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How to develop the ability of proof methods?[1]

Behnoodi, Maryam*
Department of Human Environmental Science, Faculty of Human Development, Kobe University, Kobe 657-8501, Japan; Email: Maryam.behnoodi@gmail.com

Takahashi, Tadashi
Department of Human Environmental Science, Faculty of Human Development, Kobe University, Kobe 657-8501, Japan; Email: takahasi@kobe-u.ac.jp

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The purpose of this study is to describe how dynamic geometry systems can be useful in proof activity; teaching sequences based on the use of dynamic geometry systems and to analyze the possible roles of dynamic geometry systems in both teaching and learning of proof. And also dynamic geometry environments can generate powerful interplay between empirical explorations and formal proofs. The point of this study was to show that how using dynamic geometry software can provide an opportunity to link between empirical and deductive reasoning, and how such software can be utilized to gain insight into a deductive argument.

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ZDM Classification: C30, C60
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INTRODUCTION

In recent years there has been a call to reform mathematics education to produce what the NTCM calls "mathematical literacy" for all students. One of the NTCM's Standards involves the use of problem solving as a method of learning mathematics. Proof is valuable in the school curriculum because it is instrumental in the cognitive processes required for successful problem solving.

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* Corresponding author

EMPIRICAL JUSTIFICATION

In the usage of dynamic geometry systems in teaching sequences and possible roles in both teaching and learning of proof, reviewing the Friedman's argument in Dynamics of Reason bring a good knowledge in our research.

1. There are three types of principles comprising scientific theory: mathematical, constitutive and physical.
2. The constitutive framework defines space of empirical possibilities (set of statements that could empirically be True or False) by setting general correspond-dence between the math part and the concrete empirical phenomena in such a way that empirical laws could have empirical meaning.
3. The constitutive framework, together with the mathematical part which is non-empirical, provides the very conditions for a scientific statement to be empirically 'True or False', therefore
4. The constitutive framework could not be empirically justified,
5. The constitutive framework is nevertheless justified, hence
6. The constitutive framework is justified a priori.

The constitutive framework is built up of a priori revisable principles. What are the possible types of revision available? It will be concentrated on the role of the a priori revision for reasons that would emerge further.

For example, we could take the measurements of the light paths in a space as an empirical revision of the supposed a priori principle that "The Euclidean geometry is the only true one of our world." The experiential revision demonstrates that the claim is at least partly false; in the sense that even if it is true of some regions of the world, it is not true of all of them. However, our reasons holding the axioms that Euclidean geometry is true, could not be experiential. – we do have only intuitive reasons to hold them true, we do not have and we could not have any experience of infinite extension of a line, for example, we could draw this only in our imaginative space. Upon careful investigation, it becomes clear that we have had presupposed some more primitive properties of the space in which we draw the lines (the space of our imagination), namely, curvature of the space (with value zero in the case of Euclidean geometry). We have two lines – to claim that these are properties with a priori or a posteriori origin. Convincing response to this has been given by Lukas and Cassirer that we do not have any option and actually could not change the primitive properties of our phenomenal space since we will lose our ability to imagine and therefore make use of geometrical invariants upon the most simple and primitive group transformations (like reflection, displacement and rotation), which consequently will not allow us to form the concept of a material object in the way we have it now. In this sense, a posteriori line of the origin of Euclidean geometry does not look promising. The only sense in which the empirical statements are empirical is the justificatory sense. Justificatory here would mean that the employment of the statements (principle) couldn't be justified from experience but only a priori. Further distinction is the one that divides statements with respect to the origin of their justification:

- Type C – those are justified solely from experience and cannot be justified without it. This has the consequence that the only reason for us to hold these statements true could be experiential justification. These are experiential statements.
- Type B – those could be justified (in equal or different degree) both or separately by experience and by non-experiential consideration. These are joint justificatory statements.
- Type A – those could not be justified by experience and that are still held true on some (non-experiential) grounds – these statements will owe their truth solely on rational, *i.e.*, a priori grounds. These statements will be meant as a priori statements.

