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SCHATTEN CLASSES OF MATRICES IN A GENERALIZED $\mathcal{B}(l_2)$

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Abstract. In this paper, we study a generalization of the Banach space $\mathcal{B}(l_2)$ of all bounded linear operators on l_2. Over this space, we present some reasonable ways to define Schatten-type classes which are generalizations of the classical Schatten classes of compact operators on l_2.

1. Introduction and preliminary results

For a separable Hilbert space $\mathcal{H}$ and $1 \leq p \leq \infty$, the Schatten p-class, $\mathcal{C}_p$, is the class of all compact operators A on $\mathcal{H}$ such that the sequence $\{s_n(A)\}_{n=1}^{\infty}$ of singular values of A belongs to l_p. Equipped with the norms $\|\|A\|\|_p = \|\{s_n(A)\}_{n=1}^{\infty}\|_p$, the classes $\mathcal{C}_p$ are Banach spaces. These were introduced, in [8], by von Neumann and Schatten as a completion of the tensor product $\mathcal{H} \otimes \mathcal{H}$ in various norms. Since then many mathematicians have contributed and extended their results, see [3, 4, 5] for references. In this paper, we give some reasonable ways to define Schatten-type classes which generalize the classical Schatten classes of compact operators on l_2.

Let X be a compact Hausdorff space, and let $C(X)$ be the C^*-algebra of continuous complex-valued functions on X. In this paper, we denote the norm on $C(X)$ by $\|\cdot\|_{C(X)}$. In [6], Leo Livshits, Sing-Cheong Ong, and Sheng-Wang Wang defined the sequence spaces $l_2(C(X))$ and $l_2^b(C(X))$ as follows:

$$l_2(C(X)) = \left\{ \{f_k\}_{k=1}^{\infty} : f_k \in C(X)\ \forall k,\ \left\{ \sum_{k=1}^{n} |f_k|^2 \right\}_{n=1}^{\infty} \text{ converges in } C(X) \right\},$$

$$l_2^b(C(X)) = \left\{ \{f_k\}_{k=1}^{\infty} : f_k \in C(X)\ \forall k,\ \left\{ \sum_{k=1}^{n} |f_k|^2 \right\}_{n=1}^{\infty} \text{ is bounded in } C(X) \right\}.$$

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In [7], J. Rakbud and P. Chaisuriya extended to $1 \le p < \infty$:

$$l_p(C(X)) = \left\{ \{f_k\}_{k=1}^\infty : f_k \in C(X) \ \forall k, \ \left\{ \sum_{k=1}^n |f_k|^p \right\}_{n=1}^\infty \text{ converges in } C(X) \right\},$$

$$l_p^b(C(X)) = \left\{ \{f_k\}_{k=1}^\infty : f_k \in C(X) \ \forall k, \ \left\{ \sum_{k=1}^n |f_k|^p \right\}_{n=1}^\infty \text{ is bounded in } C(X) \right\}$$

and proved that $l_p(C(X))$ and $l_p^b(C(X))$ endowed with the norm

$$\|\{f_k\}_{k=1}^\infty\|_p := \sup_{t \in X} \left(\sum_{k=1}^\infty |f_k(t)|^p \right)^{\frac{1}{p}}$$

are Banach spaces.

It is clear that $l_p(C(X)) \subseteq l_p^b(C(X))$. For the case where X is finite, we have $l_p(C(X)) = l_p^b(C(X))$. If X is a singleton, then $l_p(C(X)) = l_p^b(C(X)) = l_p$. The following example shows that the inclusion $l_p(C(X)) \subseteq l_p^b(C(X))$ can be proper.

Example 1.1 ([6, 7])**.** $l_p(C(X)) \subsetneq l_p^b(C(X))$. Let $X = [0,1]$ and for each $k \in \mathbb{N}$, let $f_k(t) = (t^k - t^{k+1})^{\frac{1}{p}}$ for all $t \in [0,1]$. Let $f\langle p \rangle = \{f_k\}_{k=1}^\infty$. Then $f\langle p \rangle$ belongs to $l_p^b(C([0,1]))$, but does not belong to $l_p(C([0,1]))$.

Proposition 1.2 ([7])**.** *Let $f = \{f_k\}_{k=1}^\infty$ be a sequence over $C(X)$ with $f[t] := \{f_k(t)\}_{k=1}^\infty \in l_p$ for all $t \in X$. Then the following are equivalent.*

(1) $f \in l_p(C(X))$.
(2) *The function $t \mapsto f[t]$ from X into l_p is continuous.*
(3) *The function $t \mapsto \|f[t]\|_p$ from X into $[0,\infty)$ is continuous.*

Theorem 1.3 ([6, 7])**.** *Let $g = \{g_k\}_{k=1}^\infty$ be a sequence in $C(X)$. Then*

(1) *$\{f_k g_k\}_{k=1}^\infty \in l_1(C(X))$ for all $f = \{f_k\}_{k=1}^\infty \in l_2(C(X))$ if and only if $g \in l_2^b(C(X))$. If $g \in l_2^b(C(X))$, then $\|g\|_2 = \sup\{\|\{g_k f_k\}_{k=1}^\infty\|_1 : f = \{f_k\}_{k=1}^\infty \in l_2(C(X)), \|f\|_2 \le 1\}$.*
(2) *$\{f_k g_k\}_{k=1}^\infty \in l_1(C(X))$ for all $f = \{f_k\}_{k=1}^\infty \in l_2^b(C(X))$ if and only if $g \in l_2(C(X))$. If $g \in l_2(C(X))$, then $\|g\|_2 = \sup\{\|\{g_k f_k\}_{k=1}^\infty\|_1 : f = \{f_k\}_{k=1}^\infty \in l_2^b(C(X)), \|f\|_2 \le 1\}$.*

