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COMMON STABILIZATIONS OF HEEGAARD SPLITTINGS OF KNOT EXTERIORS

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Abstract. In [4], we gave a condition for a pair of unknotting tunnels of a non-trivial tunnel number one link to give a genus three Heegaard splitting of the link complement. In this paper we prove the corresponding result for tunnel number one knots.

1. Introduction

For a closed 3-manifold M, a decomposition $M = H_1 \cup_S H_2$, which is a gluing of two handlebodies H_1 and H_2 along their common boundary S, is called a *Heegaard splitting* of M.

To describe Heegaard splitting of 3-manifold with non-empty boundary, we use compression bodies instead of handlebodies. A *compression body* H is a 3-manifold obtained from a closed surface S by attaching 2-handles to $S \times I$ on $S \times \{1\}$ and capping off any resulting 2-sphere boundary components with 3-balls. $S \times \{0\}$ is denoted by $\partial_+ H$ and $\partial H - \partial_+ H$ is denoted by $\partial_- H$. Then for a 3-manifold M with non-empty boundary, a Heegaard splitting $M = H_1 \cup_S H_2$ is a gluing of two compression bodies H_1 and H_2 along their common "plus" boundary. It is known that every compact 3-manifold has a Heegaard splitting.

For a Heegaard splitting $H_1 \cup_S H_2$, let α be a properly embedded arc in H_2 which is ∂-parallel. Attach a neighborhood of α in H_2 to H_1 and remove it from H_2. The result is again a Heegaard splitting $H_1' \cup_{S'} H_2'$ with the genus increased by one. This operation is called a *stabilization*.

Suppose there are two Heegaard splittings of same genus $H_1 \cup_S H_2$ and $H_1' \cup_{S'} H_2'$ for a 3-manifold M. The two splittings become isotopic after stabilizing both sufficiently many times [8]. However, no example has been known where actually we should stabilize more than once to make them isotopic. It has been conjectured that a single stabilization always suffices [2] (Problem 3.89) and there are some results to this direction [4], [6], [7].

Very recently, in [1] Hass, Thompson and Thurston gave examples of 3-manifolds with two genus g Heegaard splittings requiring g stabilizations to

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become isotopic. We remark that the manifolds M_g in Theorem 1.1 are closed 3-manifolds.

Theorem 1.1 ([1]). *For each $g > 1$ there is a 3-manifold M_g with two genus g Heegaard splittings that require g stabilizations to become equivalent.*

Now we consider Heegaard splittings of knot exteriors and the stabilization problem. A *tunnel system* (or *tunnels*) of a knot (or link) K in S^3 is a collection of properly embedded arcs $t_1, t_2, \ldots, t_n$ in the exterior of K such that $H = \overline{S^3 - N(K \cup \bigcup_{i=1}^n t_i)}$ is a genus $n+1$ handlebody. The tunnel system gives rise to a Heegaard splitting of the exterior of K

$$\overline{S^3 - N(K)} = H \bigcup_{\partial H} \overline{N(K \cup \bigcup_{i=1}^{n} t_i) - N(K)} \ ,$$

where $N(K)$ is in the interior of $N(K \cup \bigcup_{i=1}^n t_i)$. The minimum of such number n is called the *tunnel number* of K. If the tunnel number of K is 1, the tunnel is called an *unknotting tunnel* for K.

For 3-manifolds with boundary, the stabilization conjecture can possibly fail also as in Theorem 1.1. In [3], candidates for counterexamples of genus three Heegaard splittings of knot exteriors are given.

From now we focus on the genus two case. Consider two disjoint unknotting tunnels t_1 and t_2 of a tunnel number one knot (or link) K. Suppose $H = \overline{S^3 - N(K \cup t_1 \cup t_2)}$ is a genus three handlebody. Then by the uniqueness of Heegaard splittings of handlebody [5], $S = \partial H$ is a Heegaard surface which is a common stabilization of Heegaard splittings induced by t_1 and induced by t_2.

However $H = \overline{S^3 - N(K \cup t_1 \cup t_2)}$ may not be a handlebody, hence ∂H is not a Heegaard surface even if t_1 and t_2 are isotopic tunnels. For example, take t_2 as a parallel copy of t_1. Pull a part of t_2 in a complicated way and hook it to t_1. This construction does not give a genus three Heegaard surface (Fig. 1). So there must be some restrictions on the choice of unknotting tunnels.