HOW TO DEVELOP THE ABILITY OF PROOF METHODS?

As we know geometry is an important part of mathematics curriculum. However, students are not demonstrating strong conceptual knowledge of this subject. My experiences as a geometry teacher have shaped my beliefs about the teaching and learning of geometry.

I believe that it is important for students to understand why a particular postulate theorem is true. Formal proofs are important. Discovery and exploration in a dynamic environment can be an important first step in the proving process. I have had positive experiences using dynamic geometry software as a mathematics teacher.

Interactive geometry software has variety of functions and purposes. It is used in conjecture writing and proof as a problem-solving tool, so that geometric constructions could perform the geometric transformations as a demonstration tool. In this case, we concentrate on proof.

What is proof and don't you think that 'is this simple term so confusing to math educator and mathematics students? If a young child is asked to prove a mathematical fact, he/she would be happy to show many examples of how it works. This does not constitute a proof, but it is a step in right direction. If it were to be asked to a high school student or a first year college student to do a proof, it will most likely meet groans and feelings of disgust.

Students at this age have probably encountered proof in a geometry class where they were expected to follow a strict format without much freedom to express proofs on their own.

But, if it were to be asked to a mathematician about proof, they would begin to tell you how beautiful proof in mathematics can be.

Proof has always been an interesting topic to me. In high school geometry class, I either did not understand proof. I felt, like many other students, frustrated by the fact that we were asked to prove theorems that the textbook had already told us were true. It was as though the instructor was playing magical games on the chalkboard and all of a sudden we had a proof. However, as time passed by, I began to see the beauty of proof parallel to learning computer software. Then, mathematical induction introduced me to the power of proof.

For example, a group of students who had taken one year of high school geometry were asked to use a straight-edge and compass to show how to construct the circle that is tangent to two intersecting lines, with one point of tangency being a given point *P* on one of the lines. In the same problem session, the students had already proved that the center of a circle tangent to two given lines lies at the intersection of the angle bisector, and the angle was formed by the lines and the perpendiculars through the lines at the points of tangency. Despite this finding, 30 percent of students proposed that the center of the circle was the midpoint of the segment that joined P and its counterpart on the other line.

Evidently, either the students' proof activity did not help them understand or the knowledge itself was classified and was not accessible as a part of constructions. Even after finding or learning a correct proof for a statement, High school students still remained surprises was possible. For most geometry students, deduction and experimental methods are separate domains with different ways to establish correctness.

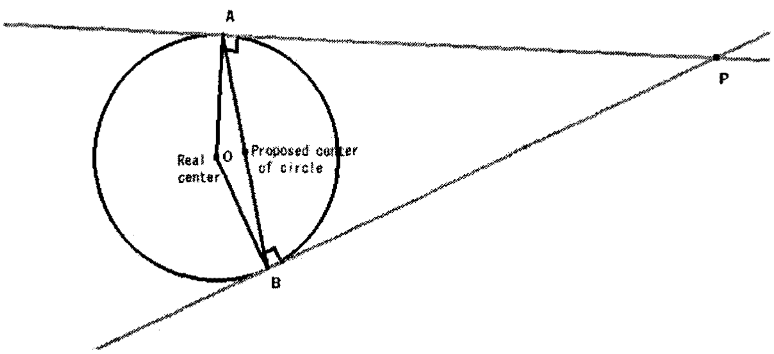

Figure 0. After learning a correct proof for statement, misunderstanding may remain.

HOW TO DEVELOP THE CONCEPTS OF PROOF

How do students develop the ability to prove ideas formally? Piaget described how this development occurs without considering curricula; Van Hiele (*cf.* Senk, 1989) analyzed progress with curricula. Both views shed insight into how students can develop the ability to use proof and to judge where in this development our students might be.

Piaget's stages

According to Piaget, developing the ability to construct a proof as logical necessity passes through several stages.