2. A generalization of $\mathcal{B}(l_2)$

We say that a matrix $A = [a_{jk}]$ with entries from $C(X)$ defines a linear operator on $l_2(C(X))$ if for every $f = \{f_k\}_{k=1}^\infty \in l_2(C(X))$, the series $\sum_{k=1}^\infty a_{jk} f_k$ converges in $C(X)$ for all j, and the sequence $\{\sum_{k=1}^\infty a_{jk} f_k\}_{j=1}^\infty$ belongs to $l_2(C(X))$. We denote the sequence $\{\sum_{k=1}^\infty a_{jk} f_k\}_{j=1}^\infty$ by Af for all $f = \{f_k\}_{k=1}^\infty \in l_2(C(X))$ and call the operator $f \mapsto Af$ the linear operator defined by A. Let $\mathcal{B}(l_2(C(X)))$ be the set of all matrices A over $C(X)$ such that A defines a linear operator on $l_2(C(X))$. For any matrix $A = [a_{jk}]$ over $C(X)$, we let, for each $n \in \mathbb{N}$, $A_{n\lrcorner}$ be the matrix which agrees with A on the upper left $n \times n$ block and is 0 on all other entries. For each $t \in X$, we let $A[t] := [a_{jk}(t)]$. For $f = \{f_k\}_{k=1}^{\infty} \in l_2(C(X))$, we let, for each $t \in X$, $f[t] := \{f_k(t)\}_{k=1}^{\infty}$. It is clear that $f[t] \in l_2$ for all t and $\|f\|_2 = \sup\limits_{t\in X} \|f[t]\|_2$.

Proposition 2.1 ([6]). *If $A \in \mathcal{B}(l_2(C(X)))$, then the linear operator defined by A is bounded.*

If $A \in \mathcal{B}(l_2(C(X)))$, we define the norm $\|A\|$ to be the norm of the linear operator defined by A.

Proposition 2.2. *Let A be a matrix with entries from $C(X)$.*

(1) *If $A \in \mathcal{B}(l_2(C(X)))$, then $\sup_{t\in X} \|A[t]\| < \infty$. Moreover, $\|A\| = \sup_{t\in X} \|A[t]\|$.*
(2) *If $A \in \mathcal{B}(l_2(C(X)))$, then $\|A_{n\lrcorner}\| \nearrow \|A\|$.*

Proof. (1). Let $f = \{f_k\}_{k=1}^{\infty} \in l_2(C(X))$ with $\|f\|_2 \le 1$. Then we get for each $t \in X$ that $\|(A[t])f[t]\|_2 \le \|A[t]\|$. So

$$\|A\| = \sup\left\{\sup_{t\in X} \|(A[t])f[t]\|_2 : f \in l_2(C(X)), \|f\|_2 \le 1\right\} \le \sup_{t\in X} \|A[t]\|.$$

Let $x = \{\xi_k\}_{k=1}^{\infty} \in l_2$ with $\|x\|_2 \le 1$. For each k, we put $f_k(t) = \xi_k$ for all $t \in X$ and $f_x = \{f_k\}_{k=1}^{\infty}$. Then $f_x \in l_2(C(X))$ and $\|f_x\|_2 = \|x\|_2 \le 1$. Thus, for each $t \in X$, $\|A[t]x\|_2 = \|A[t]f_x[t]\|_2 \le \|A\|$. This implies that $\|A[t]\| \le \|A\|$ for all t. The proof is complete.

(2). Suppose that $A \in \mathcal{B}(l_2(C(X)))$. Then by the assertion (1) above, $A[t] \in \mathcal{B}(l_2)$ for all $t \in X$. So, we have $A_{n\lrcorner}[t] \nearrow A[t]$ for all t. Hence we get by (1) again that $\|A_{n\lrcorner}\| \le \|A_{n+1\lrcorner}\| \le \|A\|$ for all n. This implies that $A_n \nearrow \sup_n \|A_{n\lrcorner}\|$ and $\sup_n \|A_{n\lrcorner}\| \le \|A\|$. To see that $\|A\| \le \sup_n \|A_{n\lrcorner}\|$, let $\epsilon > 0$ be given. Then by (1), there exists $s \in X$ such that $\|A\| < \|A[s]\| + \epsilon$. This implies that there is a positive integer n_0 such that

$$\|A\| < \left\|A_{n_0\lrcorner}[s]\right\| + \epsilon \le \left\|A_{n_0\lrcorner}\right\| + \epsilon \le \sup_n \|A_{n\lrcorner}\| + \epsilon.$$

Since ϵ is arbitrary, $\|A\| \le \sup\limits_n \|A_{n\lrcorner}\|$. □

The following example shows that for a matrix A over $C(X)$, the finiteness of $\sup_{t\in X} \|A[t]\|$ does not necessarily imply the boundedness of A.

Example 2.3. Let $X = [0, 1]$ and let A be the matrix whose the first column is the sequence $f\langle 2\rangle$ given in Example 1.1 and all other columns 0. Then $\sup_{t\in X} \|A[t]\| \le 1$, but A does not define a linear operator on $l_2(C([0, 1]))$ since $Ax = f\langle 2\rangle$, where $x = \{1, 0, 0, 0, \ldots\}$, does not belong to $l_2(C([0, 1]))$.

Lemma 2.4. *If A is a matrix with entries from $C(X)$ and the set $\{\|A_{n\lrcorner}\| : n \in \mathbb{N}\}$ is bounded, then Af is a sequence in $C(X)$ for all $f \in l_2(C(X))$.*

Proof. Let $M = \sup_{n\in\mathbb{N}} \|A_{n\lrcorner}\|$ and $f = \{f_k\}_{k=1}^{\infty} \in l_2(C(X))$. For any $n > m \in \mathbb{N}$, we let $f_{[m,n]} := \{0, 0, \ldots, 0, f_m, f_{m+1}, \ldots, f_n, 0, 0, \ldots\}$, clearly, $f_{[m,n]} \in l_2(C(X))$. Let $j \in \mathbb{N}$ and $\epsilon > 0$. Then there exists a positive integer N such that

$$\left\|\sum_{k=m}^{n}|f_k|^2\right\|_{C(X)} < \left(\frac{\epsilon}{M}\right)^2 \text{ for all } n > m > N.$$