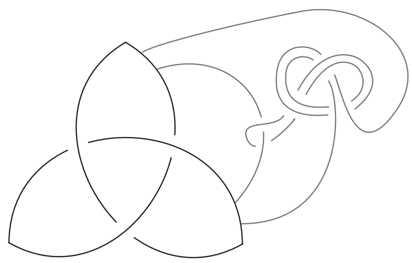

FIGURE 1. Embedding of tunnels which does not give a genus three splitting

In [4], we gave a sufficient condition for a pair of unknotting tunnels of a non-trivial tunnel number one link to give a genus three Heegaard splitting of the link exterior.

Theorem 1.2 ([4]). *Let $L = K_1 \cup K_2$ be a non-trivial tunnel number one link and t_1 and t_2 be two disjoint unknotting tunnels of L such that a meridian disk D of the genus two handlebody $H_1 = \overline{S^3 - N(L \cup t_1)}$ does not intersect t_2. Then $\overline{S^3 - N(L \cup t_1 \cup t_2)}$ is a genus three handlebody.*

Some arguments in the proof of Theorem 1.2 worked only for 2-component links. In this paper we prove the corresponding result for tunnel number one knots.

Theorem 1.3. *Let K be a non-trivial tunnel number one knot and t_1 and t_2 be two disjoint unknotting tunnels of K such that an essential disk D of the genus two handlebody $H_1 = \overline{S^3 - N(K \cup t_1)}$ does not intersect t_2. Then $\overline{S^3 - N(K \cup t_1 \cup t_2)}$ is a genus three handlebody.*

2. Proof of Theorem 1.3

Let H_2 be the genus two handlebody $\overline{S^3 - N(K \cup t_2)}$. ∂H_1 (respectively ∂H_2) consists of two parts as in the Fig. 2—twice punctured torus $T_{1,K}$ (respectively $T_{2,K}$) and an annulus A_1 (respectively A_2). Let E_1 and E_2 be two non-separating essential disks of H_2 which are not parallel to each other. Then $E_1 \cup E_2$ cuts H_2 into a 3-ball. We may choose such E_1 and E_2 so as to satisfy $\partial T_{1,K} \cap E_i = \emptyset$ $(i = 1, 2)$ and minimize $|D \cap (E_1 \cup E_2)|$ which is the number of components of $D \cap (E_1 \cup E_2)$.

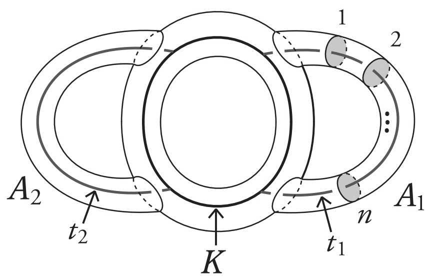

FIGURE 2. $N(K \cup t_1) \cap \text{int}(E_1 \cup E_2)$

Lemma 2.1. *$\partial D \cap A_1$ and $\partial E_i \cap A_2$ $(i = 1, 2)$ are non-empty. We may assume that $\partial D \cap T_{1,K}$, $\partial D \cap A_1$ consists of essential arcs and also $\partial E_i \cap T_{2,K}$, $\partial E_i \cap A_2$ $(i = 1, 2)$ consists of essential arcs.*

Proof. Suppose $\partial D \cap A_1 = \emptyset$. Then ∂D is in $\partial N(K)$. If ∂D is a meridian of $\partial N(K)$, then a punctured $S^2 \times S^1$ is in S^3, a contradiction. If ∂D is a longitude of $\partial N(K)$, then K is a trivial knot, a contradiction. If ∂D winds around $\partial N(K)$ longitudinally more than once, then a punctured lens space is in S^3, a contradiction. This proves $\partial D \cap A_1 \neq \emptyset$. Similarly, $\partial E_i \cap A_2 \neq \emptyset$ $(i = 1, 2)$.

If ∂D meets A_1 in an essential loop, then a punctured $S^2 \times S^1$ is in S^3, a contradiction. Therefore $\partial D \cap A_1$ has no essential loops. Similarly $\partial E_i \cap A_2$ $(i = 1, 2)$ has no essential loops.