Stage 1. The child's thinking is not reflective, unsystematic, and illogical. Various pieces of data collected or examples examined are treated as separate and/or unrelated events. Exploration proceeds randomly without a plan.

Stage 2. Students not only use their experimental results to make predictions but try to justify their predictions. They anticipate results in their searches for information and think logically only about the premises that they believe in.

Stage 3. Does the student progress beyond a belief that something is simply always true to making a logical conclusion that it must be true? The student is capable of formal deductive reasoning based on some assumptions and so is capable of operating explicitly on a mathematical system.

How do students progress through the stages? At what point does the need for verification arise? "Surely it must be the shock of our thought coming into contact with that of others, which produces doubt and the desire to prove...Proof is the outcome of the argument" (Piaget, 1928, p. 204). Owing to contact with others, the student becomes ever more conscious of his/her own thought, becoming "conscious of his/her own thought, and becoming conscious of the definitions of the concepts he/she is using" and acquiring "a partial aptitude for introspecting his/her own mental experiments" (p. 243). The student becomes increasingly able to take the perspective of others. Finally, with the onset of formal thought, experiments ideas are tested for by mental reproducing sequences of events as they occur or are imagined to occur are replaced by logical experiments, on which the actual construction is reflected and evaluated for inconsistencies.

Piaget's theory, on the one hand, describes how thinking in general progresses from being non-reflective and unsystematic to empirical, and finally to logical-deductive. The theory of van Hiele (1989), on the other hand, deals specifically with geometric thought as it develops through several levels of sophistication under the influence of a school curriculum (Clements & Battista, 1996)

The levels of van Hiele

The van Hiele's theory of geometry learning explains geometry understanding as a series of more and more sophisticated ways to reason geometrically. The theory is known for its use in guiding K-12 geometry instruction. The van Hiele's levels are numbered differently in various sources. However, all references to levels are specific to this article's numbering system.

Level 1 visual: Students reason about geometric shapes on the basis of their appearance and the visual transformations that they perform on images of these shapes.

Level 2 descriptive/analytic: Students reason experimentally; they establish properties of shapes by observing, measuring, drawing, and making models.

Level 3 abstract/relational: Students reason logically. They can form abstract definitions, distinguish between necessary and sufficient conditions for a concept, and understand and sometimes even present logical arguments.

Level 4 formal deduction: Students reason formally by logically interpreting geometric statements, such as axioms, definitions, and theorems.

Level 5 rigor/meta-mathematical: Students reason formally about mathematical systems rather than just within them.

De Villiers (1987) suggests that deductive reasoning in geometry first occurs at level 3 when the network of logical relations among properties of concepts is established. He claims that because students at level 1 or 2 do not doubt the validity of their empirical observations, formal proof is meaningless to them; they see it as justifying the obvious. Van Dormolen (1997) argues that at the visual level, single cases are justified and conclusions are restricted to the specific example which the justification is given, so to say, for a particular rectangle. At the descriptive/analytic level, justifications and conclusions may be made simply for specific cases but referred to collections of similar objects such as a class of rectangles. Only following level 3 can students justify statements by forming arguments that conform with accepted norms, that is, give formal proofs.

Research by Senk (1989) supports the notion that a proof-oriented geometry course requires thinking at least at level 3 in the van Hiele's hierarchy. She found that less than 22% of students below level 3, but 57%, 85%, and 100% at levels 3, 4, and 5, respectively, mastered proof writing. Thus, at van Hiele's level 4, students master proof, with level 3 being a transitional level. Unfortunately, over 70 percent of students begin high school geometry at level 1 or below, and only those students who enter at level 2 or higher become competent with proof by the end of the course (Senk 1989; Shaughnessy & Burger, 1985).