So, if $n > m > \max\{j, N\}$, we get that

$$\begin{aligned}\left\|\sum_{k=m}^{n}a_{jk}f_k\right\|_{C(X)} &= \sup_{t\in X}\left|\sum_{k=m}^{n}a_{jk}(t)f_k(t)\right| \le \left\|A_{n\lrcorner}f_{[m,n]}\right\|_2 \\ &\le \|A_{n\lrcorner}\| \left\|f_{[m,n]}\right\|_2 \le M\left\|\sum_{k=m}^{n}|f_k|^2\right\|_{C(X)}^{\frac{1}{2}} \\ &< M\left(\frac{\epsilon}{M}\right) = \epsilon.\end{aligned}$$

Hence $\{\sum_{k=1}^{n} a_{jk}f_k\}_{n=1}^{\infty}$ is a Cauchy sequence in $C(X)$, so it is convergent. □

Remark 2.5. If X is a singleton, then the assumption of the above lemma implies that $A \in \mathcal{B}(l_2(C(X)))$. This is not true in general. Indeed, from Example 2.3, we also have by Proposition 2.2(1) that $\|A_{n\lrcorner}\| \le 1$ for all n, but A does not belong to $\mathcal{B}(l_2(C(X)))$. The following proposition tells us when the boundedness of the set $\{\|A_{n\lrcorner}\| : n \in \mathbb{N}\}$, which is clearly equivalent to the finiteness of $\sup_{t\in X}\|A[t]\|$, implies the boundedness of the matrix A.

Proposition 2.6. *Suppose that A is a matrix over $C(X)$ with $A[t] \in \mathcal{B}(l_2)$ for all $t \in X$ and the function $t \mapsto A[t]$ from X into $\mathcal{B}(l_2)$ is continuous. Then $A \in \mathcal{B}(l_2(C(X)))$.*

Proof. Since the function $t \mapsto A[t]$ is continuous and X is compact, $\sup_{t\in X}\|A[t]\| < \infty$. Let $M = \sup_{t\in X}\|A[t]\|$. Then for each n, $\|A_{n\lrcorner}[t]\| \le \|A[t]\| \le M$ for all t. Thus, by the Proposition 2.2(1) and Lemma 2.4, Af is a sequence in $C(X)$ for all $f \in l_2(C(X))$. We now want to show that $Af \in l_2(C(X))$ for all $f \in l_2(C(X))$. Let $f \in l_2(C(X))$. Then by Proposition 1.2, the function $t \mapsto f[t]$ from X into l_2 is continuous. For each $t \in X$, we have $A[t] \in \mathcal{B}(l_2)$. So $Af[t] = A[t]f[t] \in l_2$ for all t. It follows that the function $t \mapsto Af[t]$ from X into l_2 is well defined. For any $s, t \in X$, we have

$$\begin{aligned}\|Af[s] - Af[t]\|_2 &= \|A[s]f[s] - A[t]f[t]\|_2 \\ &\le \|A[s]f[s] - A[s]f[t]\|_2 + \|A[s]f[t] - A[t]f[t]\|_2 \\ &\le \|A[s]\|\,\|f[s] - f[t]\|_2 + \|A[s] - A[t]\|\,\|f[t]\|_2 \\ &\le M\,\|f[s] - f[t]\|_2 + \|A[s] - A[t]\|\,\|f\|_2\,.\end{aligned}$$

Hence, by the continuity of the functions $t \mapsto A[t]$ and $t \mapsto f[t]$, we obtain that the function $t \mapsto Af[t]$ is continuous. So, by Proposition 1.2, $Af \in l_2(C(X))$. □

Theorem 2.7. *$\mathcal{B}(l_2(C(X)))$ equipped with the operator norm is a Banach space. Furthermore, $\mathcal{B}(l_2(C(X)))$ contains the identity operator, and for $A = [a_{ji}], B = [a_{ik}] \in \mathcal{B}(l_2(C(X)))$, the matrix*

$$AB := \left[\sum_{i=1}^{\infty} a_{ji} b_{ik}\right]$$

belongs to $\mathcal{B}(l_2(C(X)))$ and $(AB)f = A(Bf)$ for all $f \in l_2(C(X))$. In other words, $\mathcal{B}(l_2(C(X)))$ is a Banach subalgebra with identity of the Banach algebra of all bounded linear operators on $l_2(C(X))$.

Proof. Let $\{A_n = [a_{jk}^{(n)}]\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{B}(l_2(C(X)))$. By Proposition 2.2(1), we have for each $(j,k) \in \mathbb{N} \times \mathbb{N}$ and $t \in X$ that

$$(*) \qquad \left|a_{jk}^{(n)}(t) - a_{jk}^{(m)}(t)\right| \leq \|A_n[t] - A_m[t]\| \leq \|A_n - A_m\| \text{ for all } n, m.$$

Thus, for any (j,k), $\left\|a_{jk}^{(n)} - a_{jk}^{(m)}\right\|_{C(X)} \leq \|A_n - A_m\|$ for all n, m. So $\{a_{jk}^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $C(X)$ for all (j,k). Hence, by completeness of $C(X)$, there exists $a_{jk} \in C(X)$ such that $a_{jk}^{(n)} \to a_{jk}$ as $n \to \infty$. Put $A = [a_{jk}]$. We will show that $A \in \mathcal{B}(l_2(C(X)))$ and $A_n \to A$ as $n \to \infty$. Let $\nu \in \mathbb{N}$ and $x = \{\xi_k\}_{k=1}^{\infty} \in l_2$ with $\|x\|_2 \leq 1$. Let $M = \sup_n \|A_n\|$. Then for each $t \in X$,

$$\begin{aligned} \|A_{\nu\lrcorner}[t]x\|_2^2 &= \sum_{j=1}^{\nu} \left|\sum_{k=1}^{\nu} a_{jk}(t)\xi_k\right|^2 \\ &\leq 4\sum_{j=1}^{\nu} \left|\sum_{k=1}^{\nu} \left(a_{jk}^{(n)}(t) - a_{jk}(t)\right)\xi_k\right|^2 + 4M^2 \text{ for all } n. \end{aligned}$$

