_2 := c_0x_{N-3}^2 + c_1x_{N-3}x_{N-1} + c_2x_{N-3}x_N + c_3x_{N-2}^2 + c_4x_{N-2}x_{N-1} \\ + c_5x_{N-2}x_N + c_6x_{N-1}^2 + c_7x_{N-1}x_N + c_8x_N^2. \end{aligned}$$

Because of our choice of coordinates that have specific vanishing orders at q and r, it follows that

(A) $T_qE = \{x_0 = \cdots = x_{N-4} = x_{N-2} = x_N = 0\}$, $q = [0, \ldots, 0, 1, 0, 0, 0]$,

(B) $T_rE = \{x_0 = \cdots = x_{N-4} = x_{N-3} = x_N = 0\}$, $r = [0, \ldots, 0, 0, 1, 0, 0]$.

(A) implies that $c_0 = c_1 = 0$ and $c_2 \neq 0$ while (B) forces $c_3 = c_4 = 0$ and $c_5 \neq 0$. Moreover, for E to be smooth, c_6 must not be zero.

Taking all these into account, the generic form of f_2 is as follows.

$$f_2 = x_{N-1}^2 + x_{N-3}x_N + x_{N-2}x_N + c_1 x_{N-1}x_N + c_2 x_N^2, \quad c_i \in k.$$

The j-invariant of E can be computed by realizing it as a double cover of $\mathbb{P}^1$ ([7]) via $E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\pi} \mathbb{P}^1$, where π is the projection to one of the factors:

$$j(E) = \frac{-2^8 3^3 (c_1^2 - 12c_2)^3}{4(c_1^2 - 12c_2)^3 + 27(2c_1^3 - 72c_1c_2 - 2^4 3^3)}.$$

Let $\rho : \mathbb{G}_m \to \mathrm{GL}_{N+1}(k)$ be a one-parameter subgroup defined by the diagonal matrix

$$\rho(t) = \begin{pmatrix} t^2 & & & & \\ & \ddots & & & \\ & & t^2 & & \\ & & & t & \\ & & & & 1 \end{pmatrix}. \tag{3.4}$$

To compute $\mu([C]_m, \rho)$, we shall first compute the ideal of the flat limit C_0 of the family $\rho(t).C$.

For fixed $t \neq 0$, the two generators f_1 and f_2 of the ideal of C give rise to

$$\begin{aligned} \widetilde{f_1}(x_0, \dots, x_N) &= t^4 f_1(t^{-2}x_0, t^{-2}x_1, \dots, t^{-2}x_{N-2}, t^{-1}x_{N-1}, x_N) \\ &= x_{N-3}x_{N-2} - t^3 x_{N-1}x_N \rightsquigarrow x_{N-3}x_{N-2} = in_{\prec_\rho}(f_1), \\ \widetilde{f_2}(x_0, \dots, x_N) &= t^2 f_1(t^{-2}x_0, t^{-2}x_1, \dots, t^{-2}x_{N-2}, t^{-1}x_{N-1}, x_N) \\ &= x_{N-1}^2 + x_{N-3}x_N + x_{N-2}x_N + tc_1 x_{N-1}x_N + t^2 c_2 x_N^2 \\ &\rightsquigarrow x_{N-1}^2 + x_{N-3}x_N + x_{N-2}x_N = in_{\prec_\rho}(f_2). \end{aligned}$$

Let $I' = \langle x_{N-3}x_{N-2}, x_{N-1}^2 + x_{N-3}x_N + x_{N-2}x_N \rangle \supset in_{\prec_\rho}(I_E)$, where $I_E = \langle f_1, f_2 \rangle$ is the homogeneous ideal of E. The Hilbert polynomial of $\mathrm{Proj}\,(S/I')$ is $P(m) = 4m$ which is the same as the Hilbert polynomial $m \cdot \deg \mathcal{O}_E(2q + 2r) + 1 - 1$ of the flat limit. Since $I' \supset in_{\prec_\rho}(I_E)$, we conclude that I' is equal to $in_{\prec_\rho}(I_E)$, the ideal defining the flat limit.

The curve E' of arithmetic genus 1 defined by I' consists of two conics

$$\begin{aligned} C_1' &= \{x_{N-3} = 0, x_{N-1}^2 + x_{N-2}x_N = 0\}, \\ C_2' &= \{x_{N-2} = 0, x_{N-1}^2 + x_{N-3}x_N = 0\} \end{aligned} \tag{3.5}$$

meeting in a tacnode $p' = [0, \dots, 0, 1]$. The flat limit of $\rho(t).D$ is obviously D itself since ρ acts trivially on D.