In summary, both Piaget's and van Hiele's theories suggest that students must pass through lower levels of geometric thought before they can attain higher levels and that this passage takes a considerable amount of time. The van Hiele theory suggests that instruction should help students gradually progress through lower levels of geometric thought before they begin a proof-oriented study of geometry. Because students cannot bypass levels and achieve understanding, premature dealing with formal proof can lead students only to attempts at memorization and to confusion about the purpose of proof. Furthermore, both theories suggest that students can understand and explicitly work with axiomatic systems only after they have reached the highest levels in both hierarchies. Thus, the explicit study of axiomatic systems is unlikely to be productive for the vast majority of students in high school geometry.

Computer software one of the solutions for problem understanding

Students generally have a problem understanding that different representations can refer to the same mathematic expression. In addition, students may have a stronger understanding of one representation. Using applications that show multiple representations at once can help students make a connection between different representations. Being able to change one representation and seeing its effects in the other representations would be even more powerful.

Consistent with the alternatives to axiomatic approaches, the focus of computer construction programs such as The Geometric Supposer software series, The Geometer's Sketchpad (*cf.* Jackiw, 1991; Albrecht *et al.*, 2001) and Cabri 3-D and many other mathematics software facilitate students' making and testing conjectures. The Supposer programs allow students to choose a primitive shape, such as a triangle or quadrilateral, and to perform measurement operations and geometric constructions on it. These record sequences of constructions perform on shapes and can repeat the action on other shapes. This application allows students to develop mini-programs that draw shapes for geometry. In this way, parameters can be used or changed, and different shapes of the same class are drawn. This enables students to understand that the rules of geometry apply to all shapes in a certain class, not just to the one triangle drawn in a book. All variations of a shape that is asked for are displayed at once, solidifying the concept.

Research has demonstrated the effectiveness of such construction programs. In one evaluation showed students who used the Supposer software that they performed (as well as or better than those who didn't use the Supposer) counterparts on geometry examinations (Chazan, 2002). Students' learning went beyond standard geometry content as they invented definitions, made conjectures, posed and solved significant problems, and devised original proofs. Making conjectures was not easy for students, but eventually nearly all students made conjectures and justified their generalizations. The Supposer-based activities helped students move away from considering measurement evidence as proof. Unlike the case with textbook theorems which lead students to assume as true "because they are in the book," students believed that theorems generated with the Supposer software needed to be proved before they could be accepted as true. Students using this software also made gains in understanding diagrams. They were flexible in their approach to diagrams, treated a single diagram as a model for a class of shapes, and were aware of this model for a class of shapes and that this model contained characteristics, not representative of the class (Yerushalmy & Chazan, 1988).

Recommendations for classroom

The high school geometry curriculum should be appropriate for the various thought levels through which students pass in a year-long study of the topic. It should guide students to learn about significant and interesting concepts. It should permit students to use visual justification and empirical thinking, because such thinking is the foundation for higher levels of geometric thought. The curriculum should require students to explain and justify their ideas. It should encourage students to refine their thinking, gradually leading them to understand the shortcomings of visual and empirical justifications so that they discover and begin to use some of the critical components of formal proof. But it is the meaningful justification of ideas to establish their validity while making sense in the mathematics classroom, and that should be a major goal of the geometry curriculum. Formal proof is appropriate only to the extent that students can use it as a way to justify ideas meaningfully. We shall describe an example to illustrate how geometric ideas can be investigated at increasing levels of sophistication for thinking about and justifying ideas.

Locus

The purpose of this snapshot is to provide an example of how Dynamic Geometry can support the development of the proof where empirical evidence is used as the source of insight and inspiration for deductive argument. In the discussion that follows, we proceed through a number of steps:

- Experimental Results (In which I explore the problem empirically using Cabri Geometry).
- Towards a Proof (In which I use Cabri Geometry to make observations that will lead to deductive proof).

- Proof (In which proved results about the locus deductively using all the experimental results acquired from Cabri Geometry).
- Some generalizations (In which I extend the proof using additional observations supported by Cabri Geometry)

The Problem

What is the locus of points determined by the intersections of the medians of a triangle as one of the vertices of the triangle traverses a circle?