By taking the limit as $n \to \infty$, we get $\|A_{\nu\lrcorner}[t]x\| \leq 2M$ for all t. Thus $\|A_{\nu\lrcorner}[t]\| \leq 2M$ for all t. It follows from Proposition 2.2(1) that $\|A_{\nu\lrcorner}\| \leq 2M$ for all ν. Hence, by Lemma 2.4, Af is a sequence in $C(X)$ for all $f \in l_2(C(X))$. Let $\epsilon > 0$ be given. Since $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there exists a positive integer N such that $\|A_n - A_m\| < \frac{\epsilon}{2}$ for all $n, m \geq N$. By $(*)$, we also have that $\{A_n[t]\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}(l_2)$ for all t. Hence, for each t, there exists $B[t] = [b_{jk}(t)] \in \mathcal{B}(l_2)$ such that $A_n[t] \to B[t]$ as $n \to \infty$. For each (j,k), we have for every t that $|a_{jk}^{(n)}(t) - b_{jk}(t)| \leq \|A_n[t] - B[t]\| \to 0$ as $n \to \infty$. It follows that $A[t] = B[t]$ for all t. Let $f = \{f_k\}_{k=1}^{\infty} \in l_2(C(X))$ with $\|f\|_2 \leq 1$. Then

$$\begin{aligned} \sup_{t \in X} \|(A_n[t] - A_m[t])f[t]\|_2 &= \|(A_n - A_m)f\|_2 \leq \|A_n - A_m\| \\ &< \frac{\epsilon}{2} \quad \text{for all } n, m \geq N. \end{aligned}$$

By taking the limit as $m \to \infty$, we get

$$(**) \qquad \|(A_n - A)f\|_2 \leq \frac{\epsilon}{2} \quad \text{for all } n \geq N.$$

This gives us that $Af \in l_2^b(C(X))$ and $A_n f \to Af$ in $l_2^b(C(X))$. Since $A_n \in \mathcal{B}(l_2(C(X)))$ for all n, $A_n f \in l_2(C(X))$ by closedness of $l_p(C(X))$ in $l_p^b(C(X))$. Hence $Af \in l_2(C(X))$. Thus $A \in \mathcal{B}(l_2(C(X)))$. Since $(**)$ holds for arbitrary $f \in l_2(C(X))$, $\|A_n - A\| < \epsilon$ for all $n \geq N$. Consequently, $A_n \to A$ as $n \to \infty$.

It is obvious that the linear operator defined by the matrix with entries in the main diagonal 1 and all other entries 0 is exactly the identity operator on $l_2(C(X))$. Let $A = [a_{ji}], B = [b_{ik}] \in \mathcal{B}(l_2(C(X)))$. Then for each k, $\{\sum_{i=1}^{\infty} a_{ji}b_{ik}\}_{j=1}^{\infty} = A(Be_k) \in l_2(C(X))$, where e_k is the sequence with kth coordinate 1 and all other coordinates 0. Thus the series $\sum_{i=1}^{\infty} a_{ji}b_{ik}$ converges in $C(X)$ for all (j,k). So the matrix $AB = [\sum_{i=1}^{\infty} a_{ji}b_{ik}]$ is well defined. We will show that AB defines a linear operator on $l_2(C(X))$ and $(AB)f = A(Bf)$ for all $f \in l_2(C(X))$. Let $f = \{f_k\}_{k=1}^{\infty} \in l_2(C(X))$. Then we have for every n that

$$\begin{aligned}
\|(AB)_{n\lrcorner} f\|_2 &= \sup_{t\in X}\left(\sum_{j=1}^{n}\left|\sum_{k=1}^{n}\sum_{i=1}^{\infty} a_{ji}(t)b_{ik}(t)f_k(t)\right|^2\right)^{1/2} \\
&= \sup_{t\in X}\left(\sum_{j=1}^{n}\left|\sum_{i=1}^{\infty}\sum_{k=1}^{n} a_{ji}(t)b_{ik}(t)f_k(t)\right|^2\right)^{1/2} \\
&= \|A_{n\lrcorner}(Bf_{n\lrcorner})\|_2 \leq \|A_{n\lrcorner}\|\,\|B\|\,\|f_{n\lrcorner}\|_2 \leq \|A\|\,\|B\|\,\|f\|_2\,.
\end{aligned}$$

It follows that $\|(AB)_{n\lrcorner}\| \leq \|A\|\,\|B\|$ for all n. Hence, by Lemma 2.4, we obtain that $(AB)f$ is a sequence in $C(X)$. Since $A[t]$ and $B[t]$ belong to $\mathcal{B}(l_2)$ for all t, $(AB)[t]f[t] = A[t](B[t]f[t])$. This implies that $(AB)f = A(B(f))$, so $(AB)f \in l_2(C(X))$. The proof is complete. □

For $A \in \mathcal{B}(l_2(C(X)))$, we have by Proposition 2.2(1) that $A[t] \in \mathcal{B}(l_2)$ for all $t \in X$. So the function $c_A : X \to \mathcal{B}(l_2)$ defined by $c_A(t) = A[t]$ for all $t \in X$ is well defined. Let $\mathcal{B}_c(l_2(C(X)))$ be the set of matrices A in $\mathcal{B}(l_2(C(X)))$ such that the function c_A is continuous. For the case where X is a singleton, we have that $\mathcal{B}_c(l_2(C(X))) = \mathcal{B}(l_2(C(X))) = \mathcal{B}(l_2)$.