If any of $\partial D \cap T_{1,K}$, $\partial D \cap A_1$, $\partial E_i \cap T_{2,K}$, $\partial E_i \cap A_2$ $(i = 1, 2)$ has inessential arcs, we can remove them by isotopy. So we may assume that all the intersections are essential arcs. □

Label the arcs $\partial D \cap T_{1,K}$, $\partial D \cap A_1$ of ∂D with K, t_1, respectively. Also label the arcs $\partial E_i \cap T_{2,K}$, $\partial E_i \cap A_2$ with K, t_2, respectively. Let us assume that t_1 intersects $E_1 \cup E_2$ transversely in n points (possibly n can be zero) and number the meridian disks $N(K \cup t_1) \cap \text{int}(E_1 \cup E_2)$ of t_1 consecutively along $N(K \cup t_1)$ (Fig. 2).

Since D does not intersect t_2 by hypothesis, $D \cap (E_1 \cup E_2)$ consists of loops and properly embedded arcs in D. If there is a loop component of $D \cap (E_1 \cup E_2)$ in D, by cutting and pasting $E_1 \cup E_2$ along the innermost disk in D, we can reduce $|D \cap (E_1 \cup E_2)|$.

So we may assume that $D \cap (E_1 \cup E_2)$ consists of properly embedded arcs in D. An endpoint of an arc of $D \cap (E_1 \cup E_2)$ which is in an arc of ∂D labelled t_1 corresponds to a meridian disk $N(K \cup t_1) \cap \text{int}(E_1 \cup E_2)$ of t_1. Label that endpoint with the number given to the corresponding meridian disk. Fig. 3 shows the intersection $D \cap (E_1 \cup E_2)$ and $D \cap E_i$ $(i = 1, 2)$ on D and E_i $(i = 1, 2)$, respectively.

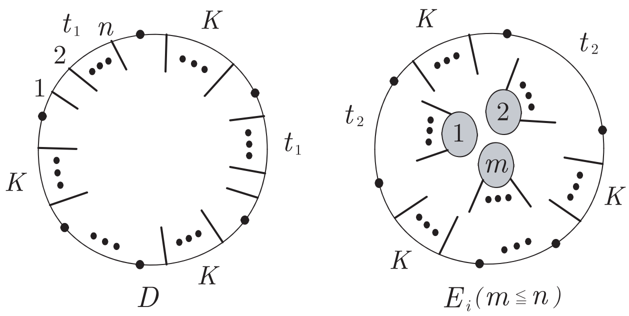

Figure 3. $D \cap (E_1 \cup E_2)$ and $D \cap E_i$ $(i = 1, 2)$ with labels

Lemma 2.2. *Any arc of ∂D labelled K whose endpoints are in the same puncture of $T_{1,K}$ has (nonempty) intersection with $E_1 \cup E_2$.*

Proof. Suppose an arc a labelled K whose endpoints are in the same puncture of $T_{1,K}$ has no intersection with $E_1 \cup E_2$. Let b be the subarc of the puncture of $T_{1,K}$ joining two endpoints of ∂a (Fig. 4). Then $a \cup b$ is a closed curve.

If $a \cup b$ bounds a disk Δ in $\partial N(K)$, then Δ should contain the other puncture since a is an essential arc in $T_{1,K}$. Then the number of intersection points of (one puncture of $T_{1,K}) \cap \partial D$ and (the other puncture of $T_{1,K}) \cap \partial D$ differs by two, a contradiction. Hence $a \cup b$ is an essential loop in $\partial N(K)$.

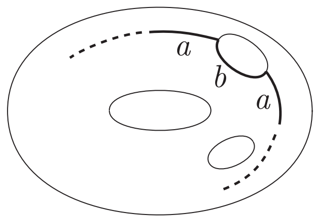

FIGURE 4. $a \cup b$ bounds a disk

Cut the genus two handlebody $H_2 = \overline{S^3 - N(K \cup t_2)}$ by $E_1 \cup E_2$. Then we get a 3-ball. Since $a \cup b$ does not intersect $E_1 \cup E_2$, it bounds a properly embedded disk in the 3-ball. If $a \cup b$ is a meridian of $\partial N(K)$, a punctured $S^2 \times S^1$ is embedded in S^3, a contradiction. If $a \cup b$ is a longitude of $\partial N(K)$, then K is a trivial knot, a contradiction. If $a \cup b$ winds around $\partial N(K)$ longitudinally more than once, then a punctured lens space is embedded in S^3, a contradiction. □

Consider an outermost arc a of $D \cap (E_1 \cup E_2)$ in D. Let Δ be the outermost disk of D corresponding to a. Without loss of generality, we may assume that $a \subset D \cap E_1$. Let $\partial a = \{p, q\}$. There are several cases according to the labels of the arcs of ∂D containing p and q.