Figure 1 is the representation of the problem. Which can be stated now as "What is the locus of G when A traverses the circle O?" When this problem was presented to students in my course, they usually conjectured; they could not easily justify their answers nor could they explicate the features of the locus.

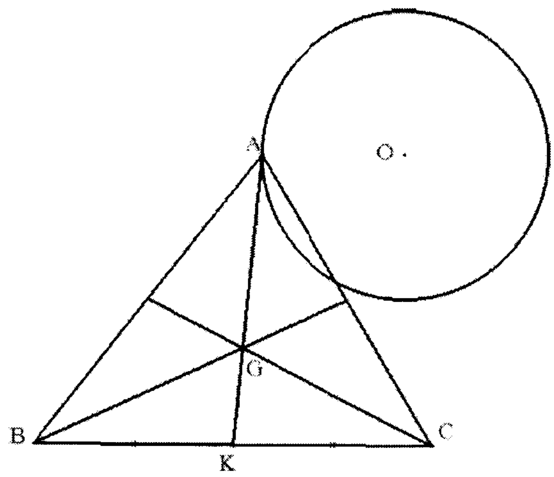

Figure 1. Representation of the problem

Experimental Results

Before deductive acquiring a proof about the locus of point *G*, it is decided to trace out the locus of *G* and to make some measurements on it. To do this, it is decided to turn to dynamic geometry, in the hope that it might help us to obtain the locus of point *G* and gather some information about it.

It is found the locus of point *G*, in Cabri, by constructing the centroid of triangle *ABC*, which was done by determining a point *A* as the traversing vertex and by selecting point *G* as the locus point.

As seen in Figure 2, Cabri shows that the locus of the centroid (according to point *A*) is a circle whose radius is smaller than the radius of circle *O*. At this point, we may wonder whether there is a relationship between the radius of the locus circle and radius of the circle *O*. To determine this relationship (if one exists), we should find the center of the circle *O* and make some comparison between the two. We know the theorem that the perpendicular bisector of a chord passes through the center of a circle. Using this theorem, we can take two chords of the locus circle and draw their bisectors, knowing that they intersect with the center of the locus circle.

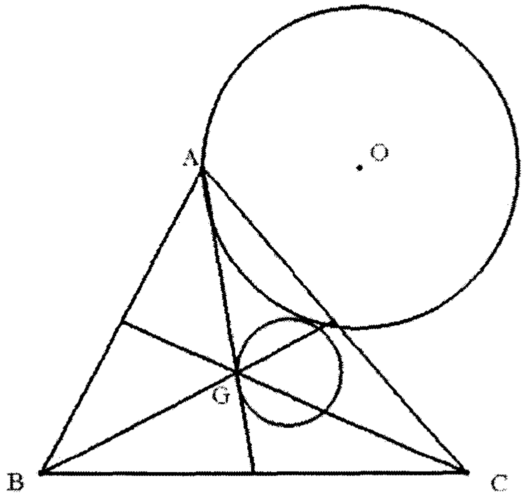

Figure 2. Cabri shows that the locus of the centroid (according to point *A*) is a circle

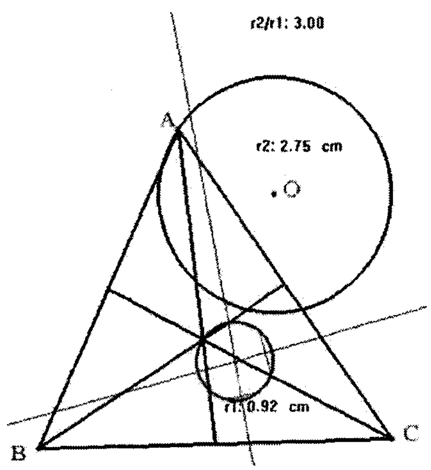

Figure 3. Finding a ratio between the data obtained by 'dragging'

By using this method and utilizing Cabri Geometry, it is determined the center of the locus circles as in Figure 3. The next step is to find a ratio between the data obtained by 'dragging' then it suggest that the length of the radius of circle *O* (r2) is three times than the length of radius of locus circle (r1).