Proposition 2.8. *The inclusion $\mathcal{B}_c(l_2(C(X))) \subseteq \mathcal{B}(l_2(C(X)))$ can be proper.*

Proof. Let $X = [0,1]$ and A be the matrix with the first row the sequence $f\langle 2\rangle = \{f_k\}_{k=1}^{\infty}$ defined in Example 1.1 and all other rows 0. Since $f\langle 2\rangle \in l_2^b(C(X))$, by Theorem 1.3, we get that $A \in \mathcal{B}(l_2(C(X)))$. Let $t_n = 1 - \frac{1}{n}$ for $n = 1,2,3,\ldots$. We have that $t_n \to 1$ as $n \to \infty$ and $\sum_{k=n}^{2n} |f_k(t_n)|^2 = \left(1-\frac{1}{n}\right)^n - \left(1-\frac{1}{n}\right)^{2n+1} \to \frac{1}{e} - \frac{1}{e^2}$ as $n \to \infty$. Obviously, $A[1] = 0$. We claim that $A[t_n]$ does not converge to $A[1]$. Suppose that $A[t_n] \to A[1]$ as $n \to \infty$. Fix $0 < \epsilon < \frac{1}{e} - \frac{1}{e^2}$, then there exists a positive integer N such that $\sum_{k=n}^{2n} |f_k(t_n)|^2 \leq \|f\langle 2\rangle[t_n]\|_2^2 = \|A[t_n]\|^2 < \epsilon$ for all $n \geq N$. By letting $n \to \infty$, we obtain that $\frac{1}{e} - \frac{1}{e^2} \leq \epsilon$, which is a contradiction. □

Theorem 2.9. *$\mathcal{B}_c(l_2(C(X)))$ is a Banach subalgebra with identity of $\mathcal{B}(l_2(C(X)))$.*

Proof. To see that $\mathcal{B}_c(l_2(C(X)))$ is a Banach space, we will show that it is a closed subspace of $\mathcal{B}(l_2(C(X)))$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{B}_c(l_2(C(X)))$ and $A \in \mathcal{B}(l_2(C(X)))$. Suppose that $A_n \to A$ as $n \to \infty$. We want to show that $A \in \mathcal{B}_c(l_2(C(X)))$. Let $\{t_\alpha\}$ be a net in X and suppose that $t_\alpha \to t$ for some $t \in X$. Let $\epsilon > 0$ be given. Then there exists a positive integer N such that $\|A_N - A\| < \frac{\epsilon}{3}$. Since $A_N \in \mathcal{B}_c(l_2(C(X)))$, $A_N[t_\alpha] \to A_N[t]$. Hence there exists γ such that $\|A_N[t_\alpha] - A_N[t]\| < \frac{\epsilon}{3}$ for all $\alpha \succeq \gamma$. So, for $\alpha \succeq \gamma$,

$$\begin{aligned} \|A[t_\alpha] - A[t]\| &\leq \|A_N[t_\alpha] - A[t_\alpha]\| + \|A_N[t] - A[t]\| + \|A_N[t_\alpha] - A_N[t]\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

It follows that $A \in \mathcal{B}_c(l_2(C(X)))$. Therefore $\mathcal{B}_c(l_2(C(X)))$ is a Banach subspace of $\mathcal{B}(l_2(C(X)))$. It is clear that the identity matrix belongs to $\mathcal{B}_c(l_2(C(X)))$. For $A, B \in \mathcal{B}_c(l_2(C(X)))$, we have for all $t \in X$ that $\|A[t]\| \leq \|A\|$, $\|B[t]\| \leq \|B\|$ and $c_{AB}(t) = AB[t] = A[t]B[t] = c_A(t)c_B(t)$. Since $\mathcal{B}(l_2)$ is a Banach algebra under the composition operation, c_{AB} is continuous. □

The following are consequences of Proposition 2.6.

Proposition 2.10. *$\mathcal{B}_c(l_2(C(X)))$ is equal to the set of all matrices A over $C(X)$ such that $A[t] \in \mathcal{B}(l_2)$ for all $t \in X$ and the function $t \mapsto A[t]$ from X into $\mathcal{B}(l_2)$ is continuous.*

Proof. It follows directly from Proposition 2.6. □

For $A = [a_{jk}] \in \mathcal{B}(l_2(C(X)))$, we let $A^* = [c_{jk}]$, where $c_{jk} = \overline{a_{kj}}$ for all j, k. In the case where X is a singleton, we have that A^* is exactly the adjoint of A, so it belongs to $\mathcal{B}(l_2(C(X)))$. In general, this is not true: for example, consider the matrix A with the first row the sequence $f\langle 2\rangle$ given in Example 1.1 and all other rows 0. We have seen from Proposition 2.8 and Example 2.3 that $A \in \mathcal{B}(l_2(C(X)))$ and $A^* \notin \mathcal{B}(l_2(C(X)))$.

Proposition 2.11. *If $A \in \mathcal{B}_c(l_2(C(X)))$, then $A^* \in \mathcal{B}_c(l_2(C(X)))$.*

Proof. It follows immediately from the continuity of the function $B \mapsto B^*$ on $\mathcal{B}(l_2)$ and Proposition 2.6. □

Corollary 2.12. *$\mathcal{B}_c(l_2(C(X)))$ equipped with the involution $A \mapsto A^*$ is a C^*-algebra with identity.*

Proposition 2.13. *$\mathcal{B}_c(l_2(C(X)))$ is a Banach algebra (without identity) under the Schur product.*

Proof. Let $A, B \in \mathcal{B}_c(l_2(C(X)))$. Then by Schur-Bennett's theorem [1, 9]: $\mathcal{B}(l_2)$ is a Banach algebra under the Schur product, we obtain that the function $c_{A \bullet B}$ is well defined. Since the functions c_A and c_B are continuous, and we have $\|A[t]\| \leq \|A\|$ and $\|B[t]\| \leq \|B\|$ for all t, the function $c_{A \bullet B}$ is continuous. Thus, by Proposition 2.6, $A \bullet B \in \mathcal{B}_c(l_2(C(X)))$. By Schur-Bennett's theorem again, we obtain $\|(A \bullet B)[t]\| = \|A[t] \bullet B[t]\| \leq \|A[t]\| \, \|B[t]\| \leq \|A\| \, \|B\|$ for all t. It follows from Proposition 2.2(1) that $\|A \bullet B\| \le \|A\|\,\|B\|$. So $\mathcal{B}_c(l_2(C(X)))$ is a Banach algebra under the Schur product. $\square$

We do not however know if $\mathcal{B}(l_2(C(X)))$ is closed under the Schur product.