Case 1. p and q are in one arc labelled t_1, and the two numbers labelled to p and q are i and $i+1$, respectively (Fig. 5).

Pushing E_1 along Δ removes a, so we can reduce $|D \cap (E_1 \cup E_2)|$.

Case 2. p and q are in one arc labelled K (Fig. 6).

By cutting and pasting E_1 along Δ, we can get two disks E_1' and E_1''. By pushing slightly, we can make E_1' and E_1'' disjoint from E_1 and E_2.

If E_1' (respectively E_1'') is isotopic to E_1, we can reduce $|D \cap (E_1 \cup E_2)|$ by replacing E_1 with E_1' (respectively E_1''). This occurs when E_1'' (respectively E_1') is ∂-parallel.

Suppose E_1' and E_1'' are not ∂-parallel and not isotopic to E_1. If E_1' (respectively E_1'') is isotopic to E_2, then E_1'' (respectively E_1') should be isotopic to one of the two in Fig. 7 since it cannot happen that both E_1' and E_1'' are

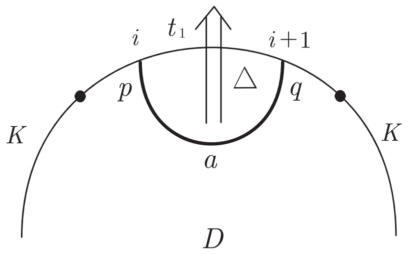

FIGURE 5. p and q are in one arc labelled t_1

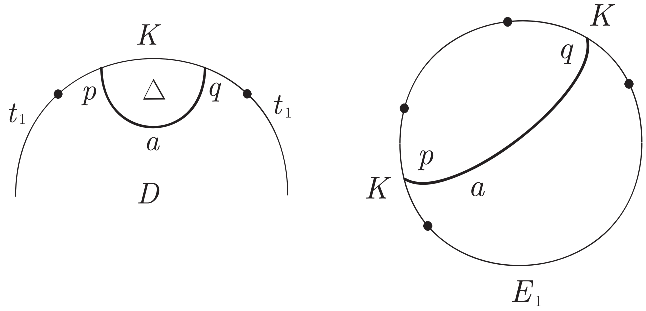

FIGURE 6. p and q are in one arc labelled K

isotopic to E_2. But we can observe that the left one in Fig. 7 cannot happen and in the right one in Fig. 7, $E_1'' \cup E_2$ (respectively $E_1' \cup E_2$) cuts H_2 into a 3-ball. Therefore $|D \cap (E_1 \cup E_2)|$ can be reduced by replacing E_1 with E_1'' (respectively E_1').

Now we consider the case that none of E_1' and E_1'' is isotopic to E_i $(i = 1, 2)$. Since E_1 is non-separating, at least one of E_1' and E_1'', say E_1', is non-separating and it is again the right one in Fig. 7. Then we can reduce $|D \cap (E_1 \cup E_2)|$ by replacing E_1 with E_1'.

In any case, we can see that the replaced E_1 and E_2 satisfy $\partial T_{1,K} \cap E_i = \emptyset$ $(i = 1, 2)$.

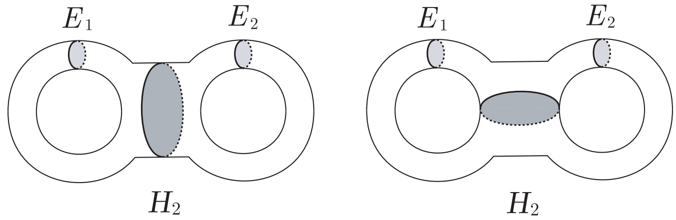

FIGURE 7. Two essential disks in H_2 not isotopic to E_i $(i = 1, 2)$

Case 3. p is in an arc labelled K and q is in an adjacent arc labelled t_1 (Fig. 8).