Following a proof method

To get a deductive proof analytically, some observations regarding the structure of the configuration need to be made.

Observation 1: When the OK line is drawn, it is observed that the center *O* focus circle (*M*) is continually on this line (point *O, M, K* are collinear) as seen in Figure 4.

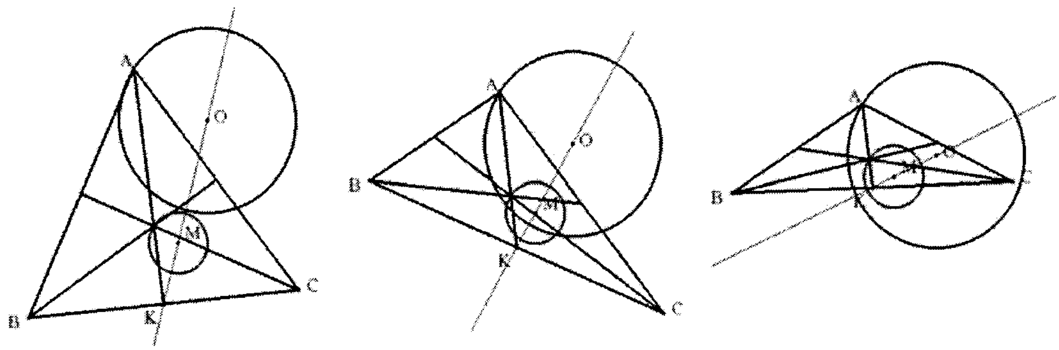

Figure 4. Point *O, M, K* are collinear

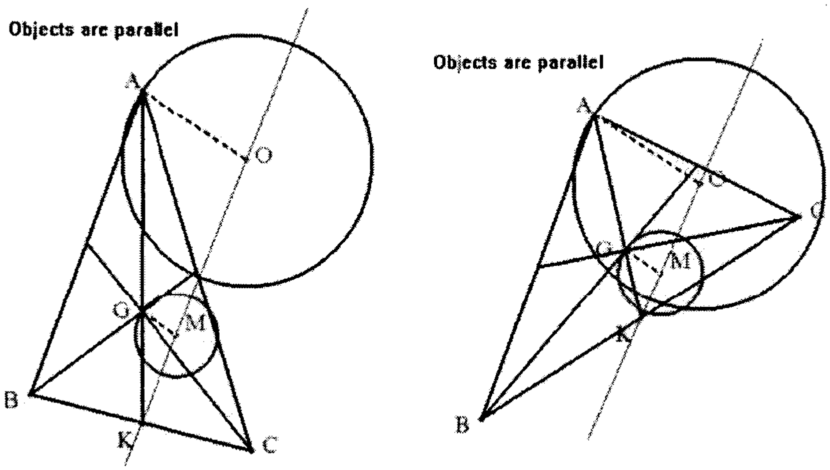

Figure 5. The segments *GM* and *AO* are parallel to each other

Observation 2: When the segments *GM* and *AO* was examined it is observed that these segments are parallel to each other as seen in Figure 5 it can be observed this result for different positions of triangle *ABC*.

Because segments *GM* and *AO* are parallel, triangles *GKM* and *AKO* are similar. Using this similarity and considering that point *G* is a centroid, we can write the following proportions; these observations are of course not a proof as they are based on empirical data. However, it is gained of some insight into aspects of the situation that enable to get a deductive proof, which became apparent after experimentation of the kind the dynamic geometry environments can support.

A Proof by software

Now it begins to prove the following results obtained using Cabri Geometry.

1. Why is the locus of point *G* (the centroid of triangle *ABC*) a circle?
2. Why is there 1/3 ratio between the radius of the locus circle and the center of the locus circle *O*?