3. Schatten classes of matrices in $\mathcal{B}_c(l_2(C(X)))$

Let $\mathcal{M}_0$ be the set of matrices over $C(X)$ having finitely many nonzero entries, and $\mathcal{K}(C(X))$ be the closure of $\mathcal{M}_0$ in $\mathcal{B}(l_2(C(X)))$.

Proposition 3.1. $\mathcal{K}(C(X)) \subsetneq \mathcal{B}_c(l_2(C(X)))$.

Proof. It is easy to see that $\mathcal{M}_0 \subseteq \mathcal{B}_c(l_2(C(X)))$. Hence, by Theorem 2.9, $\mathcal{K}(C(X)) \subseteq \mathcal{B}_c(l_2(C(X)))$. Since the identity matrix does not belong to $\mathcal{K}(C(X))$, the inclusion is proper. $\square$

Proposition 3.2. *$A \in \mathcal{K}(C(X))$ if and only if $\|A_{n_\lrcorner} - A\| \to 0$ as $n \to \infty$.*

Proof. Suppose that $A \in \mathcal{K}(C(X))$ and let $\epsilon > 0$. Then there exists $B \in \mathcal{M}_0$ such that $\|A - B\| < \frac{\epsilon}{2}$. Let N be a positive integer such that $B_{N_\lrcorner} = B$. Then for $n \ge N$, $A_{n_\lrcorner} - B = (A - B)_{n_\lrcorner}$. Hence, by Proposition 2.2(2), we have

$$\begin{aligned} \|A_{n_\lrcorner} - A\| &\le \|A - B\| + \|A_{n_\lrcorner} - B\| = \|A - B\| + \|(A - B)_{n_\lrcorner}\| \\ &\le 2\,\|A - B\| < \epsilon \quad \text{for all } n \ge N. \end{aligned}$$

The converse is obvious. $\square$

Let $\mathcal{K}$ be the class of compact operators on l_2. If X is a singleton, then $\mathcal{K}(C(X)) = \mathcal{K}$.

Proposition 3.3. $\mathcal{K}(C(X)) = \{A \in \mathcal{B}_c(l_2(C(X))) : A[t] \in \mathcal{K} \text{ for all } t \in X\}$.

Proof. Suppose that $A \in \mathcal{B}_c(l_2(C(X)))$ with $A[t] \in \mathcal{K}$ for all $t \in X$. Let $\epsilon > 0$ be given. Then by the continuity of the function c_A, we get for each $t \in X$ that there exists an open set $U(t)$ in X such that $t \in U(t)$ and

$$\|A[t] - A[s]\| < \frac{\epsilon}{4} \text{ for all } s \in U(t).$$

Since X is compact, there exist $t_1, t_2, \dots, t_m \in X$ such that $X = U(t_1) \cup U(t_2) \cup \cdots \cup U(t_m)$. Since $A[t_i] \in \mathcal{K}$ for all $i \in \{1, 2, \dots, m\}$, there exists, for each i, a positive integer N_i such that

$$\|A_{n_\lrcorner}[t_i] - A[t_i]\| < \frac{\epsilon}{4} \text{ for all } n \ge N_i.$$

Put $N = \max\{N_1, N_2, \dots, N_m\}$ and let $s \in X$. Then there exists $i_0 \in \{1, 2, \dots, m\}$ such that $s \in U(t_{i_0})$. Hence if $n \ge N$, we have that

$$\begin{aligned} \|A_{n_\lrcorner}[s] - A[s]\| &\le \|A_{n_\lrcorner}[t_{i_0}] - A_{n_\lrcorner}[s]\| + \|A_{n_\lrcorner}[t_{i_0}] - A[t_{i_0}]\| + \|A[t_{i_0}] - A[s]\| \\ &\le 2\,\|A[t_{i_0}] - A[s]\| + \|A_{n_\lrcorner}[t_{i_0}] - A[t_{i_0}]\| < \frac{2\epsilon}{4} + \frac{\epsilon}{4} = \frac{3\epsilon}{4}. \end{aligned}$$

Thus, by Proposition 2.2(1), we obtain that

$$\|A_{n_\lrcorner} - A\| < \epsilon \text{ for all } n \ge N.$$

So $A \in \mathcal{K}(C(X))$. The reverse inclusion follows immediately from Proposition 3.1 and Proposition 2.2(1). The proof is complete. □

Corollary 3.4. *$\mathcal{K}(C(X))$ is a proper closed ideal of $\mathcal{B}_c(l_2(C(X)))$.*

Proof. If $A \in \mathcal{K}(C(X))$ and $B \in \mathcal{B}_c(l_2(C(X)))$, then $AB[t]$ and $BA[t]$ are elements of $\mathcal{K}$ for all t. Hence, by proposition above, both AB and BA belong to $\mathcal{K}(C(X))$. Consequently, $\mathcal{K}(C(X))$ is an ideal of $\mathcal{B}_c(l_2(C(X)))$. □

The following example shows us that $\mathcal{K}(C(X))$ may not be an ideal of $\mathcal{B}(l_2(C(X)))$.

Example 3.5. Let $X = [0,1]$, let A be the matrix whose $(1,1)$ entry is 1 and all other entries 0, and let B be the matrix with the first row the sequence $f\langle 2\rangle$ given in Example 1.1 and all other rows 0. Clearly, $AB = B$. We have seen from Proposition 2.8 that $B \in \mathcal{B}(l_2(C([0,1]))) \setminus \mathcal{B}_c(l_2(C([0,1])))$.