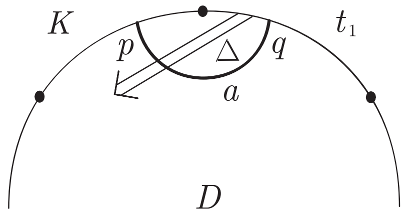

FIGURE 8. p is in an arc labelled K and q is in an adjacent arc labelled t_1

Note that the number labelled to q is 1 or n. Suppose the number labelled to q is 1. Then the meridian disk of t_1 with label 1 cuts $N(t_1)$ into two parts. Slide one of the parts adjacent to Δ along and then off Δ. Then $|D \cap (E_1 \cup E_2)|$ is reduced. It is similar in the case that the number labelled to q is n.

Remove all possible outermost arcs of Case 1, Case 2 and Case 3. We'll show that $t_1 \cap (E_1 \cup E_2) = \emptyset$.

Lemma 2.3. *Let an arc a of $D \cap (E_1 \cup E_2)$ divide D into two subdisks Δ_1 and Δ_2. Let $\partial a = \{p, q\}$. Let p_1 be one of the endpoints of the arc of ∂D containing p that belongs to Δ_1 and q_1 be one of the endpoints of the arc of ∂D containing q that belongs to Δ_1 (Fig. 9). Then there exists no arc a with the following properties.*

- *p and q are in different arcs of ∂D labelled t_1 and both p and q have the same labelled number, say m.*
- *p_1 and q_1 belong to different punctures of $\partial T_{1,K}$.*

Proof. Suppose such an arc a exists. Without loss of generality, we may assume that $a \subset E_1$. The meridian disk of $N(t_1) \cap E_1$ labelled m divides $N(t_1)$ into two parts A and B. Let A be the part containing p_1 and B be the part containing q_1. If we move from p to q along the arc a, we can see that the position of two parts A and B on both sides of the meridian disk labelled m is exchanged. This is a contradiction, hence there exist no such arc a. □

If $t_1 \cap (E_1 \cup E_2) \neq \emptyset$ after removing all possible outermost arcs of Cases 1, 2 and 3, an outermost arc a with $\partial a = \{p, q\}$ would be like this—p is in an arc labelled t_1, q is in another arc labelled t_1 and there is one arc labelled K between them. Furthermore, the number labelled to p should be 1 and the number labelled to q should be n ($n \geq 2$), by Lemma 2.2 and Lemma 2.3.

If $n > 2$, consider only the arcs of $D \cap (E_1 \cup E_2)$ in D which has at least one endpoint with labelled number 2 or $n - 1$. Then among them, an outermost

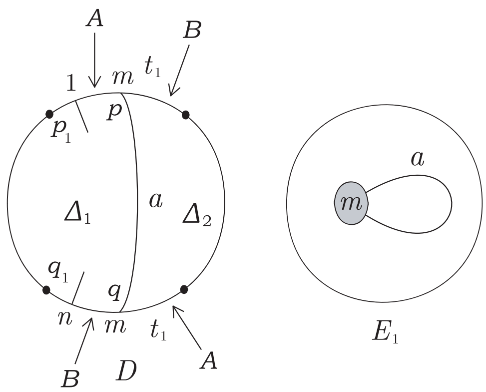

FIGURE 9. This kind of intersection does not exist

arc a with $\partial a = \{p, q\}$ would be such that p and q with labelled number 2 and $n-1$ respectively ($n \geq 4$) are in different arcs labelled t_1 and there is one arc labelled K between them, and an arc whose endpoints have labelled number 1 and n is nested, by Lemma 2.2 and Lemma 2.3. Continuing in this way, there are some nested arcs as in the Fig. 10 and $n = 2m$, an even number.

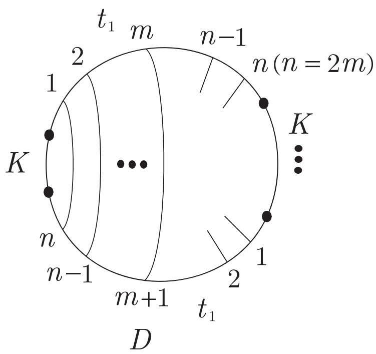

FIGURE 10. m nested arcs ($n = 2m$)

Now we concentrate on the arcs of $D \cap (E_1 \cup E_2)$ in D which has endpoint with labelled number m or $m+1$. The claim is that such an arc has one endpoint with labelled number m and the other $m+1$. The idea is that we consider only the arcs which seem to be obstacles and consider an outermost arc among them. The collection Γ of such arcs are arcs with

- each of its endpoints is in an arc labelled K, or
- one endpoint has labelled number i, and the other has labelled number not equal to $n-i+1$ or is in an arc labelled K ($1 \leq i \leq n$).