It begins by drawing the segment *OK* (because it is observed that the center of the locus circle is on this line). Figure 6 shows this representation.

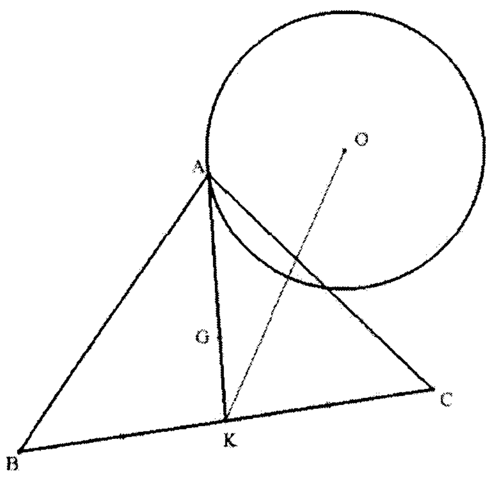

Figure 6. It is observed that the center of the locus circle is on *OK*

Subsequently, it draws segment *AO* and a parallel segment to *AO* from point *G* intersecting segment *OK* at point *P* as seen in Figure 7 (because previously it is observed that *GM* and *AO* are two parallel segments (See Figure 7).

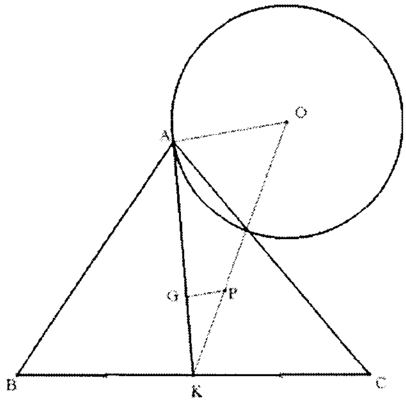

Figure 7. *GP* and *AO* are parallel because in figure 5 GM and *AO* were parallel

It will be seen in the next steps of the proof that the point *P* will be the center of the locus circle.

Since the point *G* is the centriod of triangle *ABC*, and triangles *GKP* and *AKO* are similar, Figure 8 suppose that *Á* is a special case for the point *A*.

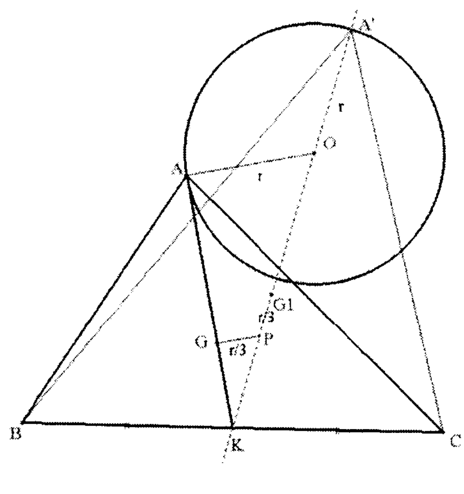

Figure 8. *G*1 is the centeriod of triangle *ÁBC*

$$\frac{\text{GP}}{\text{AO}} = \frac{1}{3} \text{ and } |\text{KP}| = \frac{\text{KO}}{3}$$

$$|\text{KG}_1 = \frac{\text{A'K}}{3} = \frac{r + |\text{KO}|}{3} = \frac{r}{3} + \frac{|\text{KO}|}{3} = \frac{r}{3} + |\text{KP}|$$

$$|PG_1| = \frac{r}{3}.$$

As seen in Figure 8 the point G1 is the centriod of this new triangle $A'BC$.