If $A \in \mathcal{K}(C(X))$, then $A[t] \in \mathcal{K}$ for all $t \in X$. For each $t \in X$, let $\{s_n(A[t])\}_{n=1}^{\infty}$ be the sequence of singular values of $A[t]$. For each $n \in \mathbb{N}$, let $\widetilde{s}_n(A) : X \to [0,\infty)$ be the function defined by

$$\widetilde{s}_n(A)(t) = s_n(A[t]) \quad \text{for all } t \in X.$$

Theorem 3.6. *For $A \in \mathcal{K}(C(X))$, the function $\widetilde{s}_n(A)$ is continuous for all n. Furthermore, $\widetilde{s}_1(A)(t) \geq \widetilde{s}_2(A)(t) \geq \cdots \geq 0$ for all $t \in X$ and $\widetilde{s}_n(A) \to 0$ as $n \to \infty$ in $C(X)$.*

Proof. Let $A \in \mathcal{K}(C(X))$. For each $n \in \mathbb{N}$, we defined a function $f_n : \mathcal{K} \to [0,\infty)$ by $f_n(B) = s_n(B)$ for all $B \in \mathcal{K}$. Clearly, $\widetilde{s}_n(A) = f_n c_A$ for all n, hence, by Proposition 3.1, the continuity of $\widetilde{s}_n(A)$ will be proved once we can show that f_n is continuous. The continuity of f_n follows directly from the fact that $|s_n(B) - s_n(C)| \leq \|B - C\|$ for all $B, C \in \mathcal{K}$. It is clear that for every $t \in X$, $\widetilde{s}_1(A)(t) \geq \widetilde{s}_2(A)(t) \geq \cdots \geq 0$. Since X is compact and $\widetilde{s}_n(A)(t) \to 0$ as $n \to \infty$ for all $t \in X$, $\|\widetilde{s}_n(A)\|_{C(X)} \to 0$ as $n \to \infty$. □

For $1 \leq p < \infty$, we define three classes of matrices in $\mathcal{K}(C(X))$ as follows:

$$\begin{aligned}
\mathcal{C}_p^b(C(X)) &= \left\{A \in \mathcal{K}(C(X)) : \{\widetilde{s}_n(A)\}_{n=1}^{\infty} \in l_p^b(C(X))\right\};\\
\mathcal{C}_p(C(X)) &= \{A \in \mathcal{K}(C(X)) : \{\widetilde{s}_n(A)\}_{n=1}^{\infty} \in l_p(C(X))\};\\
\mathcal{C}_p^c(C(X)) &= \{A \in \mathcal{K}(C(X)) : \text{the function } t \mapsto A[t] \text{ from } X \text{ into } \mathcal{C}_p\\
&\qquad \text{is continuous}\}.
\end{aligned}$$

It is clear that $A \in \mathcal{C}_p^b(C(X))$ if and only if $A \in \mathcal{K}(C(X))$ and $\sup_{t\in X} \, |||A[t]|||_p < \infty$.

Proposition 3.7. $\mathcal{C}_p^c(C(X)) \subseteq \mathcal{C}_p(C(X)) \subseteq \mathcal{C}_p^b(C(X))$.

Proof. It is obvious that both $\mathcal{C}_p^c(C(X))$ and $\mathcal{C}_p(C(X))$ are subsets of $\mathcal{C}_p^b(C(X))$. We will show that $\mathcal{C}_p^c(C(X)) \subseteq \mathcal{C}_p(C(X))$. Suppose that $A \in \mathcal{C}_p^c(C(X))$. Then the function $t \mapsto A[t]$ from X into $\mathcal{C}_p$ is continuous. This implies that the function $t \mapsto \|\|A[t]\|\|_p = \|\{\widetilde{s}_k(A)(t)\}_{k=1}^\infty\|_p$ is continuous. It follows from Proposition 1.2 that $\{\widetilde{s}_k(A)\}_{k=1}^\infty \in l_p(C(X))$, so $A \in \mathcal{C}_p(C(X))$. □

The following example shows that the inclusion $\mathcal{C}_p(C(X)) \subseteq \mathcal{C}_p^b(C(X))$ can be proper. So the inclusion $\mathcal{C}_p^c(C(X)) \subseteq \mathcal{C}_p^b(C(X))$ can also be proper. For the space $l_p(C(X))$, we have that $f \in l_p(C(X))$ if and only if the function $t \mapsto f[t]$ from X into l_p is continuous. We expect to have a similar characterization for $\mathcal{C}_p(C(X))$, i.e., $A \in \mathcal{C}_p(C(X))$ if and only if the function $t \mapsto A[t]$ from X into $\mathcal{C}_p$ is continuous (or equivalently, $\mathcal{C}_p(C(X)) = \mathcal{C}_p^c(C(X))$). Now, we have $\mathcal{C}_p^c(C(X)) \subseteq \mathcal{C}_p(C(X))$, but we do not know if we have equality or an example of proper inclusion.

Example 3.8. $\mathcal{C}_p(C(X)) \subsetneq \mathcal{C}_p^b(C(X))$. Let $X = [0,1]$ and let A be the matrix with the main diagonal the sequence $f\langle p\rangle = \{f_k\}_{k=1}^\infty$ given in Example 1.1 and all other entries 0. It is easy to see that $A \in \mathcal{B}(l_2(C([0,1])))$. Since $f_k \to 0$ as $n \to \infty$ in $C(X)$, $A \in \mathcal{K}(C([0,1]))$. It is clear that $\widetilde{s}_k(A) = f_k$ for all k. Hence $A \in \mathcal{C}_p^b(C([0,1])) \setminus \mathcal{C}_p(C([0,1]))$.

It is easy to see that $\mathcal{C}_p^b(C(X))$ and $\mathcal{C}_p^c(C(X))$ are linear spaces. For the case where X is infinite, we do not know if $\mathcal{C}_p(C(X))$ is closed under addition.