It remains to show that $[C]_m$ is strictly semistable with respect to ρ. Equivalently, we may show that $\mu([C_0]_m, \rho) = 0$. Let I_0 denote the ideal of C_0. Recall that $\mu([C]_m, \rho) = \mu([C_0]_m, \rho)$, the right hand side of which we shall compute by using the formula

$$-(N+1) \sum_{i=1}^{P(m)} wt_\rho(x^{a(i)}) + mP(m) \sum_{i=0}^{N} r_i,$$

where $x^{a(1)}, \ldots, x^{a(P(m))}$ are degree m monomials not in $in_\prec(I_0)$, and $r_i = wt_\rho(x_i)$.

First, we shall consider the second Hilbert point of C_0. We need to sort out the degree 2 monomials (of weight < 4) that are not in $in_\prec(I')$. The following are the degree 2 monomials with ρ-weight less than 4:

ρ-weight	
3	$x_j x_{N-1}$, $j \le N-2$
2	x_{N-1}^2, $x_j x_N$, $j \le N-2$
1	$x_{N-1}x_N$
0	x_N^2

Among these, clearly $x_j x_N$ and $x_j x_{N-1}$, $j = 0, \ldots, N-4$, are of weight < 4 and in $in_\prec(I_0)$ since they are in I_0. Therefore, the only degree 2 monomials of weight < 4 that are possibly not in $in_\prec(I_0)$ are

(3.6)

ρ-weight	
3	$x_{N-3}x_{N-1}$, $x_{N-2}x_{N-1}$
2	$x_{N-3}x_N$, $x_{N-2}x_N$, x_{N-1}^2
1	$x_{N-1}x_N$
0	x_N^2

We claim that in the table (3.6), $x_{N-3}x_N$ is the only monomial that is in $in_\prec(I_0)$. Clearly, $in_\prec(I_0) \subset in_\prec(I')$. A Gröbner basis of I' is:

$$x_{N-3}x_N + x_{N-2}x_N + x_{N-1}^2,\ x_{N-3}x_{N-2},\ x_{N-2}^2 x_N + x_{N-2}x_{N-1}^2.$$

Hence the initial ideal is

$$in_\prec(I') = \langle x_{N-3}x_N, x_{N-3}x_{N-2}, x_{N-2}^2 x_N \rangle \tag{3.7}$$

and the only degree 2 monomials in $in_\prec(I')$ are $x_{N-3}x_N$ and $x_{N-3}x_{N-2}$. Hence among the monomials in the list, $x_{N-3}x_N$ is the only possible element in $in_\prec(I_0)$.

Since $f_2 = x_{N-1}^2 + (x_{N-3} + x_{N-2})x_N \in I'$ vanishes entirely on D, $f_2 \in I_0$ and

$$in_\prec(f_2) = x_{N-3}x_N \in in_\prec(I_0).$$

On the other hand, if $x_{N-3}x_{N-2} = in_\prec(f)$ for some $f \in I_0$, then f must be of the form

$$a x_{N-3}x_{N-2} + b x_{N-2}^2 + x_{N-1}g_1 + x_N g_2$$

for some $a, b \in k$ and linear polynomials g_1, g_2. But this would imply that $x_{N-2} = 0$ or $a x_{N-3} + b x_{N-2} = 0$ entirely on D, which contradicts that D is nondegenerate in $\{x_{N-1} = x_N = 0\}$. Hence $x_{N-3}x_{N-2} \notin in_\prec(I_0)$. Therefore, the total weight $\sum_{i=1}^{P(2)} wt_\rho x^{a(i)}$ is

$$\begin{aligned} \textstyle\sum_{i=1}^{P(2)} wt_\rho(x^{a(i)}) &= 2 + 2\cdot 3 + 2 + 1 + 4\cdot(7g - 7 - 6) \\ &= 11 + 28g - 52 = 28g - 41. \end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{2\cdot P(2)}{3g-3}\sum_{i=0}^{N} r_i &= \frac{2\cdot 7(g-1)}{3(g-1)}\cdot(1+2(3g-5))\\ &= 28g-42.\end{aligned}$$

Hence

$$\begin{aligned}\mu([C]_2,\rho) &= \mu([C_0]_2,\rho)\\ &= -(3g-3)\cdot(28g-41-(28g-42))\\ &= -3(g-1)<0.\end{aligned}$$

It follows that ρ destabilizes the 2nd Hilbert point of C.