Now suppose that A'' is a new special case of point A as in Figure 9. Point G_2 is the centriod of this triangle $A''BC$. S

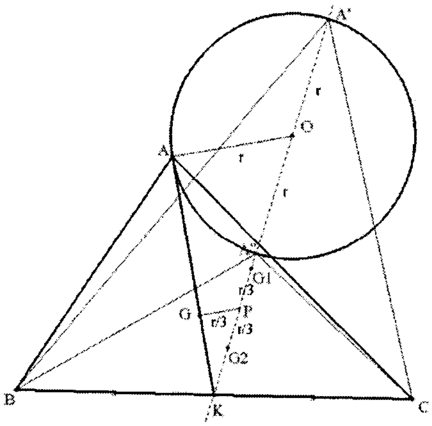

Figure 9. Point G_2 is the centriod of this triangle $A''BC$.

Now we know that G, G_1 and G_2 are equidistance from point P. Because three points determined only one circle, we can draw a circle P that pass through G, G_1 and G_2 as seen in Figure 10.

It obtained a circle, yet we do not know whether this circle is the locus of the point G. For this, it has to show that the centroid of triangle ABC is on the circle P for an arbitrary location of point A on the circle O.

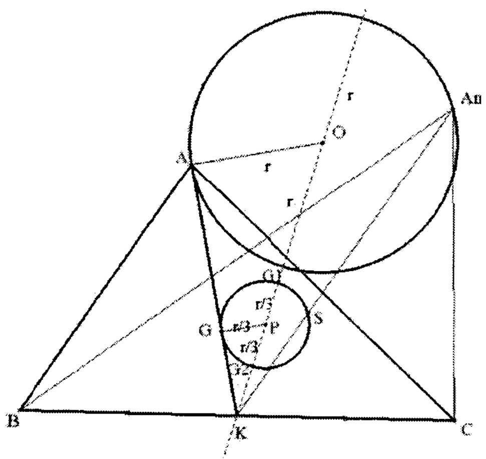

Figure 10. G, G_1 and G_2 are equidistance from point P

If it takes an arbitrary location for the point A, call it A'', on the circle O and draw segment $A''K$, this segment and the circle P intersect at a point S (see the Figure 11).

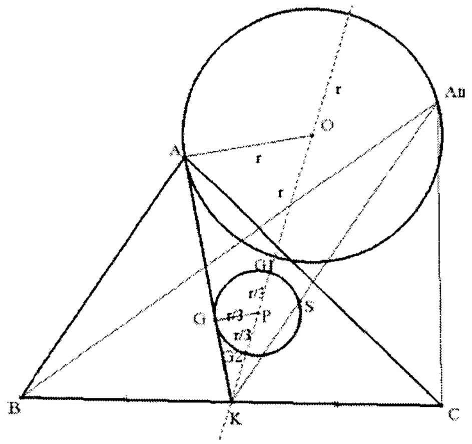

Figure 11. AK and the circle P intersect at a point S

It has to show that point S is the centroid of triangle $A''BC$. Next, it should draw segments PS and OA'' as seen in Figure 11. The segment $A''K$ and the circle P intersect at two different points. Because

$$\frac{|KP|}{|KO|}=\frac{1}{3}, \quad \frac{|PS|}{|OA''|}=\frac{1}{3}$$

and angle PKS is common in triangle KPS and KOA'', it is obvious that one of these two intersection points (for example S) must ensure the condition

$$\frac{|KP|}{|KO|}=\frac{1}{3}.$$

Therefore, S is the centriod of triangle KOA''. The locus of G determines a circle *QED*.

CONCLUSION

In typical use of dynamic geometry environments, attention tends not to be focused on proving and proof. But rather, it is focused on the software's potential in aiding the transition from particular to general cases, since specific instances can be easily varied by direct manipulation or text-based commands and the text "seen" on the computer screen. However, dynamic geometry environments can generate powerful interplay between empirical explorations and formal proofs as presented in this snapshot. It must be emphasized that the experiments presented herein didn't add new facts to the world of mathematics. Of course this was not the point. The point, rather, was to show that how using dynamic geometry software can provide an opportunity to link between empirical and deductive reasoning, and how such software can be utilized to gain insight into a deductive argument.

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