Let a be an outermost arc among the arcs in Γ with $\partial a = \{p, q\}$ and Δ be the outermost disk corresponding to a. Note that an arc not belonging to Γ has one endpoint with labelled number i and the other $n - i + 1$ $(1 \leq i \leq n)$.

Case I. p and q are in the arcs labelled K.

Since we removed all outermost arc of Cases 1, 2 and 3, there are at least two arcs labelled t_1 in $\partial\Delta$, and at least one arc labelled K having no intersection with $E_1 \cup E_2$ in $\partial\Delta$. Remember that an arc labelled K with no intersection with $E_1 \cup E_2$ has endpoints in different punctures of $\partial T_{1,K}$ by Lemma 2.2. Cut D by $E_1 \cup E_2$. Then we can obtain a disk like the one in Fig. 11 in a 3-ball B. The boundary of the disk is an alternating sequence of arcs with endpoints labelled m and $m + 1$ and arcs in ∂B, and the appearance of m and $m + 1$ is alternating. But this is a contradiction because there cannot be a disk with its boundary running on the neighborhood of a curve longitudinally more than once in S^3, which results in a punctured lens space in S^3.

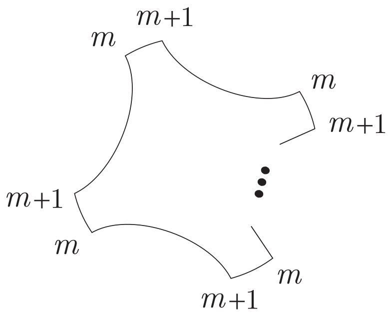

FIGURE 11. One of the disks D cut by $E_1 \cup E_2$

Case II. p is in an arc labelled t_1 and q is in an arc labelled K.

As in the Case I, consider the disk which has labels m and $m + 1$ on the boundary as in the Fig. 11. If each of the labels m and $m+1$ appears only once in the boundary, then the disk is the one in Fig. 5 of Case 1, which contradicts that we removed all the arcs of Cases 1, 2 and 3. If each of m and $m + 1$ appears more than once, it is a contradiction by the arguments of Case I.

Case III. p has labelled number i and q has labelled number not equal to $n - i + 1$ $(1 \leq i \leq n)$.

If there is only one arc labelled K in $\partial\Delta$, the numbers on the arcs labelled t_1 of $\partial\Delta$ would not be matched correctly. Otherwise, there is at least one arc of $D \cap (E_1 \cup E_2)$ in Δ with one endpoint labelled m and the other $m+1$. Then considering the disk like the one in Fig. 11, we get a contradiction by Case I and Case II.

Now we may assume $t_1 \cap (E_1 \cup E_2) = \emptyset$. Let an arc a with $\partial a = \{p, q\}$ be an outermost arc of $D \cap (E_1 \cup E_2)$ in D and Δ be the outermost disk corresponding to a. Obviously p and q are in the arcs labelled K. Cut D by $E_1 \cup E_2$. If there is at least one arc labelled K having no intersection with $E_1 \cup E_2$ in $\partial\Delta$, there are at least two arcs labelled t_1 in $\partial\Delta$. Then we get a contradiction on Δ by similar arguments as in the Case I.

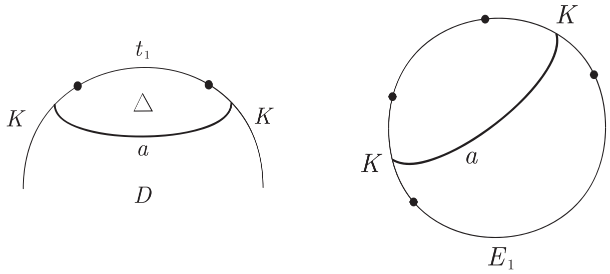

FIGURE 12. Stabilizing disk for t_1

If there is no arc labelled K having no intersection with $E_1 \cup E_2$ in $\partial\Delta$, there is one arc labelled t_1 in $\partial\Delta$. By cutting and pasting $E_1 \cup E_2$ along Δ, we get a stabilizing disk for t_1 (Fig. 12). This completes the proof of Theorem 1.3.

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