Now let's consider the 3rd Hilbert point of C. The degree 3 monomials of ρ-weight less than 6 are

ρ-weight	
5	$x_ix_jx_{N-1}$, $i,j\le N-2$
4	$x_jx_{N-1}^2$, $x_ix_jx_N$, $i,j\le N-2$
3	x_{N-1}^3, $x_ix_{N-1}x_N$, $i\le N-2$
2	$x_{N-1}^2x_N$, $x_jx_N^2$, $j\le N-2$
1	$x_{N-1}x_N^2$
0	x_N^3

Among these monomials, $x_ix_jx_{N-1}$, $x_jx_{N-1}^2$, $x_ix_jx_N$ and $x_jx_N^2$, are obviously contained in $in_\prec(I_0)$ if $i\le N-4$ or $j\le N-4$ since they are in I_0. Hence we need to consider

(3.8)

ρ-weight	
5	$x_{N-3}^2x_{N-1}$, $x_{N-3}x_{N-2}x_{N-1}$, $x_{N-2}^2x_{N-1}$
4	$x_{N-3}^2x_N$, $x_{N-3}x_{N-2}x_N$, $x_{N-2}^2x_N$, $x_{N-3}x_{N-1}^2$, $x_{N-2}x_{N-1}^2$
3	x_{N-1}^3, $x_{N-3}x_{N-1}x_N$, $x_{N-2}x_{N-1}x_N$
2	$x_{N-1}^2x_N$, $x_{N-3}x_N^2$, $x_{N-2}x_N^2$
1	$x_{N-1}x_N^2$
0	x_N^3

First, note that $x_{N-3}x_{N-2}x_{N-1}$ and $x_{N-3}x_{N-2}x_N$ are in $in_\prec(I_0)$ since they are in I_0. Then we argue similarly as in the 2nd Hilbert point case. By examining the initial ideal (3.7), we deduce that among the monomials in (3.8), the following monomials are the only possible elements in $in_\prec(I_0)$:

$$x_{N-3}x_{N-2}x_{N-1}, x_{N-3}^2x_N, x_{N-3}x_{N-2}x_N, x_{N-3}x_{N-2}x_{N-1}, x_{N-2}x_{N-1}^2, x_{N-3}x_N^2.$$

We have

$$\begin{aligned}g_1 &= x_{N-2}\cdot(x_{N-1}^2+(x_{N-3}+x_{N-2})x_N)-x_N\cdot(x_N-3x_{N-2})\\ &= x_{N-2}^2x_N+x_{N-2}x_{N-1}^2\in I_0.\end{aligned}$$

Hence $x_{N-2}^2x_N=in_\prec(g_1)\in in_\prec(I_0)$. Therefore, the total weight is

$$\begin{aligned}\textstyle\sum_{i=1}^{P(3)} wt_\rho(x^{a(i)}) &= 2\cdot 5+2\cdot 4+2\cdot 3+2\cdot 2+1\cdot 1+6\cdot(11(g-1)-10)\\ &= 29+6(11g-21)=66g-97.\end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{3 \cdot P(3)}{3g-3} \sum_{i=0}^{N} r_i &= \frac{3 \cdot 11(g-1)}{3(g-1)} \cdot (1 + 2(3g-5)) \\ &= 66g - 99. \end{aligned}$$

Hence

$$\begin{aligned} \mu([C]_3, \rho) &= \mu([C_0]_3, \rho) \\ &= -(3g-3) \cdot (66g - 97 - (66g - 99)) \\ &= -6(g-1) < 0. \end{aligned}$$

It follows that ρ destabilizes the 3rd Hilbert point of C. From $\mu([C_0]_2, \rho)$ and $\mu([C_0]_3, \rho)$, we obtain the filtered Hilbert function

$$\begin{aligned} P_{C_0,\rho}(m) &= (m-1)[-3(g-1)(3-m) - 6(g-1)(m/2-1)] \\ &= -3(g-1)(m-1). \end{aligned}$$

This has negative values for all $m \geq 2$, and it follows that $[C]_m$ is unstable with respect to ρ for all $m \geq 2$. □

Corollary 6. *Let C_0 and ρ be as in Proposition 5. Then $\mu(Ch(C_0), \rho) = 0$.*

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