Theorem 3.9. *$\mathcal{C}_p^b(C(X))$ and $\mathcal{C}_p^c(C(X))$ equipped with the norm*

$$\|\|A\|\|_p := \sup_{t\in X} \|\|A[t]\|\|_p$$

are Banach spaces.

Proof. Let $\{A_n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{C}_p^b(C(X))$. Note that for any $B \in \mathcal{C}_p$, $\|B\| = s_1(B) \leq \|\|B\|\|_p$. This gives us that $\{A_n\}_{n=1}^\infty$ is also a Cauchy sequence in $\mathcal{K}(C(X))$. So there exists A in $\mathcal{K}(C(X))$ such that $A_n \to A$ in $\mathcal{K}(C(X))$. We will show that $A \in \mathcal{C}_p^b(C(X))$ and $A_n \to A$ as $n \to \infty$. Since $\{A_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{C}_p^b(C(X))$, $\{A_n[t]\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{C}_p$ for all $t \in X$. Thus, for each t, we have by completeness of $\mathcal{C}_p$ that there exists A_t in $\mathcal{C}_p$ such that $A_n[t] \to A_t$ as $n \to \infty$. From this, we have $A_n[t] \to A_t$ in $\mathcal{K}$ for all t. Since $A_n \to A$ in $\mathcal{K}(C(X))$, $A_n[t] \to A[t]$ in $\mathcal{K}$ for all t. Hence $A[t] = A_t$ for all t. Let $\epsilon > 0$ be given. Then there exists a positive integer N such that for any t, $\|\|A_n[t] - A_m[t]\|\|_p \leq \|\|A_n - A_m\|\|_p \leq \frac{\epsilon}{2}$ for all $n, m \geq N$. By taking the limit as $m \to \infty$, we obtain for each t that $\|\|A_n[t] - A[t]\|\|_p \leq \frac{\epsilon}{2}$ for all $n \geq N$. It follows that $\|\|A_n - A\|\|_p = \sup_{t\in X} \|\|A_n[t] - A[t]\|\|_p \leq \frac{\epsilon}{2} < \epsilon$ for all $n \geq N$. This gives us that $A \in \mathcal{C}_p^b(C(X))$ and $A_n \to A$ as $n \to \infty$. Accordingly, $\mathcal{C}_p^b(C(X))$ is a Banach space.

To see that $\mathcal{C}_p^c(C(X))$ is a Banach space, we will show that it is a closed subspace of $\mathcal{C}_p^b(C(X))$. Let $\{A_n\}_{n=1}^\infty$ be a sequence in $\mathcal{C}_p^c(C(X))$ and $A \in$

$\mathcal{C}_p^b(C(X))$. Suppose that $A_n \to A$ as $n \to \infty$. Let $\{t_\alpha\}$ be a net in X such that $t_\alpha \to t$ for some $t \in X$. We want to show that $\|A[t_\alpha] - A[t]\| \to 0$. Let $\epsilon > 0$. Then there exists a positive integer N such that $\|A_N - A\|_p < \frac{\epsilon}{3}$. Since $A_N \in \mathcal{C}_p^c(C(X))$, there is γ such that $\|A_N[t_\alpha] - A_N[t]\|_p < \frac{\epsilon}{3}$ for all $\alpha \succeq \gamma$. So

$$\begin{aligned}\|A[t_\alpha] - A[t]\|_p &\le \|A_N[t_\alpha] - A[t_\alpha]\|_p + \|A_N[t] - A[t]\|_p + \|A_N[t_\alpha] - A_N[t]\|_p \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for all } \alpha \succeq \gamma.\end{aligned}$$

The proof is complete. □

Proposition 3.10. *$\mathcal{C}_p^b(C(X))$ and $\mathcal{C}_p^c(C(X))$ are ideals of $\mathcal{B}_c(l_2(C(X)))$.*

Proof. We will first show that $\mathcal{C}_p^b(C(X))$ is an ideal of $\mathcal{B}_c(l_2(C(X)))$. Let $A \in \mathcal{C}_p^b(C(X))$ and $B \in \mathcal{B}_c(l_2(C(X)))$. Then by Corollary 3.4, both AB and BA belong to $\mathcal{K}(C(X))$. Since $A[t] \in \mathcal{C}_p$ and $B[t] \in \mathcal{B}(l_2)$ for all $t \in X$, $\|(AB)[t]\|_p \le \|A[t]\|_p \|B[t]\| \le \|A\|_p \|B\|$. Similarly, we have $\|(BA)[t]\|_p \le \|A\|_p \|B\|$ for all t. This implies that both AB and BA belong to $\mathcal{C}_p^b(C(X))$, so $\mathcal{C}_p^b(C(X))$ is an ideal of $\mathcal{B}_c(l_2(C(X)))$.

To show that $\mathcal{C}_p^c(C(X))$ is an ideal of $\mathcal{B}_c(l_2(C(X)))$, suppose that $A \in \mathcal{C}_p^c(C(X))$ and $B \in \mathcal{B}_c(l_2(C(X)))$. We will show that $AB \in \mathcal{C}_p^c(C(X))$. By the fact above, we have $AB \in \mathcal{C}_p^b(C(X))$. For any $s, t \in X$, we have

$$\begin{aligned}\|AB[s] - AB[t]\|_p &= \|A[s]B[s] - A[t]B[t]\|_p \\ &\le \|A[s]B[s] - A[s]B[t]\|_p + \|A[s]B[t] - A[t]B[t]\|_p \\ &\le \|A[s]\|_p \|B[s] - B[t]\| + \|A[s] - A[t]\|_p \|B[t]\| \\ &\le \|A\|_p \|B[s] - B[t]\| + \|A[s] - A[t]\|_p \|B\|.\end{aligned}$$

So, by the assumption, the function $t \mapsto AB[t]$ is continuous. This means that $AB \in \mathcal{C}_p^c(C(X))$. By using a similar argument, we also have that $BA \in \mathcal{C}_p^c(C(X))$. It follows that $\mathcal{C}_p^c(C(X))$ is an ideal of $\mathcal{B}_c(l_2(C(X)))$. □

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